Reconstruction of Radar Reflectivity Densities in a Narrowband Regime

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Abstract

This paper is concerned with reconstruction problems arising in the context of radar signal analysis. The goal in radar is to obtain information about objects by emitting certain signals and analyzing the reflected echoes. In this paper, we shall focus on the narrowband model for radar echoes and on the case of continuously distributed objects D (reflectivity density). Following the approach of H. Naparst [14] and L. Rebolla et al. [16] we suggest to treat this problem by transmitting not just one signal but a family of signals. Indeed, a reconstruction formula can be derived, provided that the set of outgoing signals forms a frame.

Keywords: Radar, reflectivity density, narrowband regime, frames, wavelets.

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1 Introduction

In recent years, wavelet analysis has been very successfully applied to many problems in signal/image analysis/compression and also in numerical analysis. Moreover, since the pioneering work of Naparst [14, 15], it is well–known that the specific features of wavelets can also be very efficiently used to treat certain problems in the context of radar signal analysis.

There one tries to gain information about an object by analyzing waves reflected from it. In practice, it may very often happen that the received echo can be described by means of a certain *reflectivity density*. Consequently, one task is often to reconstruct the density from the received echo. In Section 2, it turns out that the reflectivity density can be interpreted as the wavelet transform of some echo where the role of the analyzing wavelet is played by the transmitted signal. However, this interpretation does not tell the whole truth. Unfortunately, the wavelet transform in not surjective and therefore it is only invertible on its image. To our knowledge, there exists no physical principle that guarantees that the density is in fact contained in the image of the wavelet transform. Inspired by these problems, Naparst [14, 15] was the first one who suggested not to transmit just *one* signal but a *family* of signals. In his fundamental work, Naparst primarily studied the case that the transmitted signals form an orthonormal basis. However, this assumption is very restrictive in practice. Therefore, quite recently, Rebollo–Neira, Platino and Fernandez–Rubio generalized Naparst's approach to the case of *frames* which is usually a much weaker setting, [16]. The present study is very much inspired by their results, however, we show that their approach can be applied to the *narrow–band regime*. In this regime we also present significantly shorter proofs of the basic reconstruction formulas.

This paper is organized as follows. In Section 2, we introduce the wide–band and the narrow–band regime. By the theory of groups both regimes can be related. Then, in Section 3, we introduce basics of frames. Finally, in Section 4 we establish a reflectivity density reconstruction formula for the narrow–band regime.

2 Radar Modeling

Suppose we want to find the location and velocity of an object, such as an airplane. One way to achieve this is to send out an electromagnetic wave in the direction of the object and observe the echo that is reflected to the source. As we shall now explain, a comparison between the outgoing signal and its echo allows an approximate determination of the distance R of the object and the radial velocity v along the line-of-sight. This is the problem of radar in its most elementary form. For a thorough treatment, we refer for instance to the classical book of Borden and Rihaczek [1, 17]. The wavelet–based analysis has been initiated by Naparst [14, 15]. In this paper, our major reference will always be the book of Kaiser [8].

To explain the basic radar setting, let us first assume that the object under consideration can be described as a point, moving with constant velocity v towards or away from a given source. The distance between object and source at time $t = 0$ is denoted by R. The emitted signal is denoted by $h(t)$, then the *wide-band model* for the received echo $f(t)$ is given by

$$f(t) \;=\; \sqrt{|s|}\; h(s(t-\tau)), \tag{2.1}$$

where the *Doppler scale factor* s is obtained from the speed of light c and the object velocity v as

$$s = \frac{c-v}{c+v}, \tag{2.2}$$

and the delay τ is determined by the distance R between the object and the source as

$$\tau = \frac{2R}{c-v}. \tag{2.3}$$

The so-called Doppler coordinates (s, τ) are in one–to–one correspondence to the desired values v and R. The multiplicative factor $\sqrt{|s|}$ is chosen such that the energy is conserved, i.e., we assume a perfectly reflecting object. For further information, the reader is referred, e.g., to [8, 9, 10].

In the presence of many objects the total echo is modeled as the superposition of the single echoes. More general, if we assume that we want to observe a reflecting continuum with varying reflectivity as described in Doppler coordinates by a *reflectivity density* $D(\tau, 1/s)$, then the total echo is given by

$$f(t) = \int_{\mathbf{R}} \int_{\mathbf{R}\setminus\{0\}} D(\tau, s)|s|^{-1/2} h\left(\frac{t-\tau}{s}\right) \frac{ds d\tau}{s^2}. \tag{2.4}$$

Consequently, the task is to reconstruct the density $D(\tau, s)$ from the received echo. To treat this problem, let us first remark that formula (2.4) can be reinterpreted in the context of wavelet analysis: In general, the *continuous wavelet transform* $W_\psi(F)$ of a function $F \in L_2(\mathbf{R})$ is given by

$$(W_\psi F)(a,b) := \int_{\mathbf{R}} F(x)\ |a|^{-1/2} \overline{\psi\left(\frac{x-b}{a}\right)} dx. \tag{2.5}$$

This transformation is well–defined, provided that the analyzing wavelet ψ satisfies the admissibility condition

$$C_\psi \;=\; \int_{\mathbf{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty, \tag{2.6}$$

where, as usual, the Fourier transform of a function $g \in L_1(\mathbf{R})$ is defined by

$$\hat{g}(\xi) = \int_{\mathbf{R}} g(x) e^{-i\xi x} dx.$$

The wavelet transform W_ψ is a multiple of an isometry whose inverse is given by the adjoint wavelet transform

$$\begin{aligned} F(x) &= W_\psi^* \left(W_\psi F(a,b)\right)(x) \qquad (2.7) \\ &= \frac{1}{C_\psi} \int_{\mathbf{R}} \int_{\mathbf{R}\setminus\{0\}} (W_\psi F)(a,b)\ |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) \frac{da}{a^2} db, \end{aligned}$$

see, e.g., [4, 8, 11] for details. Therefore a comparison of (2.7) with (2.4) yields the well–known and basic identity which links wide-band radar echoes to wavelet analysis, see e.g. [14, 13]: the echo f is identical with the inverse wavelet transform of the searched reflectivity distribution D where the transmitted signal h plays the role of the analyzing wavelet. This suggests to recover D by computing the wavelet transform of the echo f:

$$D(\tau, s) := \frac{1}{C_h} [(W_h f)(\tau, s)]. \tag{2.8}$$

However, the null space $\mathcal{N}$ of an inverse wavelet transform is non trivial. Hence by this procedure one can only recover the component of D which lies in the orthogonal complement of $\mathcal{N}$ or equivalently one can recover the component of D in the range of the wavelet transform W_h. To our knowledge, there exists no physical principle that guarantees that D is in fact contained in the range of W_h, so that (2.8) only describes one part of the desired density D. Therefore the general solution reads as follows.

Theorem 2.1 *The most general solution $D(s,\tau)$ in $L_2(ds_0 d\tau_0/s_0^2)$ with respect to the total echo* (2.4) *is given by*

$$D(s,\tau) = D(s,\tau)^{\|} + D(s,\tau)^{\perp}\ ,$$

where $D(s,\tau)^\perp$ *is any function in* $\mathrm{rg}(W_h)^\perp$*, i.e.,*

$$\int\int R(s,\tau,s_0,\tau_0)D(s,\tau)^\perp \frac{ds_0 d\tau_0}{s_0^2} = 0 \ ,$$

where $R(s,\tau,s_0,\tau_0) = \langle h_{s,\tau}, h_{s_0,\tau_0}\rangle$ *is the so-called wide-band self–ambiguity function of* h*.*

Inspired by these problems, Naparst [14, 15] was the first one who suggested not to transmit just *one* signal but a *family* of signals. In his fundamental work, Naparst primarily studied the case that the transmitted signals form an orthonormal basis. However, this assumption is very restrictive in practice. Therefore, quite recently, Rebollo–Neira, Platino and Fernandez–Rubio generalized Naparst's approach to the case of transmitting a *frame* of signals, which is a much weaker restriction, [16].

However, the wide-band model, which describes echoes for arbitrary signals h, can be simplified for most real–life situations. The commonly used *narrow–band* approximation deals with signals of the form

$$h(t) = e^{-i\omega_c t}\eta(t) \ ,$$

where the carrier frequency ω_c is assumed to be much larger then the comparatively narrow-banded frequencies of the modulation function η.
Furthermore, most objects of interest in radar travel with a speed much smaller than light. Thus $|v|/c << 1$ and

$$\tau \sim \frac{2R}{c}. \tag{2.9}$$

Now we have to treat the positive and negative frequencies of η separately. To this end, we introduce the projectors $Q_{\mathcal{P}}$ and $Q_{\mathcal{R}}$ by

$$(Q_{\mathcal{P}}\eta)^\wedge(\omega) := \hat{\eta}_{\mathcal{P}}(\omega) = \hat{\eta}(\omega)\chi_{[0,\infty)}(\omega), \qquad (Q_{\mathcal{R}}\eta)^\wedge(\omega) := \hat{\eta}_{\mathcal{R}}(\omega) = \hat{\eta}(\omega)\chi_{(-\infty,0]}(\omega). \tag{2.10}$$

Removing the carrier frequency ω_c from both, the signal η and the echo f, and neglecting time independent scale factors leads to the standard narrowband model for the echo f of a single moving object

$$f(t) = c\left\{\eta_{\mathcal{P}}(t-\tau)e^{-i\phi t} + \eta_{\mathcal{R}}(t-\tau)e^{+i\phi t}\right\} \ ,$$

where $\phi = 2\omega_c v/c$, see e.g. the classical textbooks [1, 17, 18] and [8] for details. The variables (τ,ϕ) are called the narrowband Doppler coordinates.

Consequently, the narrowband model for the echo produced by a reflectivity density $D_{NB}(\phi, \tau)$ is given by

$$f(t) = \int_{\mathbf{R}} \int_{\mathbf{R}} \left\{ \eta_{\mathcal{P}}(t-\tau)e^{-i\phi t} + \eta_{\mathcal{R}}(t-\tau)e^{+i\phi t} \right\} D_{NB}(\phi, \tau) d\phi d\tau \ . \tag{2.11}$$

We may decompose the space $L_2(\mathbf{R})$ as

$$L_2(\mathbf{R}) \simeq L_{2,\mathcal{P}}(\mathbf{R}) \oplus L_{2,\mathcal{R}}(\mathbf{R}) \tag{2.12}$$

where

$$L_{2,\mathcal{P}}(\mathbf{R}) \ := \ \{f \in L_2(\mathbf{R}) \mid \text{supp } \hat{f} \subset [0, \infty), \text{ i.e., } Q_{\mathcal{P}} f = f\}, \tag{2.13}$$

$$L_{2,\mathcal{R}}(\mathbf{R}) \ := \ \{f \in L_2(\mathbf{R}) \mid Q_{\mathcal{R}} f = f\}. \tag{2.14}$$

Then the goal is clearly to reconstruct the density $D_{NB}(\phi, \tau)$ from the received echoes. It turns out that a suitable reconstruction formula can indeed be derived, provided that the families of outgoing signals form frames in the spaces $L_{2,\mathcal{P}}(\mathbf{R})$ and $L_{2,\mathcal{R}}(\mathbf{R})$, respectively. As in the wide-band case, a general echo is modeled by assuming a distribution of reflectors, now described as a function of the delay and Doppler shifts

$$f(t) = \int \int \Psi_{\phi,\tau}(t) D_{NB}(\phi, \tau) d\phi d\tau \ , \tag{2.15}$$

where $\Psi_{\phi,\tau}(t) = e^{i\phi t}\Psi(t-\tau)$. Equation (2.15) poses an inverse problem: given f, find D_{NB}. As wavelet analysis was used for the wide-band inverse problem, so can time-frequency analysis be used to do the same for the narrowband problem. Based on the windowed Fourier transform we obtain

$$D_{NB}^{||}(\phi, \tau) = \int f(t)\bar{\Psi}(t-\tau)e^{-i\phi t} dt \ . \tag{2.16}$$

Hence, we can rewrite Theorem 2.1 for the narrowband case.

Theorem 2.2 *The most general solution $D_{NB}(\phi, \tau)$ in $L_2(d\phi_0 d\tau_0)$ with respect to the echo* (2.15) *is given by*

$$D_{NB}(\phi, \tau) = D_{NB}^{||}(\phi, \tau) + D_{NB}(\phi, \tau)^{\perp} \ ,$$

where $D_{NB}(\phi, \tau)^{\perp}$ is any function in the orthogonal complement of the range of the windowed Fourier transform, i.e.,

$$\int \int R(\phi, \tau, \phi_0, \tau_0) D_{NB}(\phi, \tau)^{\perp} d\phi_0 d\tau_0 = 0.$$

As for the wide-band regime in [14, 15] and [16] derived, we are able to establish a similar reconstruction formula of D_{NB}, which is presented in Section 4.

The similar reconstruction results for the wide–band and the narrow–band regime can be explained by the similarities between the two radar models given by the theory of groups. To show the common group theoretical treatment one may start by a unitary irreducible representation $\mathcal{U}$ of a group G in a Hilbert space H. This representation is called *square integrable* if there exists a vector ψ in H such that

$$\int_G |\langle \psi, \mathcal{U}(g)\psi\rangle_H|^2 \, d\mu(g) < \infty \,, \tag{2.17}$$

where $d\mu$ denotes the Haar measure on G. A vector satisfying (2.17) is called *admissible.* If ψ is admissible, then the mapping

$$\begin{aligned} T_\psi \, : \, H \quad &\rightarrow \quad L_2(G, d\mu) \\ f \quad &\mapsto \quad \langle f, \mathcal{U}(g)\psi\rangle_H \end{aligned} \tag{2.18}$$

is well-defined, see [6] for details. The 1-D continuous wavelet transform is performed by means of the so-called *affine group* $\mathcal{A}$ given by

$$\mathcal{A} : \big(\{(a,b)\,|\,(a,b) \in \mathbf{R}^2, a \neq 0\}, \circ\big) \quad \text{with} \quad (a,b)\circ(a',b') = (aa', b + ab') \ . \tag{2.19}$$

$\mathcal{A}$ possesses a square integrable representation on $L_2(\mathbf{R})$ given by

$$\mathcal{U}(a,b)f(x) := |a|^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right) \,, \tag{2.20}$$

and every admissible vector $\psi \in L_2(\mathbf{R})$ induces a well-defined *continuous wavelet transform*, see, e.g., [4, 8, 11] for details. Representation (2.20) coincides with the wide–band model.

Another important example is the famous *Weyl–Heisenberg group*

$$G := (\{(\omega, b, \tau)\,|\, b, \omega \in \mathbf{R},\, \tau \in \mathbf{C},\, |\tau| = 1\}, \circ) \quad \text{with} \tag{2.21}$$

$$(\omega, b, \tau) \circ (\omega', b', \tau') = (\omega + \omega', b + b', \tau + \tau' + b\omega' \text{mod } 2\pi) \ .$$

The Weyl–Heisenberg group possesses a unitary irreducible representation U in $L_2(\mathbf{R})$ which is given by

$$U(\omega, b, \tau) f(x) := e^{i(\tau - \omega(x-b))} f(x-b), \tag{2.22}$$

then definition (2.18) leads us to the *windowed Fourier transform*

$$(G_\psi f)(\omega, b) := \int_{\mathbf{R}} f(x)\psi(x-b)e^{-i\omega x}\,dx\ , \tag{2.23}$$

where we have neglected the toral component. In contrast to the affine group, which leads to the wide–band model, the Weyl-Heisenberg group leads by (2.22) to the narrow–band model.

3 Frame Theory

In this section, we briefly recall the general theory of frames as far as it is needed for our purposes. In general, a system $\{h_m\}_{m\in\mathbf{Z}}$ of functions is called a *frame* if there exist constants A and B, $0 < A \le B < \infty$, such that

$$A\|F\|^2_{L_2(\mathbf{R})} \le \sum_{m\in\mathbf{Z}} |\langle F, h_m\rangle|^2 \le B\|F\|^2_{L_2(\mathbf{R})}. \tag{3.24}$$

The numbers A, B are called the *frame bounds.* Given a frame $\{h_m\}_{m\in\mathbf{Z}}$, we may define the *frame operator* T as

$$T(F) := \sum_{m\in\mathbf{Z}} \langle F, h_m\rangle h_m. \tag{3.25}$$

For later use, let us recall the following fundamental theorem which was proved in [5].

Theorem 3.1 *Let $\{h_m\}_{m\in\mathbf{Z}}$ be a frame in $L_2(\mathbf{R})$. Then the following holds.*

i) T is invertible and $B^{-1}I \le T^{-1} \le A^{-1}I$.

ii) $\{h^m\}_{m\in\mathbf{Z}}$, $h^m := T^{-1}h_m$ is a frame with bounds A^{-1}, B^{-1}, called the dual frame *of $\{h_m\}_{m\in\mathbf{Z}}$.*

iii) Every $F \in L_2(\mathbf{R})$ can be written as

$$F = \sum_{m\in\mathbf{Z}} \langle F, h^m\rangle h_m = \sum_{m\in\mathbf{Z}} \langle F, h_m\rangle h^m. \tag{3.26}$$

4 Reconstruction Formula for the Narrowband Approach

The goal is clearly to reconstruct the density $D_{NB}(\phi, \tau)$ from the received echoes. It turns out that a suitable reconstruction formula can indeed be derived, provided that the families of outgoing signals form frames in the spaces $L_{2,\mathcal{P}}(\mathbf{R})$ and $L_{2,\mathcal{R}}(\mathbf{R})$, respectively.

Theorem 4.1 *Let $\{h_m\}_{m\in\mathbf{Z}}, \{g_m\}_{m\in\mathbf{Z}}$ be sets of outgoing signals in $L_{2,\mathcal{P}}(\mathbf{R})$ and $L_{2,\mathcal{R}}(\mathbf{R})$, respectively. Let us furthermore assume that $\{h_m\}_{m\in\mathbf{Z}}$, $\{g_m\}_{m\in\mathbf{Z}}$ form frames in these spaces, and let $\{h^m\}_{m\in\mathbf{Z}}$ and $\{g^m\}_{m\in\mathbf{Z}}$ denote the corresponding dual frames. The echoes of $\{h_m\}_{m\in\mathbf{Z}}$ are denoted by $f_{\mathcal{P},m}$, the echoes of $\{g_m\}_{m\in\mathbf{Z}}$ are denoted by $f_{\mathcal{R},m}$. Let us assume that the reflectivity density $D_{NB}(\phi,\tau)$ satisfies the following conditions*

$$h_m(t-\tau)D_{NB}(\phi,\tau) \in L_1(d\phi d\tau), \quad \widehat{D_{NB}(\cdot,\tau)}(\omega) \in L_1(d\omega). \tag{4.27}$$

Then $D_{NB}(\phi,\tau)$ can be reconstructed as follows

$$\begin{aligned} D_{NB}(\phi,\tau) &= \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\int_{\mathbf{R}} f_{\mathcal{P},m}(t)h^m(t-\tau)e^{i\phi t}dt \\ &\quad + \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\int_{\mathbf{R}} f_{\mathcal{R},m}(-t)g^m(-t-\tau)e^{i\phi t}dt \ . \end{aligned} \tag{4.28}$$

Proof: We proceed by following the lines of the proof of Theorem 2.2 in [3]. Using (2.11) yields

$$\begin{aligned} f_{\mathcal{P},m}(t) &= \int_{\mathbf{R}}\int_{\mathbf{R}} e^{-i\phi t}h_m(t-\tau)D_{NB}(\phi,\tau)d\phi d\tau \\ &= \int_{\mathbf{R}} h_m(t-\tau)\left(\int_{\mathbf{R}} e^{-i\phi t}D_{NB}(\phi,\tau)d\phi\right)d\tau \\ &= \int_{\mathbf{R}} h_m(t-\tau)\widehat{D_{NB}(\cdot,\tau)}(t)d\tau \\ &= \int_{\mathbf{R}} h_m(\tau)D_{NB}\widehat{(\cdot,t-\tau)}(t)d\tau \\ &= \langle h_m(\cdot), (\widetilde{D_{NB}}(t,\cdot))_P\rangle_{L_2(\mathbf{R})}, \end{aligned}$$

where

$$\widetilde{D_{NB}}(t,\tau) := D_{NB}\widehat{(\cdot,t-\tau)}(t). \tag{4.29}$$

Consequently, by using the reciprocal frame $\{h^m\}_{m\in\mathbf{Z}}$ and exploiting the fundamental reconstruction formula (3.26), we obtain

$$\begin{aligned} (\widetilde{D_{NB}}(t,\tau))_{\mathcal{P}} &= \sum_{m\in\mathbf{Z}}\langle(\widetilde{D_{NB}}(t,\cdot))_{\mathcal{P}}, h_m(\cdot)\rangle_{L_2(\mathbf{R})}h^m(\tau) \\ &= \sum_{m\in\mathbf{Z}} f_{\mathcal{P},m}(t)h^m(\tau), \end{aligned}$$

and therefore

$$(\widehat{D_{NB}(\cdot,\tau)}(t))_{\mathcal{P}} = \sum_{m\in\mathbf{Z}} f_{\mathcal{P},m}(t)h^m(t-\tau). \tag{4.30}$$

A similar calculation yields

$$\begin{aligned}(\overline{D}_{NB}(t,\tau))_{\mathcal{R}} &= \sum_{m\in\mathbf{Z}} \langle(\overline{D}_{NB}(t,\cdot))_{\mathcal{R}}, g_m(\cdot)\rangle_{L_2(\mathbf{R})} g^m(\tau) \\ &= \sum_{m\in\mathbf{Z}} f_{\mathcal{R},m}(t) g^m(\tau)\end{aligned}$$

where

$$\overline{D}_{NB}(t,\tau) := D_{NB}\widehat{(\cdot, t-\tau)}(-t). \tag{4.31}$$

Consequently, we obtain

$$(D_{NB}\widehat{(\cdot,\tau)}(-t))_{\mathcal{R}} = \sum_{m\in\mathbf{Z}} f_{\mathcal{R},m}(t) g^m(t-\tau) \tag{4.32}$$

so that

$$(D_{NB}\widehat{(\cdot,\tau)}(t))_{\mathcal{R}} = \sum_{m\in\mathbf{Z}} f_{\mathcal{R},m}(-t) g^m(-t-\tau) \tag{4.33}$$

and

$$\begin{aligned}D_{NB}\widehat{(\cdot,\tau)}(t) &= (D_{NB}\widehat{(\cdot,\tau)}(t))_{\mathcal{R}} + D_{NB}\widehat{(\cdot,\tau)}(t)_{\mathcal{P}} \\ &= \sum_{m\in\mathbf{Z}} f_{\mathcal{P},m}(t) h^m(t-\tau) + \sum_{m\in\mathbf{Z}} f_{\mathcal{R},m}(-t) g^m(-t-\tau).\end{aligned}$$

Now the result follows by applying the inverse Fourier transform

$$\begin{aligned}D_{NB}(\phi,\tau) &= \frac{1}{2\pi}\int_{\mathbf{R}} D_{NB}\widehat{(\cdot,\tau)}(t) e^{i\phi t} dt \\ &= \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\int_{\mathbf{R}} f_{\mathcal{P},m}(t) h_m(t-\tau) e^{i\phi t} dt \\ &\quad + \frac{1}{2\pi}\sum_{m\in\mathbf{Z}}\int_{\mathbf{R}} f_{\mathcal{R},m}(-t) g^m(-t-\tau) e^{i\phi t} dt.\end{aligned}$$

■

We finish this section by giving some examples of outgoing signals meeting the conditions of Theorem 4.1.

i) The first example is given by means of the Haar function

$$H(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x).$$

It is well-known that the set of all dilated and integer translated versions of H form an orthonormal basis for $L_2(\mathbf{R})$, hence, the sets

$$\{H_{m(j,k)} = 2^{j/2} H(2^j \cdot -k)\}_{k\geq 0, j\in\mathbf{Z}} \quad \text{and} \quad \{H_{m(j,k)} = 2^{j/2} H(2^j \cdot -k)\}_{k<0, j\in\mathbf{Z}}$$

form orthonormal bases and therewith frames in $L_2(\mathbf{R}_+)$ and $L_2(\mathbf{R}_-)$, respectively. Then, frames for $L_{2,\mathcal{P}}(\mathbf{R})$ and $L_{2,\mathcal{R}}(\mathbf{R})$ can be obtained by applying the inverse Fourier transform

$$\{h_{m(j,k)} = 2\pi\mathcal{F}^{-1}(H_{m(j,k)})\}_{k\geq 0, j\in\mathbf{Z}} \quad \text{and} \quad \{g_{m(j,k)} = 2\pi\mathcal{F}^{-1}(H_{m(j,k)})\}_{k<0, j\in\mathbf{Z}}.$$

In this case, primal and dual frame clearly coincide.

ii) For the second example we choose the Cauchy kernel

$$C(x) = (1 - ix)^{-1}.$$

The Fourier transform of C belongs to $L_{2,\mathcal{P}}$. Then, for some $l > 0$ and a suitable choice of a_0 and b_0 the family

$$\{h_{m(n,k)} = a_0^{-n/2} C^{(l)}(a_0^{-n} \cdot - b_0 k)\}_{n,k\in\mathbf{Z}}$$

forms a frame for $L_{2,\mathcal{P}}(\mathbf{R})$, see, e.g., [4]. By involving a reflexion in Fourier space we obtain a frame $\{g_m\}$ for $L_{2,\mathcal{R}}(\mathbf{R})$. The dual frames $\{h^m\}$ and $\{g^m\}$ can be constructed by applying the inverse frame operator T^{-1} as outlined in Theorem 3.1.

iii) In principle, further examples can be obtained by constructing boundary adapted biorthogonal or frame systems on $\mathbf{R}_+$ and $\mathbf{R}_-$, respectively, see, e.g., [2], followed by applying the inverse Fourier transform.

Finally, we want to emphasize that the approach outlined here has at least the potential for practical realizations. Indeed, in contrary to earlier developments, we can work with a very small frequency band, and, moreover, it is only necessary to transmit a frame of signals which is much less restrictive than the approach suggested by Naparst [14, 15].

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