Stable Multiscale Bases and Local Error Estimation for Elliptic Problems

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Abstract

This paper is concerned with the analysis of adaptive multiscale techniques for the solution of a wide class of elliptic operator equations covering, in principle, singular integral as well as partial differential operators. The central objective is to derive reliable and efficient a-posteriori error estimators for Galerkin schemes which are based on stable multiscale bases. It is shown that the locality of corresponding multiresolution processes combined with certain norm equivalences involving weighted sequence norms leads to adaptive space refinement strategies which are guaranteed to converge in a wide range of cases, again including operators of negative order.

Key words: Stable multiscale bases, norm equivalences, elliptic operator equations, Galerkin schemes, a-posteriori error estimators, convergence of adaptive schemes

AMS subject classification: 65N55, 65N30, 65N38, 65N12

1 Introduction

The increasing importance of adaptive techniques in large scale computation is reflected by a vast amount of recent literature on this topic primarily in connection with finite element schemes (see e.g. [BEK, BM, BR, BW, J, Ve]). How to fit such adaptive techniques into the context of stable splittings for fast multilevel Schwarz type preconditioners for elliptic problems has been briefly indicated in [O]. On the other hand, there have been several attempts to apply wavelet concepts to the solution of differential and more generally pseudo-differential equations. As the above mentioned multilevel

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schemes these concepts hinge upon making successive corrections of current solutions when progressing to finer scales of discretization. However, the wavelet methodology differs from the finite element techniques in that direct use of bases is made which span the complements between successive trial spaces. It is clear that the construction of such bases may be a prohibitive task by itself. On the other hand, several cases have been studied where such bases are available and have proved to offer significant advantages. For instance, pre-wavelets have been shown to yield robust preconditioners in combination with sparse grid discretizations for two- and three-dimensional anisotropic problems [GO]. Divergence free wavelets with small support have been applied to the Stokes problem where again the change from two to three spatial variables does not cause any problem. More generally, there are examples of wavelet bases with built in Ladyzhenskaya-Babuška-Brezzi-condition for various types of saddle point problems. Finally, such bases give rise to matrix compression techniques when dealing with discretizations of pseudo-differential or singular integral operators as they arise, for instance, in connection with boundary element methods. Wavelet bases defined on two dimensional manifolds in $\mathbb{R}^3$ satisfying all the requirements which guarantee optimal compression and convergence rates have now become available [DS1]. This is an important aspect, since for such equations on two dimensional manifolds the fact that corresponding system matrices are not sparse is the major computational bottleneck. So far the compression techniques apply to Galerkin or collocation schemes based on essentially uniformly refined trial spaces.

In view of the availability of the various instances of promising stable multiscale bases for PDE as well as integral equation problems, the question arises how to design and analyse adaptive strategies in connection with such multiscale bases oriented methods. Therefore the objective of this paper is to discuss some basic concepts and ideas which we feel are crucial for the understanding of adaptive techniques in connection with multiscale bases, and to relate them to existing techniques in more conventional settings. A particular motivation is that for integral equations the understanding of local error estimators is comparatively less developed than for partial differential equations. To our knowledge the results of this paper offer for the first time *reliable* and *efficient* a-posteriori error estimators also for integral operators in the sense that the current error is bounded from above and below by expression involving computable local quantities. It will be seen that on one hand, unlike the finite element case local error estimators arise in a fairly unified fashion essentially as coefficients of corresponding multiscale expansions. On the other hand, as mentioned before, these facts can be established for a rather wide class of problems involving differential and integral operators. We also emphasize that the a-posteriori error estimators are not confined to symmetric problems.

We will focus here on (analogues for) energy norm estimates in terms of residuals. Thus our starting point is similar to the observations made in [O, R] for more special situations. The main problem treated here is then to analyse further the resulting error terms which still contain infinitely many terms. It will be shown that these expressions can be reduced in the general case to efficient and reliable error bounds involving finitely many terms. It will be seen that these estimates give rise to adaptive space refinement techniques which are guaranteed to converge without assuming beforehand the so called *saturation property.* We wish to mention that this work has been inspired to some degree by the results in [Do] for the technically quite different setting of piecewise linear finite elements for Laplace's equation and by recent studies in [Be] concerning wavelet related error estimators for univariate two point boundary value problems.

The layout of the paper is as follows. In Section 2 we describe a general framework for the type of problems to be studied and list a few examples. In Section 3 we collect some relevant facts concerning multiscale bases. In Section 4 we collect some prerequisites about Galerkin schemes, in particular, pertaining to their stability and preconditioning of resulting matrices. Section 5 is devoted to a-posteriori residual estimates with respect to energy (-like) norms and their algorithmic consequences.

2 A Class of Problems

We will be concerned with linear operator equations

$$Au = f \tag{2.1}$$

where A will be assumed to be a boundedly invertible operator from some Hilbert spaces H_1 into another Hilbert space H_2, i.e.,

$$\|Au\|_{H_2} \sim \|u\|_{H_1}, \quad u \in H_1, \tag{2.2}$$

where '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise ' $\lesssim$ ' indicates inequalities up to constant factors. We will write out such constants explicity only when their value matters.

To get an idea of the range of problems we have in mind, one can follow [DPS1], and view A as a classical pseudo-differential operator. This covers a wide range of classical differential and (singular) integral operators. It is known that when A is injective and its symbol is strongly elliptic a Gårding inequality holds which implies (2.2) for Sobolev spaces $H_1 = H^s$, $H_2 = H^{s-\rho}$, say (where the order ρ of A is determined by the homogeneity of the symbol). A typical example of this sort may be described as follows (see [CS]).

Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded simply connected Lipschitz domain, $\Omega_2 := \mathbb{R}^2 \setminus \bar{\Omega}_1$ its complement and $\Gamma = \partial\Omega_1$ its boundary. Then for a given $f \in H^{\frac{1}{2}}(\Gamma)$ and $g \in H^{-\frac{1}{2}}(\Gamma)$ one seeks $(u_1, u_2) \in H^1(\Omega_1) \times H^1_{loc}(\Omega_2)$ satisfying

$$\begin{aligned} \Delta u_j &= 0 \quad \text{in } \Omega_j, \quad \text{for j = 1,2} \\ u_1 &= u_2 + f, \quad \frac{\partial u_1}{\partial n_1} = \frac{\partial u_2}{\partial n_2} + g \quad \text{on } \Gamma \end{aligned} \tag{2.3}$$

and the radiation condition

$$\lim_{|x|\to\infty} (u_2(x) - \frac{b}{2\pi} \log|x|) = 0$$

for some $b \in \mathbb{R}$.

To present an equivalent integral equation formulation with respet to Γ, we introduce the single layer potential

$$V\Phi(x) = -\frac{1}{\pi}\int_\Gamma \Phi(y)\log|x-y|ds_y,$$

the double layer potential

$$K\Phi(x) = -\frac{1}{\pi}\int_\Gamma \Phi(y)\frac{\partial}{\partial n_y}\log|x-y|ds_y,$$

its adjoint K', and the hypersingular operator

$$W\Phi(x) = -\frac{\partial}{\partial n_x}K\Phi(x).$$

Then the operator A defined by

$$\begin{aligned} A : H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) &\to H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) \\ (u,v) &\mapsto A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -Ku & Vv \\ Wu & K'v \end{pmatrix} \end{aligned}$$

is for Lipschitz domains linear, bounded and boundedly invertible, i.e.,

$$\left\| A\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)} \sim \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)}$$

with

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|^2_{H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)} = \|u\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|v\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

Then (2.3) is equivalent to the integral equation [CS]

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}(\begin{pmatrix} f \\ g \end{pmatrix} + A\begin{pmatrix} f \\ g \end{pmatrix}).$$

In this case one has $H_1 = H_2 = H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$.

In general the spaces $H_i, i = 1,2$, will be elements of a whole scale of spaces H^s or products of such, where H^s will, for instance, be Sobolev spaces relative to some domain $\Omega \subseteq \mathbb{R}^d$ or relative to some possibly closed piecewise smooth manifold as above (see e.g. [DPS1]). The Sobolev spaces can be defined with the aid of a partition of unity and an atlas. A typical case is that Ω is the boundary of some domain $\hat{\Omega} \subset \mathbb{R}^{d+1}$. Thus $H^s(\Omega)$ could be taken as the trace space of $H^{s+\frac{1}{2}}(\hat{\Omega})$. Specially, when $\Omega \subset \mathbb{R}^n$ is a bounded domain we denote by $H^s(\Omega)$ (for $s > 0$) the usual Sobalev space on Ω and by $H^s_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to $\|\cdot\|_{H^s(\Omega)}$. When s is negative H^s is to be understood as the dual of H^{-s}. We have spaces H^s in mind which satisfy $H^s_0(\Omega) \subset H^s \subset H^s(\Omega)$, e.g. prescribing homogeneous Dirichlet boundary conditions on part of $\partial\Omega$.

Symmetry selects an important subclass of operators. By this we mean operators A such that for some $\rho \in \mathbb{R}$

$$a(u,v) = (Au, v)_0, \quad u, v \in H^{\frac{\rho}{2}} \tag{2.4}$$

is a symmetric bilinear form, where $(\cdot,\cdot)_0$ denotes the standard inner product on $H^0 = L_2(\Omega)$. Moreover, we will assume that A is elliptic in the sense that

$$a(u,u) \sim \|u\|^2_{H^{\frac{\rho}{2}}(\Omega)}, \quad u \in H^{\frac{\rho}{2}}. \tag{2.5}$$

It is clear that in this case (2.2) holds with $H_1 = H^{\frac{\rho}{2}}$, $H_2 = H^{-\frac{\rho}{2}}$.

Of course, the simplest examples of this type are $\Omega \subset \mathbb{R}^d$, $Au = -\Delta u$ or $Au = -\Delta u + cu$ where $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian and $c > 0$. Here $\rho = 2$ and $H^1 = H_0^1(\Omega)$ or $H^1(\Omega)$ respectively.

In order to focus on the main ideas, we will confine the following analysis to the technically somewhat simpler case $H_1 = H^t, H_2 = H^{-t}$ where $t = \frac{\rho}{2}$, although the arguments extend to the situation considered in the first example as well.

3 Multiresolution

Our goal is to employ Galerkin methods for the approximate solution of (2.1). It is well known that the most efficient ways of solving the resulting systems of equations exploit the interaction of several scales of discretization. To correspond to the above scope of applications we formulate the relevant facts for the following general framework.

Suppose H is a Hilbert space (of functions defined on Ω, say) with inner product $(\cdot,\cdot)$. Again typical examples are $H = L_2(\Omega)$, $H = H^s(\Omega)$ or products of such spaces. Let $\mathcal{S} = \{S_j\}_{j=0}^\infty$ be a sequence of closed nested subspaces of H whose union is dense in H. We will always assume that S_j is spanned by $\Phi_j = \{\phi_{j,k} : k \in I_j\}$ where these bases are uniformly stable, i.e.,

$$\|\mathbf{c}\|_{\ell_2(I_j)} \sim \left\| \sum_{k \in I_j} c_k \phi_{j,k} \right\|_H \tag{3.1}$$

uniformly in $j \in \mathbb{N}_0$. Here we denote as usual $\|\cdot\|_H^2 = (\cdot,\cdot)$ and $\|\mathbf{c}\|^2_{\ell_2(I_j)} = \sum_{k \in I_j} |c_k|^2$.

Successively updating a current approximation in S_{j-1} to a better one in S_j can be facilitated if stable bases

$$\Psi_j = \{\psi_{j,k} : k \in J_j\}$$

for some complement W_j of S_{j-1} in S_j are available. Defining for convenience $\Psi_0 = \Phi_0$, $W_0 := S_0$, any $v_n = \sum_{k \in I_n} c_k \phi_{n,k} \in S_n$ has then an alternative *multiscale* representation

$$v_n = \sum_{j=0}^n \sum_{k \in J_j} d_{j,k} \psi_{j,k}$$

which corresponds to the direct sum decomposition

$$S_n = \bigoplus_{j=0}^n W_j.$$

Let $\mathbf{T}_n$ denote the transformation that takes the coefficents $d_{j,k}$ in the multiscale representation of v_n into the coefficients c_k of the single scale representation. It corresponds to the synthesis part of the fast wavelet transform.

It will be useful for later purposes to briefly describe the structure of $\mathbf{T}_n$. For convenience let us view Φ_j as a column vector whose components are $\phi_{j,k}$, $k \in I_j$. Nestedness and stability imply the existence of $(\#I_{j+1}) \times (\#I_j)$-matrices $\mathbf{R}_{j,0}$ such that

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{R}_{j,0}. \tag{3.2}$$

Likewise there exists a $(\#I_{j+1}) \times (\#J_j)$-matrix $\mathbf{R}_{j,1}$ such that

$$\Psi_{j+1}^T = \Phi_{j+1}^T \mathbf{R}_{j,1} \tag{3.3}$$

and it is known that uniform stability of the complement bases Ψ_j is equivalent to the uniform boundedness of the composed matrices $\mathbf{R}_j = (\mathbf{R}_{j,0}, \mathbf{R}_{j,1})$ as well as their inverses as mappings from $\ell_2(I_{j+1})$ into itself [CDP]. It is easy to see that then $\mathbf{T}_n$ has the form

$$\mathbf{T}_n = \hat{\mathbf{R}}_0 \cdots \hat{\mathbf{R}}_{n-1}, \quad \hat{\mathbf{R}}_\ell = \begin{pmatrix} \mathbf{R}_\ell & 0 \\ 0 & \mathbf{I} \end{pmatrix}. \tag{3.4}$$

The application of $\mathbf{T}_n$ requires $\mathcal{O}(\dim S_n)$ operations if the number of nonzero entries in each row and column of the $\mathbf{R}_j$ remain uniformly bounded.

To avoid loss of accuracy when executing $\mathbf{T}_n$ it is important that $\mathbf{T}_n$ are well conditioned, i.e.,

$$\|\mathbf{T}_n\| \left\|\mathbf{T}_n^{-1}\right\| = \mathcal{O}(1), \quad n \to \infty, \tag{3.5}$$

where $\|\cdot\|$ denotes the spectral norm. It is well-known that this is equivalent to the fact that $\Psi = \bigcup_{j \in \mathbb{N}_0} \Psi_j$ forms a *Riesz-basis* of H, i.e. every $v \in H$ has a unique expansion

$$v = \sum_{j=0}^{\infty} \sum_{k \in J_j} (v, \tilde{\psi}_{j,k}) \psi_{j,k} \tag{3.6}$$

such that

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \sum_{k \in J_j} |(v, \tilde{\psi}_{j,k})|^2 \right)^{\frac{1}{2}}, \quad v \in H, \tag{3.7}$$

where $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : k \in J_j, j \in \mathbb{N}_0\}$ forms a biorthogonal system

$$(\psi_{j,k}, \tilde{\psi}_{j',k'}) = \delta_{j,j'} \delta_{k,k'}, \quad j, j' \in \mathbb{N}_0, \quad k \in J_j, \quad k' \in J_{j'} \tag{3.8}$$

and is in fact also a Riesz-basis for H (cf. [D]). It is clear that when the complements W_j are *orthogonal* the stability of each Ψ_j suffices to ensure (3.7) and hence (3.5). However, in many practical cases orthogonality is difficult to realize, in particular, when the functions in Φ_j and Ψ_j are to have small supports which in turn will be essential for our applications and for the efficiency of $\mathbf{T}_n$.

Assuming henceforth the stability of Ψ_j for each level j the additional information which will be needed to ensure the stability (3.7) across levels in the biorthogonal case can conveniently be described in terms of the projectors

$$Q_n v := \sum_{j=0}^{n} \sum_{k \in J_j} (v, \tilde{\psi}_{j,k}) \psi_{j,k}, \quad Q_n' v := \sum_{j=0}^{n} \sum_{k \in J_j} (v, \psi_{j,k}) \tilde{\psi}_{j,k}$$

which are obviously adjoints of each other. Note that

$$Q_j Q_n = Q_j, \quad \text{for } j \le n. \tag{3.9}$$

It is useful to keep the following facts in mind (cf. [D]).

Remark 3.1 *(3.9) is equivalent to either of the following statements*

1. *The mappings $Q_n - Q_{n-1}$ are also projectors and*

$$(Q_n - Q_{n-1})(Q_\ell - Q_{\ell-1}) = \delta_{n,\ell}(Q_n - Q_{n-1})$$

2. *The ranges $\tilde{S}_n$ of the adjoints Q'_n are also nested.*

By (3.7) the Q_n (and hence the Q'_n) are uniformly bounded so that

$$\|v - Q_n v\|_H \lesssim \operatorname{dist}_H(v, S_n), \quad v \in H, \tag{3.10}$$

and likewise for the Q'_n. Thus $\tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j=0}^{\infty}$ is also a nested dense sequence of closed subspaces of H.

Evidently, when the Ψ_j are stable (3.7) (and hence (3.5)) is equivalent to

$$\|v\|_H \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_H^2 \right)^{\frac{1}{2}}, \quad v \in H, \tag{3.11}$$

where $Q_{-1} \equiv 0$. It is shown in [D] that certain (mild) regularity and approximation properties of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ in addition to (3.9) guarantee (3.11). When $H = L_2(\Omega)$ these conditions can be formulated in terms of an inverse estimate

$$\|v_n\|_{H^s(\Omega)} \lesssim 2^{ns} \|v_n\|_{L_2(\Omega)}, \quad v_n \in S_n, \tag{3.12}$$

for $s < \gamma$, some $\gamma > 0$, and a direct estimate

$$\inf_{v_n \in S_n} \|v - v_n\|_{L_2(\Omega)} \lesssim 2^{-sn} \|v\|_{H^s(\Omega)}, \quad v \in H^s(\Omega), \tag{3.13}$$

for $s \le m$, some $\gamma \le m (\in I\!N)$.

Theorem 3.1 *Let $Q = \{Q_j\}_{j=0}^{\infty}$ satisfy (3.9) and assume that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ both satisfy (3.12) and (3.13) relative to some $\gamma, \gamma' > 0$, $\gamma \le m$, $\gamma' \le m'$, then*

$$\begin{aligned} \|v\|_{H^s(\Omega)} &\sim \left(\sum_{j=0}^{\infty} 2^{2js} \|(Q_j - Q_{j-1})v\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \\ &\sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_{H^s(\Omega)}^2 \right)^{\frac{1}{2}}, \quad v \in H^s(\Omega), \end{aligned} \tag{3.14}$$

holds for $s \in (-\gamma', \gamma)$. Moreover, the projectors Q_j, Q'_j are uniformly bounded in $H^s(\Omega)$, $s \in (-\gamma', \gamma)$.

Remark 3.2 *Instead of powers of 2 in (3.12)–(3.14) we could have used powers of a for some $a > 1$ which reflects the subdivision rate of successive refinement levels. Since this entails no essential differences we will stick in the following with halving mesh sizes.*

Note that the projectors Q_n may be also represented as

$$Q_n v = \sum_{k \in I_n} (v, \tilde{\phi}_{n,k}) \phi_{n,k} \tag{3.15}$$

where $\tilde{\Phi}_n = \{\tilde{\phi}_{n,k} : k \in I_n\}$ is a stable basis for $\tilde{S}_n$ which is *dual* to Φ_n, i.e.

$$(\phi_{n,k}, \tilde{\phi}_{n,k'}) = \delta_{k,k'}, \quad k, k' \in I_n. \tag{3.16}$$

Usually the projectors are more easily available in this form where, according to Remark 3.1 one has to find dual collections $\tilde{\Phi}_j$ that are also refinable. For ways of deriving from (3.15) corresponding multiscale bases $\Psi, \tilde{\Psi}$ we refer to [CDP].

For our applications it will be important to work with *local bases*, i.e., we will always assume that

$$\text{diam}(\text{supp}\phi_{n,k}), \quad \text{diam}(\text{supp}\psi_{n,k}) \sim 2^{-n}, \quad n \in I\!N. \tag{3.17}$$

Furthermore, it is desirable that the $\tilde{\phi}_{n,k}$, $\tilde{\psi}_{n,k}$ have the same property

$$\text{diam}(\text{supp}\tilde{\phi}_{n,k}), \quad \text{diam}(\text{supp}\tilde{\psi}_{n,k}) \sim 2^{-n}, \quad n \in I\!N. \tag{3.18}$$

A sufficient condition for the direct estimate (3.13) to hold for $s \leq m$ is then that all polynomials of degree $\leq m - 1$ are (locally) contained in each S_j, $j \in I\!N$. Moreover, when (3.17) and (3.18) hold, one easily verifies then local estimates of the form

$$\|Q_n v - v\|_{L_2(D)} \lesssim 2^{-nm} \|v\|_{H^m(\hat{D}_n)}, \quad v \in H^m_{loc} \tag{3.19}$$

where for any $D \subset\subset \Omega$, $\hat{D}_n$ is also a domain in Ω satisfying

$$D \subset \hat{D}_n \subset \Omega, \quad \text{dist}(D, \partial\hat{D}_n) \lesssim 2^{-n}, \quad n \in I\!N. \tag{3.20}$$

Throughout the rest of the paper we will assume that the bases Φ_j, $\tilde{\Phi}_j$, Ψ^j, $\tilde{\Psi}^j$ satisfy the above assumptions with respect to parameters γ, γ', m, m' which have to be tuned to the particular application (see [DS1] for examples of such bases defined on two dimensional (closed) manifolds in $I\!R^3$).

Finally, it will be important to make use of so called moment conditions. If Ψ is a stable multiscale basis for $L_2(\Omega)$ where Ω is a domain in $I\!R^d$, Ψ is said to have vanishing moments of order m' if

$$\int_\Omega P(x)\psi_{j,k}(x)dx = 0 \tag{3.21}$$

holds for all polynomials of degree less than m'. Since estimates of the form (3.13) or (3.19) usually imply that the approximation spaces contain polynomials of the order corresponding to the highest approximation order and since the $\psi_{j,k}$ are orthogonal to $\tilde{S}_j$ the moment conditions are closely related to the order of exactness of the dual multiresolution sequence $\tilde{\mathcal{S}}$. The fact that polynomials appear in (3.21) is actually not essential. What matters is that (3.21) holds for a finitely dimensional family of functions which locally approximate any smooth function well. Accordingly, this notion can be modified. For instance, when the domain Ω is a manifold represented by a family of smooth parametric mappings $\kappa_i : \Box \to \Omega_i \subset \Omega$, where $\Box$ is a fixed parameter domain, one can work with the condition

$$\int_{\Box} P(x)\psi_{j,k}(\kappa_i(x))dx = 0, \tag{3.22}$$

see [DS1].

4 Galerkin Schemes

We are interested in efficient numerical schemes for (2.1). To this end, we recall the following facts concerning Galerkin methods for (2.1) from [DPS1]. Suppose that $\mathcal{S}, \Psi, \tilde{\Psi}$ have the above properties with associated projectors Q_j. The standard Galerkin procedure requires to find $u_j \in S_j$ such that

$$(Au_j, v_j) = (f, v_j), \quad v_j \in S_j, \tag{4.1}$$

which is equivalent to

$$Q_j' A u_j = Q_j' f. \tag{4.2}$$

To simplify the notation we will write in the following $\|\cdot\|_s$ instead of $\|\cdot\|_{H^s}$ as well as $t := \rho/2$ where ρ is the order of A. Our basic assumption (2.2) reads now throughout the rest of the paper

$$\|Av\|_{-t} \sim \|v\|_t, \quad v \in H^t. \tag{4.3}$$

We will also assume that the Galerkin scheme is $(t, -t)$-stable, i.e.,

$$\|Q_j' A v_j\|_{-t} \gtrsim \|v_j\|_t, \quad v_j \in S_j. \tag{4.4}$$

Obviously (4.4) holds when A is selfadjoint and $a(\cdot,\cdot) := (A\cdot,\cdot)$ satisfies (2.5). More generally, it has been shown in [HW] that when A is strongly elliptic in the sense that $\ker A = \{0\}$ and the real part of the principle part of the symbol of A is coercive then (4.4) holds as well. More generally, one has [DPS1]

Remark 4.1 *Suppose that A satisfies (2.2) for $H_1 = H^s, H_2 = H^{s-\rho}$ and let*

$$\gamma > t, \quad \gamma' > -t. \tag{4.5}$$

If $\mathcal{S}$ satisfies (3.12) and (3.13) then for any $\rho - m \le s \le \frac{\rho}{2}$ the Galerkin scheme (4.2) is $(s, s-\rho)$-stable, i.e.,

$$\left\|Q_j' A u_j\right\|_{s-\rho} \gtrsim \|u_j\|_s, \quad u_j \in S_j. \tag{4.6}$$

Denoting by u, u_j the solution of (2.1) and (4.2), respectively, one has

$$\|u - u_j\|_s \lesssim 2^{j(s-\tau)} \|u\|_\tau \tag{4.7}$$

for $-m + \rho \le s < \gamma \le m$, $s \le \tau$, $t \le \tau \le m$.

Thus under such premises the discrete equations (4.2) possess a unique solution whose deviation from the exact solution can be estimated for a scale of norms.

It will be convenient to introduce the following notation. Let

$$J := \{\lambda = (j,k) : k \in J_j, j \in \mathbb{N}_0\} = \bigcup_{j=0}^{\infty} (\{j\} \times J_j).$$

Defining

$$|\lambda| := j \quad \text{if} \quad \lambda \in J_j,$$

let $\mathbf{A}_n := ((A\psi_{\lambda'}, \psi_\lambda))_{|\lambda|,|\lambda'|<n}$ so that (4.2) is equivalent to the linear system of equations

$$\mathbf{A}_n \mathbf{d} = \mathbf{f}_n, \tag{4.8}$$

where $\mathbf{f}_n := ((f, \psi_\lambda))_{|\lambda|<n}$. Moreover, introducing the diagonal matrix $\mathbf{D}_n^s$ defined by

$$(\mathbf{D}_n^s)_{\lambda,\lambda'} := 2^{s|\lambda|}\delta_{\lambda,\lambda'},$$

it is known [DPS1] that

$$\mathrm{cond}_2(\mathbf{D}_n^{-t}\mathbf{A}_n\mathbf{D}_n^{-t}) \sim 1. \tag{4.9}$$

Thus when $\mathbf{A}_n$ is selfadjoint positive definite (which will be referred to as the *symmetric case*), a conjugate gradient iteration applied to $\mathbf{B}_n := \mathbf{D}_n^{-t}\mathbf{A}_n\mathbf{D}^{-t}$ would perform very well. But even for certain nonsymmetric sytems stemming from the discretizing elliptic integral equations corresponding nonsymmetric versions like GMRES have proven to be very efficient.

A few more comments on the practical realization are in order. For a wide class of operators with global Schwarz kernel the matrices $\mathbf{A}_n$ are fully populated but can be approximated very well by sparse matrices in such a way that the resulting compressed systems can be handled efficiently [DPS1] while the accuracy of the corresponding solutions is still asymptotically optimal. On the other hand, when A is a differential operator it would be more efficient to store the sparse stiffness matrix $\mathbf{A}_{\Phi_n}$ relative to the fine scale basis Φ_n. Since

$$\mathbf{A}_n = \mathbf{T}_n^T\mathbf{A}_{\Phi_n}\mathbf{T}, \quad \mathbf{f}_n = \mathbf{T}_n\mathbf{f}_{\Phi_n},$$

where the components of $\mathbf{f}_{\Phi_n}$ are $(f, \phi_{j,k})$, and $\mathbf{T}_n$ is the multiscale transformation from (3.4). Thus the preconditioning can be realized by applying the fast transformation $\mathbf{T}_n$ to sparse arrays.

Note that estimates of the form (3.13) reflect that the spaces S_j arise from *uniform refinements*. However, in the present paper our main concern is not to find possibly sparse representations of the operator A relative to a-priori fixed trial spaces but to find possibly economical trial spaces leading to as small linear systems as possible in the first place. More precisely, we wish to determine step by step possibly small subspaces of the *full* spaces S_j which recover the solution as well as possible. To describe this we set for any $\Lambda \subset J$

$$S_\Lambda := \mathrm{span}\,\{\psi_\lambda : \lambda \in \Lambda\}, \quad Q_\Lambda v := \sum_{\lambda\in\Lambda}(v, \tilde{\psi}_\lambda)\psi_\lambda.$$

and

$$\mathbf{A}_\Lambda := ((A\psi_{\lambda'}, \psi_\lambda))_{\lambda,\lambda'\in\Lambda}.$$

In the symmetric case corresponding to (2.5) one still has

$$\|Q'_\Lambda A v_\Lambda\|_{-t} \sim \|v_\Lambda\|_t, \quad v_\Lambda \in S_\Lambda, \tag{4.10}$$

so that the matrices $\mathbf{A}_\Lambda$ are still nonsingular. Moreover, since for $|\Lambda| := \max\{|\lambda| : \lambda \in \Lambda\}$ the matrix $\mathbf{A}_\Lambda$ is a principal submatrix of $\mathbf{A}_{|\Lambda|}$, the diagonal preconditioning still produces uniformly bounded condition numbers.

In general, we will assume in the following that

$$Q'_\Lambda A u_\Lambda = Q'_\Lambda f \tag{4.11}$$

possesses a unique solution u_Λ in S_Λ and that (4.10) holds. By the same arguments as used in [DPS1] for the full spaces it follows from (4.10) that also

$$\mathrm{cond}_2(\mathbf{D}_\Lambda^{-t}\mathbf{A}_\Lambda\mathbf{D}_\Lambda^{-t}) \sim 1, \tag{4.12}$$

where as above $(\mathbf{D}^s_\Lambda)_{\lambda,\lambda'} := 2^{s|\lambda|}\delta_{\lambda,\lambda'}$. In fact, defining

$$\Sigma_s v := \sum_{j=0}^{\infty} 2^{sj}(Q_j - Q_{j-1})v,$$

one clearly has, in view of (3.9),

$$\Sigma_s Q_\Lambda = Q_\Lambda \Sigma_s, \quad \Sigma_s^{-1} = \Sigma_{-s}. \tag{4.13}$$

Thus Theorem 3.1 says that

$$\|\Sigma_s v\|_\tau \sim \|v\|_{s+\tau}, \qquad s+\tau \in (-\gamma', \gamma). \tag{4.14}$$

Setting $w_\Lambda := \Sigma_t v_\Lambda$ for $v_\Lambda \in S_\Lambda$ and keeping (4.5) in mind, (4.14), (4.13), and (4.10) imply

$$\begin{aligned} \|w_\Lambda\|_0 &\sim \|v_\Lambda\|_t \sim \|Q'_\Lambda A v_\Lambda\|_{-t} \\ &\sim \|\Sigma'_{-t} Q'_\Lambda A Q_\Lambda \Sigma_{-t} w_\Lambda\|_0, \end{aligned}$$

which means that the operators $\Sigma'_{-t}Q'_\Lambda A Q_\Lambda \Sigma_{-t}$ and their inverses are uniformly bounded on L_2. Moreover, it is easy to see that $\mathbf{D}_\Lambda^{-t}\mathbf{A}_\Lambda\mathbf{D}_\Lambda^{-t}$ is the matrix representation of $\Sigma'_{-t}Q'_\Lambda A Q_\Lambda \Sigma_{-t}$ which confirms (4.12).

5 Multiscale Error Estimates

5.1 Some Preliminary Remarks

In the symmetric case it is natural to estimate the accuracy of the Galerkin solution with respect to the *energy norm*

$$\|\cdot\| := a(\cdot,\cdot)^{1/2}.$$

Since by (2.5), this norm is equivalent to $\|\cdot\|_t$ we will employ this latter norm most of the time in the general (possibly non-symmetric) situation. To explain the usual starting point for adaptive strategies we focus for a moment though on the symmetric case. The basic idea is very simple (see e.g. [BEK] or [BW]). Suppose that u', u'' are Galerkin approximations to the solution u of (2.1) from spaces S', S'' respectively, where u'' is a more accurate solution, i.e., $S' \subset S''$.

Remark 5.1 *When A is selfadjoint positive definit one has for u', u'', as above*

$$\|u'' - u'\| \leq \|u - u'\| \,. \tag{5.1}$$

Moreover, one has for some $\beta < 1$

$$\|u - u'\| \leq (1-\beta^2)^{-\frac{1}{2}} \|u'' - u'\| \tag{5.2}$$

if and only if

$$\|u - u''\| \leq \beta \|u - u'\| \,. \tag{5.3}$$

The assertion is a trivial consequence of the orthogonality of the error $u - u''$ to $u' - u''$ so that $\|u-u'\|^2 = \|u-u''\|^2 + \|u''-u'\|^2$.

The relation (5.3) is often called *saturation property* requiring that the new approximation u'' is strictly better than the previous one u'. Thus if the saturation property holds the quantity $\|u' - u''\|$ provides a lower and upper bound for the true error $\|u - u'\|$. Such a-posteriori bounds are called *efficient* and *reliable*. In the finite element context S'' could be a trial space corresponding to a refined mesh or a trial space with the same mesh as S' but containing higher order trial functions. In either case the expression $u' - u''$ can be evaluated efficiently typically by solving local problems [BEK]. The resulting bounds are then usually comprised of sums of local terms whose precise form, however, depends strongly on the particular discretization at hand.

On the other hand, symmetry plays a crucial role already in the derivation of the error bounds and in many previous investigations the saturation property has to be assumed beforehand, which from a principal point of view is certainly not satisfactory.

In the following we will also study energy estimates or more generally estimates relative to the norm $\|\cdot\|_t$ but for the above general multiscale basis setting. Our goal is to develop computable efficient and reliable error bounds that lead to an adaptive strategy which ensures that the saturation property is satisfied automatically and thus can be proved to converge. These estimates will be formulated in terms of wavelet coefficients and thus take a rather unified form for a wide range of cases.

5.2 An A-Posteriori Error Estimate

Once a Galerkin approximation $u_\Lambda \in S_\Lambda$ to the solution u of (2.1) has been determined one can, in principle evaluate the *residual*

$$r_\Lambda = Au_\Lambda - f = A(u_\Lambda - u). \tag{5.4}$$

On account of (4.3), we have

$$c_1\|r_\Lambda\|_{-t} \leq \|u - u_\Lambda\|_t \leq c_2\|r_\Lambda\|_{-t}, \tag{5.5}$$

where we have now specified the constants c_1, c_2 in (4.3) for later purposes.

As above in connection with preconditioning we will next make again essential use of the norm equivalences in Theorem 3.1, recalling that under the assumption (4.5), $\|r_\Lambda\|_{-t}$ can be estimated by weighted sequence norms of the wavelet coefficients of r_Λ. In fact, specifying also the constants in (3.14) by c_3, c_4, we obtain

$$c_3\left(\sum_{\lambda\in J\setminus\Lambda} 2^{-2t|\lambda|}|(r_\Lambda,\psi_\lambda)|^2\right)^{1/2} \leq \|r_\Lambda\|_{-t} \leq c_4\left(\sum_{\lambda\in J\setminus\Lambda} 2^{-2t|\lambda|}|(r_\Lambda,\psi_\lambda)|^2\right)^{1/2}, \tag{5.6}$$

where we have used that, since u_Λ is a Galerkin solution,

$$r_\Lambda = \sum_{\lambda\in J}(r_\Lambda,\psi_\lambda)\tilde{\psi}_\lambda = \sum_{\lambda\in J\setminus\Lambda}(r_\Lambda,\psi_\lambda)\tilde{\psi}_\lambda.$$

Thus, combining (5.5) and (5.6), we obtain, in principle, an efficient and reliable error bound. However, at this stage it is practically useless since it still involves infinitely many terms. Our goal is now to replace the bounds in (5.6) by computable expressions. To this end, let us abbreviate

$$\delta_\lambda := 2^{-t|\lambda|}|(r_\Lambda,\psi_\lambda)|$$

and note that upon inserting the expansion

$$u_\Lambda = \sum_{\lambda'\in\Lambda} u_{\lambda'}\psi_{\lambda'}$$

of the Galerkin solution u_Λ yields the representation

$$\delta_\lambda = 2^{-t|\lambda|}|f_\lambda - \sum_{\lambda'\in\Lambda}(A\psi_{\lambda'},\psi_\lambda)u_{\lambda'}|, \quad f_\lambda := (f,\psi_\lambda), \tag{5.7}$$

and therefore, by (5.6),

$$c_3\left(\sum_{\lambda\in J\setminus\Lambda}\delta_\lambda^2\right)^{1/2} \leq \|r_\Lambda\|_{-t} \leq c_4\left(\sum_{\lambda\in J\setminus\Lambda}\delta_\lambda^2\right)^{1/2}. \tag{5.8}$$

Obviously, replacing the entities δ_λ by finitely many computable ones requires some information about the given data, here in terms of the right hand side f, and about the behavior of the entries $(A\psi_{\lambda'},\psi_\lambda)$.

We will first show that for almost all $\lambda \in J$ the sums $\sum_{\lambda'\in\Lambda}(A\psi_{\lambda'},\psi_\lambda)u_{\lambda'}$ can actually be neglected. To this end, it is well-known that for a large class of operators A the $(A\psi_{\lambda'},\psi_\lambda)$ decay when either the levels $|\lambda|, |\lambda'|$ or the supports $\Omega_\lambda, \Omega_{\lambda'}$ of the wavelets $\psi_\lambda, \psi_{\lambda'}$, respectively, are far apart. Thus supposing that the spatial domain of the functions in H^t is d, we will assume the following basic estimate on the entries $(A\psi_{\lambda'},\psi_\lambda)$

$$2^{-(|\lambda'|+|\lambda|)t}|(A\psi_{\lambda'},\psi_\lambda)| \lesssim \frac{2^{-||\lambda|-|\lambda'||(d/2+\sigma)}}{(1+2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+m'+2t}}, \tag{5.9}$$

where again $2t=\rho$ is the order of A, $\sigma>0$ is some fixed real number and m' is a positive integer which typically represents the order of exactness of $\tilde{\mathcal{S}}$ or the order of vanishing moments of the ψ_λ (which have to be suitably interpreted depending on the type of the underlying domain Ω, see e.g. [DS1]).

Let us briefly outline some circumstances under which estimates of the form (5.9) hold. To begin with a simple but instructive case let $A=-\Delta$, $t=1$, $H^1=H^1_0(\Omega)$, $\Omega\subset I\!R^d$, and suppose that the $\nabla\psi_\lambda$ have vanishing moments of order m', i.e.,

$$\int_\Omega P(x)\cdot\nabla\psi_\lambda(x)dx=0,$$

for any vector valued function whose components are polynomials of degree less than m'. Then one has for any such polynomial and $|\lambda|>|\lambda'|$

$$\begin{aligned}(A\psi_{\lambda'},\psi_\lambda) &= \int_\Omega \nabla\psi_{\lambda'}(x)\cdot\nabla\psi_\lambda(x)dx=\int_\Omega(\nabla\psi_{\lambda'}(x)-P(x))\cdot\nabla\psi_\lambda(x)dx\\ &\le \max_{x\in\Omega_\lambda}|\nabla\psi_{\lambda'}(x)-P(x)|\int_\Omega|\nabla\psi_\lambda(x)|dx.\end{aligned}$$

Thus when $\nabla\psi_{\lambda'}$ is still Hölder continuous with exponent $\sigma>0$ elementary calculations show that the first factor can be estimated by a constant times $2^{-|\lambda|\sigma}2^{|\lambda'|(1+\sigma)}2^{\frac{d}{2}|\lambda'|}$ while the second factor is bounded by a constant times $2^{-d|\lambda|}2^{\frac{d}{2}|\lambda|}2^{|\lambda|}$ so that overall one obtains in this case

$$|(A\psi_{\lambda'},\psi_\lambda)|\lesssim\begin{cases}2^{|\lambda|+|\lambda'|}2^{(\frac{d}{2}+\sigma)(|\lambda'|-|\lambda|)} & \text{if}\quad \Omega_\lambda\cap\Omega_{\lambda'}\ne\emptyset,\\ 0 & \text{if}\quad \Omega_\lambda\cap\Omega_{\lambda'}=\emptyset.\end{cases}$$

Obviously this is a special case of (5.9). The above estimate is still crude. In fact, a much stronger decay occurs when $|\lambda|$ is much larger than $|\lambda'|$ and Ω_λ does not intersect the singular support of $\psi_{\lambda'}$ [S].

More generally, we admit operators with Schwartz kernels of global support

$$(Av)(x)=\int_\Omega K(x,y)v(y)dy,$$

where we require that whenever $d+\rho+|\alpha|+|\beta|>0$

$$|\partial_x^\alpha\partial_y^\beta K(x,y)|\lesssim \mathrm{dist}\,(x,y)^{-(d+\rho+|\alpha|+|\beta|)} \tag{5.10}$$

holds with constants depending only on $\alpha,\beta\in\mathbb{Z}^d_+$. Estimates of the type (5.10) are known to hold for a wide range of cases including classical pseudodifferential operators and Calderón-Zygmund operators (see e.g. [DPS1, S]).

The following result has been established in [DPS1, DPS2].

Lemma 5.1 *Assume that Ψ has vanishing moments of order m' in the sense of (3.21), (3.22), respectively, and assume that for $2t=\rho\le 0$, the basis functions $\phi_{j,k}$ are Hölder continuous with exponent $\sigma\in(0,1]$, i.e.,*

$$|\phi_{\ell,k}(x)-\phi_{\ell,k}(x')|\lesssim 2^{\ell(d/2+\sigma)}[\mathrm{dist}(x,x')]^{\sigma},$$

while for $2t=\rho>0$ we require Hölder continuity of order $n+\sigma>t$, i.e.,

$$|D^{\alpha}\phi_{\ell,k}(x)-D^{\alpha}\phi_{\ell,k}(x')|\lesssim 2^{\ell(d/2+\sigma+|\alpha|)}[\mathrm{dist}(x,x')]^{\sigma}\quad\forall|\alpha|\le n.$$

Then for $d+\rho+m'>0$ and any $\lambda,\lambda'\in J$ one has

$$2^{-(|\lambda'|+|\lambda|)t}|(A\psi_{\lambda'},\psi_{\lambda})|\lesssim\begin{cases}2^{-||\lambda|-|\lambda'||(d/2+\sigma)} & \text{if}\quad \Omega_\lambda\cap\Omega_{\lambda'}\neq\emptyset,\\ \frac{2^{-||\lambda|-|\lambda'||(d/2+m'+t)}}{(1+2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+m'+2t}} & \text{if}\quad \Omega_\lambda\cap\Omega_{\lambda'}=\emptyset.\end{cases}\tag{5.11}$$

One should note that a more careful analysis leads to the following stronger estimate [PS, S]

$$|(A\psi_{\lambda'},\psi_{\lambda})|\lesssim\frac{2^{-(|\lambda|+|\lambda'|)(d/2+m')}}{(\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+2m'+\rho}}$$

provided that $\Omega_\lambda\neq\Omega_{\lambda'}$. Moreover, we emphasize that the estimates concerning overlapping supports could also be refined, especially when dealing with operators of negative order. The quantitative effect of possibly sharp estimates will certainly play an essential role in any concrete application. Here estimates of the form (5.9) or (5.11) should be viewed as representatives of a certain decay property which suffices to prove the principal fact that the bounds in (5.8) can be reduced to finitely many local quantities. The reasoning is closely related to matrix compression techniques based on estimates of the form (5.11) [DPS1].

Lemma 5.2 *Assume that (5.9) holds. Let $0<\delta<\sigma$ be fixed where σ is the constant from (5.9) and choose for $\epsilon>0$ positive numbers $\epsilon_1,\epsilon_2>0$ such that*

$$\epsilon_1^{m'+\rho}+2^{-\delta/\epsilon_2}\le\epsilon.\tag{5.12}$$

For $\lambda\in J$ define the neighborhood

$$J_{\lambda,\epsilon}:=\{\lambda'\in J:||\lambda|-|\lambda'||\le\epsilon_2^{-1}\ \text{and}\ \ 2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'})\le\epsilon_1^{-1}\}.$$

Then there exists a constant c_5 depending only on δ,m',ρ, the constant c_3 and the stability constant in (4.10) of the Galerkin scheme such that the quantities

$$e_\lambda:=\sum_{\lambda'\in\Lambda\setminus J_{\lambda,\epsilon}}(A\psi_{\lambda'},\psi_\lambda)u_{\lambda'},\quad\lambda\in J\setminus\Lambda,\tag{5.13}$$

satisfy

$$\left(\sum_{\lambda\in J\setminus\Lambda}2^{-2t|\lambda|}|e_\lambda|^2\right)^{1/2}\le c_5\epsilon\|Q'_\Lambda f\|_{-t}.\tag{5.14}$$

Proof: It has been shown in [DPS1, S], that the infinite symmetric matrix $\mathbf{R}^\epsilon = (a^\epsilon_{\lambda,\lambda'})_{\lambda,\lambda'\in J}$ defined by

$$a^\epsilon_{\lambda,\lambda'} = \begin{cases} \frac{2^{-||\lambda|-|\lambda'||(d/2+\sigma)}}{(1+2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+m'+\rho}}, & \lambda\in J\setminus\Lambda, \quad \lambda'\in J\setminus J_{\lambda,\epsilon}, \\ 0, & \text{else}, \end{cases}$$

satisfies

$$\|\mathbf{R}^\epsilon\| \lesssim \epsilon \tag{5.15}$$

where $\|\cdot\|$ denotes here the spectral norm and the constant in (5.15) depends only on the constants c_1, c_2, c_3, c_4 and ρ, δ. Therefore we infer from (5.9) that

$$\begin{aligned} \sum_{\lambda\in J\setminus\Lambda} 2^{-2t|\lambda|}|e_\lambda|^2 &\le \sum_{\lambda\in J\setminus\Lambda}\left|\sum_{\lambda'\in\Lambda\setminus J_{\lambda,\epsilon}} \frac{2^{-||\lambda|-|\lambda'||(d/2+\sigma)}}{(1+2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+m'+\rho}} 2^{t|\lambda'|}|u_{\lambda'}|\right|^2 \\ &\lesssim \epsilon^2 \sum_{\lambda'\in\Lambda\setminus J_{\lambda,\epsilon}} 2^{2t|\lambda'|}|u_{\lambda'}|^2 \\ &\lesssim \epsilon^2\|u_\Lambda\|_t^2 \lesssim \epsilon^2\|Q'_\Lambda A u_\Lambda\|_{-t}^2 = \epsilon^2\|Q'_\Lambda f\|_{-t}^2, \end{aligned}$$

where we have used (4.10) and (4.11) in the last step. ∎

Remark 5.2 *It is obvious from the above proof that the quantity $\|Q'_\Lambda f\|_{-t}$ in (5.14) can be replaced by either $\|u_\Lambda\|_t$ or $\|f\|_{-t}$ with a modified constant c_5. The latter choice has the advantage of being independent of Λ but the disadvantage that in a strict sense it is not computationally accessible while the first two choices can be estimated via the coefficients $f_\lambda, u_\lambda, \lambda\in\Lambda$ and the corresponding norm equivalences.*

The idea is now to replace for a given $\epsilon > 0$ and a given $\Lambda\subset J$ the quantities δ_λ in (5.8) by

$$d_\lambda(\Lambda,\epsilon) = d_\lambda := 2^{-t|\lambda|}|f_\lambda - \sum_{\lambda'\in\Lambda\cap J_{\lambda,\epsilon}} (A\psi_{\lambda'},\psi_\lambda)u_{\lambda'}|, \quad \lambda\in J\setminus\Lambda. \tag{5.16}$$

We will show next that the quantities d_λ give rise to a new a-posteriori estimate which is still up to any chosen tolerance efficient and reliable. For the special case of second order two point boundary value problems a similar result was obtained by S. Bertoluzza who was as far as we know the first to establish an a-posteriori estimate of the following type [Be].

Theorem 5.1 *Under the assumptions in Lemma 5.2 one has*

$$\|u-u_\Lambda\|_t \le c_2c_4\left(\sum_{\lambda\in J\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} + c_2c_4c_5\epsilon\|Q'_\Lambda f\|_{-t}, \tag{5.17}$$

as well as

$$\left(\sum_{\lambda\in J\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \le \frac{1}{c_1c_3}\|u-u_\Lambda\|_t + c_5\epsilon\|Q'_\Lambda f\|_{-t}. \tag{5.18}$$

Proof: Both inequalities follow immediately by the triangle inequality and the estimate (5.14) of Lemma 5.2. In fact, we apply $\delta_\lambda \le d_\lambda + 2^{-t|\lambda|}|e_\lambda|$ to (5.5), (5.8) and obtain (5.17). Likewise, because $d_\lambda \le \delta_\lambda + 2^{-t|\lambda|}|e_\lambda|$, we employ again (5.5), (5.8) and (5.14). ∎

A few comments on the above result are in order. It is clear that also the sums $(\sum_{\lambda\in J\setminus\Lambda} d_\lambda^2)^{1/2}$ are not finite yet. However, defining

$$N_{\Lambda,\epsilon} := \{\lambda \in J \setminus \Lambda : \Lambda \cap J_{\lambda,\epsilon} \neq \emptyset\} \tag{5.19}$$

one has, by construction,

$$\# N_{\Lambda,\epsilon} < \infty, \tag{5.20}$$

so that all but finitely many of the terms $d_\lambda(\Lambda,\epsilon)$ depend only on the right hand side f. Since $f \in H^{-t}$ the series $\sum_{\lambda\in J} 2^{-2t|\lambda|}|f_\lambda|^2$ converges so that $\sum_{\lambda\in J\setminus\ (nle\cup\Lambda)} 2^{-2t|\lambda|}|f_\lambda|^2$ can be made arbitrary small by choosing Λ appropriately. In fact, the contribution of f to $(\sum_{\lambda\in J\setminus(N_{\Lambda,\epsilon}\cup\Lambda)} d_\lambda^2)^{1/2}$ is just

$$\begin{aligned}\sum_{\lambda\in J\setminus(N_{\Lambda,\epsilon}\cup\Lambda)} 2^{-2t|\lambda|}|f_\lambda|^2 \quad &\lesssim \quad \|f - Q'_{\Lambda\cup N_{\Lambda,\epsilon}} f\|_{-t}^2 \sim \inf_{v\in\tilde S_{\Lambda\cup N_{\Lambda,\epsilon}}} \|f-v\|_{-t}^2 \\ &\le \quad \inf_{v\in\tilde S_\Lambda} \|f-v\|_{-t}^2.\end{aligned} \tag{5.21}$$

We will see that this contribution can be subsumed under the perturbation of order ϵ. This is precisely the problem of adaptively approximating an explicitly given function or distribution. Thus any singularities of f will be reflected by the initial choice of a starting set Λ. The above estimates of the quantities e_λ show how much this information is smeared due to the pseudo-locality of the elliptic operator which becomes accessible through the multiscale representation. If f is very smooth the contribution of (5.21) will be negligible and the adaptive choice of larger sets $\tilde\Lambda$ will be dominated by the behavior of the current approximation u_Λ and the action of A.

Due to the assumed compact support of the ψ_λ each of the quantities $d_\lambda(\Lambda,\epsilon)$ is a trivial *local* lower bound for the error. To refine this information one needs more knowledge about the local behavior of the residuals. We do not want to elaborate on this issue here but remark that this is a non-trivial problem for operators with negative order (cf. [R, WY, F]). To our knowledge no efficient and reliable estimators have so far been known. For example, Carstensen and Stephan proved in [CaS] a-posteriori estimates without deriving lower bounds. It is remarkable, that within our approach operators with negative order seem to have to some extent even advantages over operators with positive order, because of their smoothing property, i.e., $(A\psi_{\lambda'}, \psi_\lambda)$ decays faster for $|\lambda| \to \infty$ if $A\psi_{\lambda'}$ is smoother.

5.3 A Convergent Adaptive Strategy

The next step is to use the a-posteriori error estimates for an adaptive refinement strategy. Although totally different in a technical sense and with regard to the whole setting the results are similar in spirit to those by Dörfler [Do] who considers adaptive refinement of piecewise linear finite elements for Poisson's equation in two dimensions.

In the present setting refinement simply means adding properly selected basis functions ψ_λ to the current solution space. We will describe assumptions, that guarantee an improvement for the approximate solution after the refinement step.

Throughout the remainder of this section we adhere to the assumptions made above.

Lemma 5.3 *Suppose that $\Lambda \subset \tilde{\Lambda} \subset J$ and let $u_{\tilde{\Lambda}}$ be the Galerkin solution with respect to*

$$S_{\tilde{\Lambda}} = \mathrm{span}\{\psi_\lambda : \lambda \in \tilde{\Lambda}\}.$$

Then we have

$$\left(\sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \leq \frac{1}{c_3 c_1}\|u_{\tilde{\Lambda}} - u_\Lambda\|_t \ + \ c_5\epsilon\|Q'_\Lambda f\|_{-t}. \tag{5.22}$$

Proof: For $\lambda \in \tilde{\Lambda}$ we have

$$\sum_{\lambda'\in\Lambda}(A\psi_{\lambda'},\psi_\lambda)u_{\lambda'} = (Au_\Lambda,\psi_\lambda) = (A(u_\Lambda - u_{\tilde{\Lambda}}),\psi_\lambda) + f_\lambda$$

and therefore, by (5.13) and (5.16),

$$\begin{aligned} d_\lambda(\Lambda,\epsilon) &= 2^{-t|\lambda|}|f_\lambda - \sum_{\lambda'\in\Lambda}(A\psi_{\lambda'},\psi_\lambda)u_{\lambda'} + e_\lambda| \\ &\leq 2^{-t|\lambda|}|(A(u_\Lambda - u_{\tilde{\Lambda}}),\psi_\lambda)| + 2^{-t|\lambda|}|e_\lambda|. \end{aligned}$$

Because of (5.5) and (5.6) we obtain

$$\begin{aligned} \sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} 2^{-2t|\lambda|}|(A(u_\Lambda - u_{\tilde{\Lambda}}),\psi_\lambda)|^2 &\leq c_3^{-2}\|A(u_\Lambda - u_{\tilde{\Lambda}})\|_{-t}^2 \\ &\leq \frac{1}{c_3^2 c_1^2}\|u_\Lambda - u_{\tilde{\Lambda}}\|_t^2. \end{aligned}$$

Thus (5.14) provides

$$\left(\sum_{\tilde{\Lambda}\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \leq \frac{1}{c_3 c_1}\|u_{\tilde{\Lambda}} - u_\Lambda\|_t + c_5\epsilon\|Q'_\Lambda f\|_{-t},$$

as claimed. ∎

Our next goal is to use (5.22) for selecting a set $\tilde{\Lambda}$ containing Λ such that the saturation property is guaranteed to hold. So far all our previous estimates did not require any symmetry assumptions on the operator A. For the next step, however, it seems that more information about A is needed. The simplest setting would again be the symmetric case, i.e.,

$$a(\cdot,\cdot) := (A\cdot,\cdot) \tag{5.23}$$

defines a symmetric bilinear form such that $\|\cdot\| := a(\cdot,\cdot)^{1/2}$ satisfies

$$\|\cdot\| \sim \|\cdot\|_t. \tag{5.24}$$

We will formulate and prove the next result for this simple setting and will indicate later how to weaken the assumptions somewhat.

Theorem 5.2 *Suppose that (5.23) and (5.24) hold and let $eps > 0$ be a given tolerance. Fix any $\theta^* \in (0,1)$ and define*

$$C_e := \left(\frac{1}{c_1 c_3} + \frac{1-\theta^*}{2c_2 c_4}\right). \tag{5.25}$$

Choose any $\mu^* > 0$ such that

$$\mu^* C_e \le \frac{1-\theta^*}{2(2-\theta^*)c_2 c_4}. \tag{5.26}$$

Finally, for a given $\Lambda \subset J$ let

$$\epsilon := \frac{\mu^* eps}{c_5 \|Q'_\Lambda f\|_{-t}}. \tag{5.27}$$

Then whenever $\tilde{\Lambda} \subset J$, $\Lambda \subset \tilde{\Lambda}$ is chosen so that

$$\left(\sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \ge (1-\theta^*)\left(\sum_{\lambda\in J\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2}, \tag{5.28}$$

there exists a constant $\kappa \in (0,1)$, depending only on the constants μ^*, θ^*, the constants in (5.24) and the constants $c_i, i = 1,\ldots,4$, such that either

$$\|u - u_{\tilde{\Lambda}}\| \le \kappa \|u - u_\Lambda\| \tag{5.29}$$

or $(\sum_{\lambda\in J\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2)^{1/2} < eps$.

Proof: We first assume, that $\|u - u_\Lambda\|_t \ge \frac{\text{eps}}{C_e}$ where the constant $C_e > 0$ is defined by (5.25). When $\tilde{\Lambda}$ satisfies (5.28) we infer from (5.22), (5.17), (5.27) and (5.26)

$$\begin{aligned}
\|u_{\tilde{\Lambda}} - u_\Lambda\|_t &\ge c_1 c_3 \left(\left(\sum_{\lambda\in\tilde{\Lambda}\setminus\Lambda} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2} - c_5\epsilon\|Q'_\Lambda f\|_{-t}\right) \\
&\ge c_1 c_3 \left((1-\theta^*)\left\{\frac{1}{c_2 c_4}\|u-u_\Lambda\|_t - c_5\epsilon\|Q'_\Lambda f\|_{-t}\right\} - c_5\epsilon\|Q'_\Lambda f\|_{-t}\right) \\
&\ge c_1 c_3 \left(\frac{1-\theta^*}{c_2 c_4}\|u-u_\Lambda\|_t - (2-\theta^*)c_5\epsilon\|Q'_\Lambda f\|_{-t}\right) \\
&= c_1 c_3 \left(\frac{1-\theta^*}{c_2 c_4}\|u-u_\Lambda\|_t - (2-\theta^*)\mu^*\text{eps}\right) \\
&\ge c_1 c_3 \left(\frac{1-\theta^*}{c_2 c_4} - (2-\theta^*)\mu^* C_e\right)\|u-u_\Lambda\|_t \\
&\ge \frac{c_1 c_3 (1-\theta^*)}{2c_2 c_4}\|u-u_\Lambda\|_t.
\end{aligned} \tag{5.30}$$

By (5.24), there exists then a constant $c_6 \in (0,1)$ depending on the above constant and the constants in (5.24) such that

$$\|u_{\tilde{\Lambda}} - u_\Lambda\| \ge c_6 \|u - u_\Lambda\|. \tag{5.31}$$

At this point we exploit symmetry by applying Remark 5.1 which confirms the assertion (5.29) with $\kappa = \sqrt{1 - c_6^2}$.

On the other hand, $\|u - u_\Lambda\|_t < \frac{\text{eps}}{C_e}$ yields, in view of (5.18) and (5.27),

$$\left(\sum_{\lambda \in J\setminus\Lambda} d_\lambda(\Lambda, \epsilon)^2\right)^{1/2} \leq \frac{1}{c_1 c_3}\|u - u_\Lambda\|_t + c_5 \epsilon \|Q'_\Lambda f\|_{-t} \leq \frac{\text{eps}}{c_1 c_3 C_e} + \mu^* \cdot \text{eps} \leq \frac{\left(\frac{1}{c_1 c_3} + \mu^* C_e\right)\text{eps}}{C_e}.$$

Taking (5.25) and (5.26) into account, we see that $\frac{\left(\frac{1}{c_1 c_3} + \mu^* C_e\right)}{C_e} < 1$ so that

$$\left(\sum_{\lambda \in J\setminus\Lambda} d_\lambda(\Lambda, \epsilon)^2\right)^{1/2} < \text{eps}$$

which completes the proof. ∎

Note that $(\sum_{\lambda \in J\setminus\Lambda} d_\lambda(\Lambda, \epsilon)^2)^{1/2} < \text{eps}$ yields, by (5.17),

$$\|u - u_\Lambda\|_t \leq c_2 c_4 (1 + \mu^*) \text{eps}. \tag{5.32}$$

Remark 5.3 *By Remark 5.2, the term* $\|Q'_\Lambda f\|_{-t}$ *in (5.27) can be replaced by* $\|u_\Lambda\|_t$ *or* $\left(\sum_{\lambda \in \Lambda} 2^{2|\lambda|t} |u_\lambda|^2\right)$ *so that (5.27) can be replaced by*

$$\epsilon := \frac{\mu^* eps}{c_5 \left(\sum_{\lambda \in \Lambda} 2^{2|\lambda|t} |u_\lambda|^2\right)}. \tag{5.33}$$

The constants change then in an obvious way.

The above result may be formulated in terms of the following algorithm.

Assumptions: We assume, that the constants c_1, c_2, c_3, c_4 and c_5 or estimates for these constants are known.

Initialization: Fix $\theta^* \in (0, 1)$ and the desired accuracy eps. Compute C_e, μ^* according to (5.25) and (5.26), respectively. Choose an initial set $\Lambda \subset J$.

Algorithm A:

Step 1: Compute the Galerkin solution u_Λ with respect to Λ.

Step 2: Compute

$$\left(\sum_{\lambda \in \Lambda} 2^{2|\lambda|t} |u_\lambda|^2\right)$$

and ϵ according to (5.33). Determine

$$\eta_{\Lambda,\epsilon} := \left(\sum_{\lambda \in J \setminus \Lambda} d_\lambda(\Lambda, \epsilon)^2 \right)^{1/2}. \tag{5.34}$$

If $\eta_{\Lambda,\epsilon} < \text{eps}$ **Stop**, accept u_Λ as solution which satisfies (5.32). Otherwise, by Theorem 5.2, one has $\|u - u_{\tilde{\Lambda}}\| \le \kappa \|u - u_\Lambda\|$ with $\kappa \in (0,1)$.

Step 3: Determine an index set $\tilde{\Lambda}$, $\Lambda \subset \tilde{\Lambda} \subset J$ such that

$$\left(\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} d_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} \ge (1 - \theta^*) \eta_{\Lambda,\epsilon}.$$

Set

$$\tilde{\Lambda} \to \Lambda$$

and go to **Step 1**.

5.4 Variants and Computational Aspects

Obviously the crucial task in the above algorithm is the computation of the quantities $\eta_{\Lambda,\epsilon}$ defined in (5.34) which, however, still involve infinitely many terms. As pointed out above (see (5.20)) the set $N_{\Lambda,\epsilon}$ of indices in $J \setminus \Lambda$ for which $d_\lambda(\Lambda, \epsilon)$ depends on the current approximate solution u_Λ and on the operator A is finite so that all but finitely many of the coefficients $d_\lambda(\Lambda, \epsilon)$ actually take the form

$$d_\lambda(\Lambda, \epsilon) = 2^{-t|\lambda|} |f_\lambda|, \quad \lambda \in J \setminus (\Lambda \cup N_{\Lambda,\epsilon}). \tag{5.35}$$

Apparently this is a principal problem since in a strict sense f is generally not completely accessible. There could always be a large wavelet coefficient f_λ for very high level $|\lambda|$ which makes it impossible to realize step 3 in a strict sense.

A simple remedy is to assume that the data f possess a finite expansion in terms of $\tilde{\Psi}$. This could be viewed as solving a perturbed problem where the data are approximated by elements from the spaces in $\tilde{\mathcal{S}}$. In this case the quantities $\eta_{\Lambda,\epsilon}$ involve only finitely many computable terms which ultimately depend only on A and the current solution u_Λ.

Another reasonable assumption is that f has some extra regularity. To be specific, suppose that $f \in H^s$ for some $s > 0$, where $t + s \le m'$ say. Combining (5.21) and (5.16), and employing standard approximation estimates analogous to (3.13) which extend to negative norms as well (see e.g. [D]), yields

$$\eta_{\Lambda,\epsilon} \le \left(\sum_{\lambda \in N_{\Lambda,\epsilon}} d_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} + c_7 \, 2^{-j(\Lambda)(t+s)} \|f\|_s, \tag{5.36}$$

where $j(\Lambda) := \min\{|\lambda| : \lambda \in J \setminus (\Lambda \cup N_{\Lambda,\epsilon})\}$. If the error term involving f is small relative to $\left(\sum_{\lambda \in N_{\Lambda,\epsilon}} d_\lambda(\Lambda,\epsilon)^2\right)^{1/2}$ the latter term can be used as a basis for the determination of $\tilde{\Lambda}$ in step 3. However, when f is smooth except at certain isolated places an estimate of the above type will not help since the set $\Lambda \cup N_{\Lambda,\epsilon}$ may not be well suited to approximate f efficiently.

Therefore we will elaborate a little more on the following model where the a-priori knowledge about the given data is subsumed in the

Assumption B: *We have a way of solving the direct problem of approximating f arbitrarily well by elements of the spaces $\tilde{S}_n$. Thus we assume that (estimates for) the quantities*

$$\left(\sum_{\lambda \in J \setminus \Lambda} 2^{-2t|\lambda|}|f_\lambda|^2\right)^{1/2} \leq c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}, \quad c_5\|Q'_\Lambda f\|_{-t} \leq c'_5\|f\|_{-t} \tag{5.37}$$

are accessible for any $\Lambda \subset J$ (see Remark 5.2).

The corresponding approximation procedure itself could be adaptive or nonlinear [DJP, DV]. The point of view taken now makes the influence of f on the structure of the adapted solution spaces more transparent. To this end, define

$$a_\lambda(\Lambda,\epsilon) = a_\lambda := 2^{-t|\lambda|}\left|\sum_{\lambda' \in \Lambda \cap J_{\lambda,\epsilon}} (A\psi_{\lambda'}, \psi_\lambda) u_{\lambda'}\right|, \quad \lambda \in J \setminus \Lambda, \tag{5.38}$$

and note that, in view of (5.19),

$$a_\lambda(\Lambda,\epsilon) = 0 \quad \text{for} \quad \lambda \in J \setminus \Lambda,\ \lambda \notin N_{\Lambda,\epsilon}. \tag{5.39}$$

Since $d_\lambda \leq a_\lambda + 2^{-t|\lambda|}|f_\lambda|$, and $a_\lambda \leq d_\lambda + 2^{-t|\lambda|}|f_\lambda|$ the counterpart to Theorem 5.1 reads

Proposition 5.1 *Under the assumptions in Lemma 5.2 one has*

$$\|u - u_\Lambda\|_t \leq c_2 c_4 \left(\left(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} + c'_5\epsilon\|f\|_{-t} + c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}\right), \tag{5.40}$$

as well as

$$\left(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \leq \frac{1}{c_1 c_3}\|u - u_\Lambda\|_t + c'_5\epsilon\|f\|_{-t} + c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}. \tag{5.41}$$

Moreover, for $\Lambda \subset \tilde{\Lambda} \subset J$ and $u_\Lambda, u_{\tilde{\Lambda}}$ as in Lemma 5.3 we have

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \leq \frac{1}{c_1 c_3}\|u_{\tilde{\Lambda}} - u_\Lambda\|_t + c'_5\epsilon\|f\|_{-t} + c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}. \tag{5.42}$$

The proof follows exactly the lines of the proof for Theorem 5.1 and Lemma 5.3 taking the estimates (5.37) as well as the definition of $N_{\Lambda,\epsilon}$ (5.19) and (5.38) into account. Again employing exactly the same arguments as before in the proof of Theorem 5.2 one obtains

Theorem 5.3 *Suppose that (5.23) and (5.24) hold and let eps > 0 be a given tolerance. Fix any $\theta^* \in (0,1)$ and define*

$$C_e := \left(\frac{1}{c_1 c_3} + \frac{1-\theta^*}{2c_2 c_4}\right). \tag{5.43}$$

Choose any $\mu^* > 0$ such that

$$\mu^* C_e \leq \frac{1-\theta^*}{2(2-\theta^*)c_2 c_4} \tag{5.44}$$

and set

$$\epsilon := \frac{\mu^* eps}{2c_5' \|f\|_{-t}}. \tag{5.45}$$

Suppose that $\Lambda \subset J$ is chosen so that

$$c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t} < \frac{1}{2}\mu^* eps. \tag{5.46}$$

Then whenever $\tilde{\Lambda} \subset J$, $\Lambda \subset \tilde{\Lambda}$ is chosen so that

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \geq (1-\theta^*)\left(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2}, \tag{5.47}$$

there exists a constant $\kappa \in (0,1)$, depending only on the constants μ^*, θ^*, the constants in (5.24) and the constants $c_i, i = 1,\ldots,4$ such that either

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\| \tag{5.48}$$

or $(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2)^{1/2} = (\sum_{\lambda \in J\setminus\Lambda} a_\lambda(\Lambda,\epsilon)^2)^{1/2} < eps$.

This gives rise to the following variant of **Algorithm A**.

Initialization: Fix the desired accuracy eps > 0, $\theta^* \in (0,1)$, compute C_e, μ^* according to (5.43), (5.44).

Algorithm B:

Step 1: Compute ϵ according to (5.45).

Step 2: Determine an index set $\Lambda \subset J$ such that

$$c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t} < \frac{1}{2}\mu^* \text{eps}.$$

Step 3: Compute the Galerkin solution u_Λ with respect to Λ.

Step 4: Compute

$$\eta_{\Lambda,\epsilon} := \left(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda, \epsilon)^2 \right)^{1/2}. \tag{5.49}$$

If $\eta_{\Lambda,\epsilon} < \text{eps}$ **Stop**, accept u_Λ as solution which satisfies (5.32). Otherwise, by Theorem 5.3, one has $\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\|$ with $\kappa \in (0, 1)$.

Step 5: Determine an index set $\tilde{\Lambda}$, $\Lambda \subset \tilde{\Lambda} \subset J$ such that

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\epsilon}} a_\lambda(\Lambda, \epsilon)^2 \right)^{1/2} \geq (1 - \theta^*)\eta_{\Lambda,\epsilon}.$$

Set

$$\tilde{\Lambda} \to \Lambda$$

and go to **Step 3**.

Remark 5.4 *Again we could have used the computable quantities $\|Q'_\Lambda f\|_{-t}$ or $\|u_\Lambda\|_t$ in the definition of ϵ instead of $\|f\|_{-t}$ which would require an additional evaluation of these terms in each step as well as a possible change of ϵ.*

Remark 5.5 *Further variants suggest themselves. For instance, a completely analogous reasoning confirms that Proposition 5.1 and Theorem 5.3 remain valid for $a_\lambda(\Lambda, \epsilon)$ and $\inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}$ replaced by $d_\lambda(\Lambda, \epsilon)$ and $\inf_{v \in \tilde{S}_{\Lambda \cup N_{\Lambda,\epsilon}}} \|f - v\|_{-t}$, respectively. Since trivially*

$$\inf_{v \in \tilde{S}_{\Lambda \cup N_{\Lambda,\epsilon}}} \|f - v\|_{-t} \leq \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t} \tag{5.50}$$

the perturbation caused by f might then even be smaller. On the other hand, the practical consequences of this principal advantage might be negligible since the set Λ and not $\Lambda \cup N_{\Lambda,\epsilon}$ is the input for the next adaptive refinement step. At any rate, due to (5.50), Algorithm B works in exactly the same form for $a_\lambda(\Lambda, \epsilon)$ replaced by $d_\lambda(\Lambda, \epsilon)$.

Shooting directly for the final desired accuracy might require starting in **Algorithm B** with a rather refined set Λ. To better balance the influence of f and A it may therefore be preferable to view the above algorithms as one loop in a scheme of the following type where we assume again the respective initializations of Algorithm A or B.

(I) Choose the final desired accuracy eps_f and some initial $\mathrm{eps}_0 > 0$ (which could be much larger than eps_f). Put $\mathrm{eps} = \mathrm{eps}_0$.

(II) Apply **Algorithm A** or **B** with eps.

(III) If $\mathrm{eps} \le \mathrm{eps}_f$ **Stop**, accept u_Λ as the approximate solution. Otherwise set $\frac{\mathrm{eps}}{2} \to$ eps and go to (II).

To tie the above observations into previous studies, note that

$$a(\psi_\lambda, \psi_\lambda) \sim 2^{2t|\lambda|}. \tag{5.51}$$

In fact, by (5.23) and (5.24), one has

$$a(\psi_\lambda, \psi_\lambda) \sim \|\psi_\lambda\|_t^2 \lesssim 2^{2t|\lambda|}, \tag{5.52}$$

where we have used the inverse estimate (3.12) and the fact that stability of Ψ_j implies the normalization

$$\|\psi_\lambda\|_0 \sim 1.$$

Conversely, since $(Q_{|\lambda|} - Q_{|\lambda|-1})\psi_\lambda = \psi_\lambda$, the estimates (3.10), (3.13) yield

$$\|\psi_\lambda\|_0^2 \lesssim 2^{-2t|\lambda|}\|\psi_\lambda\|_t^2,$$

which together with (5.52) confirms (5.51). Therefore the quantities δ_λ^2 can be replaced by

$$\zeta_\lambda^2 := \frac{|(r_\Lambda, \psi_\lambda)|^2}{a(\psi_\lambda, \psi_\lambda)}.$$

Note that

$$w_{\Lambda,\lambda} := \frac{(r_\Lambda, \psi_\lambda)}{a(\psi_\lambda, \psi_\lambda)}\psi_\lambda$$

is the solution of the local problem

$$a(w_{\Lambda,\lambda}, v) = (r_\Lambda, v), \quad v \in \mathrm{span}\,\{\psi_\lambda\}.$$

We conclude this section with some comments on computational issues. Depending on how expensive the evaluation of A is, one could evaluate $(r_\Lambda, \psi_\lambda)$ and hence d_λ by quadrature. When A has a global Schwarz kernel one has to evaluate the entries $(A\psi_{\lambda'}, \psi)$ of the stiffness matrices by quadrature to compute then $d_\lambda(\Lambda, \epsilon)$ or $a_\lambda(\Lambda, \epsilon)$. For efficient ways of computing the entries $(A\psi_{\lambda'}, \psi_\lambda)$ based on adaptive quadrature we refer to [S, DS2]).

When A is a differential operator one would generally compute and store the sparser stiffness matrices relative to the single scale basis functions $\phi_{j,k}$. Suppose that $j = \max\{|\lambda| : \lambda \in \Lambda\}$ so that $u_\Lambda = \sum_{\lambda' \in \Lambda} u_{\lambda'}\psi_{\lambda'} = \sum_{k \in I_j} c_{j,k}\phi_{j,k}$ where the coefficients $c_{j,k}$ and $u_{\lambda'}$ are interrelated by the multiscale transformation $\mathbf{T}_j$ (3.4). Suppose now that $\lambda \in J \setminus \Lambda$ and $\ell = |\lambda| > j$. Since by (3.2) and (3.3) $\Phi_j^T = \Phi_\ell^T \mathbf{R}_{\ell-1,0} \cdots \mathbf{R}_{j,0}$ and $\Psi_\ell^T = \Phi_\ell^T \mathbf{R}_{\ell-1,1}$ the relation

$$(r_\Lambda, \psi_\lambda) = \sum_{k \in I_j} c_{j,k}(A\phi_{j,k}, \psi_\lambda) - (f, \psi_\lambda)$$

takes the form

$$
\begin{aligned}
(r_\Lambda, \psi_\lambda) &= \mathbf{c}_j^T (A\Phi_j, \psi_\lambda) - (f, \psi_\lambda) \\
&= \mathbf{c}_j^T \mathbf{R}_{j,0}^T \cdots \mathbf{R}_{\ell-1,0}^T (A\Phi_\ell, (\Phi_\ell^T R_{\ell-1,1})_k) - (f, (\Phi_\ell^T \mathbf{R}_{\ell-1,1})_k).
\end{aligned} \tag{5.53}
$$

When ℓ is not much larger than j the sparsity of the matrices $\mathbf{R}_{j,0}$, $\mathbf{R}_{j,1}$ ensures that these calculations are cheap.

5.5 Some Comments on the Role of Symmetry

Let us point out next that symmetry is a convenient but not quite necessary assumption in the above context. We will briefly indicate one way of weakening this hypotheses. In many cases there is a symmetric H^t-elliptic bilinearform $s(\cdot,\cdot)$ such that the difference

$$
k(u,v) := a(u,v) - s(u,v) \quad \forall u, v \in H^t \tag{5.54}
$$

is a compact bilinearform, which induces estimates by a weaker norm than $\|\cdot\|_t$. By this assumption the bilinearform $s(\cdot,\cdot)$ defines a norm

$$
\|u\| := \sqrt{s(u,u)} \quad \forall u \in H^t,
$$

which is equivalent to $\|\cdot\|_t$, i.e.

$$
\|u\| \sim \|u\|_t. \tag{5.55}
$$

Since for $\tilde{\Lambda} \subset J,\ \Lambda \subset \tilde{\Lambda}$

$$
\begin{aligned}
& \|u - u_\Lambda\|^2 - \|u - u_{\tilde{\Lambda}}\|^2 - \|u_\Lambda - u_{\tilde{\Lambda}}\|^2 \\
= \ & s(u - u_\Lambda, u - u_\Lambda) - s(u - u_{\tilde{\Lambda}}, u - u_{\tilde{\Lambda}}) - s(u_\Lambda - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}}) \\
= \ & 2s(u - u_{\tilde{\Lambda}}, u_{\tilde{\Lambda}} - u_\Lambda),
\end{aligned}
$$

the Galerkin orthogonality

$$
a(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}}) = 0,
$$

provides

$$
s(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}}) = -k(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}}), \tag{5.56}
$$

so that

$$
\|u - u_{\tilde{\Lambda}}\|^2 = \|u - u_\Lambda\|^2 - \|u_\Lambda - u_{\tilde{\Lambda}}\|^2 + 2k(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}}). \tag{5.57}
$$

Now we assume further that

$$
|k(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}})| \le \delta(\|u - u_{\tilde{\Lambda}}\|^2 + \|u - u_\Lambda\|^2) \tag{5.58}
$$

holds for some $\delta > 0$. As in the proof of Theorem 5.2 we conclude from (5.30) and (5.55) that

$$
\|u_{\tilde{\Lambda}} - u_\Lambda\|^2 \ge \kappa'^2 \|u - u_\Lambda\|^2
$$

for some $\kappa' \in (0,1)$, which, on account of (5.57) and (5.58), yields

$$\|u - u_{\widetilde{\Lambda}}\| \leq \sqrt{\frac{1-\kappa'^2+\delta}{1-\delta}}\|u - u_\Lambda\|. \tag{5.59}$$

If δ is small enough one obtains

$$\|u - u_{\widetilde{\Lambda}}\| \leq \kappa''\|u - u_\Lambda\|$$

for some $\kappa'' \in (0,1)$, i.e., we can again confirm the validity of the saturation property with respect to Λ and $\widetilde{\Lambda}$.

Let us consider next two typical examples, namely

1. a simple boundary value problem involving a symmetric elliptic second order partial differential operator and a first order term destroying symmetry,

2. an integral equation.

Throughout these examples all functions are assumed to be real-valued and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$.

1): Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and A the partial differential operator defined by

$$A = -\Delta + \partial_{x_1}.$$

The weak formulation of the homogeneous Dirichlet problem

$$\begin{aligned} Au(x) &= f(x) \quad \forall x \in \Omega \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

for a given function $f \in H^{-1}(\Omega)$ reads

$$a(u,v) := (\nabla u, \nabla v) + (\partial_{x_1} u, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

Then one can show (cf. [W]), that a Gårding inequality holds, i.e., there are constants $\rho, q > 0$ such that

$$a(u,u) \geq \rho(\nabla u, \nabla u)_0 - q(u,u)_0 \quad \forall u \in H_0^1(\Omega).$$

The bilinear form

$$s(u,v) := (\nabla u, \nabla v) \quad \forall u, v \in H_0^1(\Omega),$$

is symmetric and $H_0^1(\Omega)$-elliptic. Furthermore we have

$$k(u,v) := (\partial_{x_1} u, v).$$

Since,

$$\begin{aligned} |k(u - u_\Lambda, u_\Lambda - u_{\tilde{\Lambda}})| &\leq \|u - u_{\tilde{\Lambda}}\|_1 \|u_\Lambda - u_{\tilde{\Lambda}}\|_0 \\ &\leq (\|u_\Lambda - u\|_0 + \|u - u_{\tilde{\Lambda}}\|_0)\|u - u_{\tilde{\Lambda}}\|_1, \end{aligned}$$

condition (5.58) will indeed hold if $\|u_\Lambda - u\|_0, \|u - u_{\tilde{\Lambda}}\|_0$ are small compared to $\|u_\Lambda - u\|_1, \|u - u_{\tilde{\Lambda}}\|_1$ which is a reasonable assumption.

2): Let g be the usual fundamental solution of the Laplace operator in $\mathbb{R}^d$, i.e.

$$g(x,y) = \begin{cases} -\frac{1}{2\pi}\log|x-y| & \text{for } d=2 \\ \frac{1}{4\pi}\frac{1}{|x-y|} & \text{for } d=3. \end{cases}$$

Denoting by γ the trace operator $\gamma : H^s_{loc}(\mathbb{R}^d) \to H^{s-1/2}(\Gamma)$ for $s \in (1/2, 1]$ the single layer potential is defined by

$$V = \gamma S$$

where

$$Su(x) := \int_\Gamma g(x,y)u(y)ds_y \quad \forall x \in \mathbb{R}^d \setminus \Gamma.$$

Then a typical singular integral equation problem reads

$$Vu(x) = f(x) \qquad \forall x \in \Gamma$$

for a given function $f \in H^{1/2}(\Gamma)$.

The following theorem holds (cf. [C]).

Theorem 5.4 *For all $\sigma \in [-1/2, 1/2]$ the operators $S : H^{-1/2+\sigma}(\Gamma) \to H^{1+\sigma}_{loc}(\mathbb{R}^d)$ and $V : H^{-1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma)$ are continuous. Furthermore there exists a compact operator $K : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and a constant ρ such that*

$$\langle (V+K)u, u\rangle \geq \rho\|u\|^2_{-1/2} \quad \forall u \in H^{-1/2}(\Gamma). \tag{5.60}$$

Now we show that there is actually a symmetric compact operator K satisfying (5.60).

For an arbitrary $u \in H^{-1/2}(\Gamma)$ we consider $w = -Su \in H^1_{loc}(\mathbb{R}^d)$, choose a cut off function $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi = 1$ on a neighborhood of $\overline{\Omega}$ and define $w_1 := w|_\Omega,\ w_2 := \chi w|_{\Omega^c}$ with $\Omega^c := \mathbb{R}^d \backslash \overline{\Omega}$. By the continuity properties of the trace operator $\gamma_1 v := \partial_n v|_\Gamma$ and the jump relation $\gamma_1 w_2 - \gamma_1 w_1 = u$ we conclude that

$$\begin{aligned} \|u\|_{-1/2;\Gamma} &= \|\gamma_1 w_2 - \gamma_1 w_1\|^2_{-1/2;\Gamma} \\ &\lesssim \|w_1\|^2_{1;\Omega} + \|w_2\|^2_{1;\Omega^c} + \|\Delta w_2\|^2_{0;\Omega^c}, \end{aligned} \tag{5.61}$$

where the subscripts Γ, Ω and Ω^c indicate the respective domains.

Now Green's first forumula yields

$$0 = \|\nabla w_1\|^2_{0;\Omega} - \langle \gamma_1 w_1, \gamma_0 w_1\rangle \tag{5.62}$$

and

$$\int_{\Omega^c} w_2 \Delta w_2\, dx = \|\nabla w_2\|^2_{0;\Omega^c} + \langle \gamma_1 w_2, \gamma_0 w_2\rangle. \tag{5.63}$$

Summing (5.62), (5.63) and using again the jump relation we obtain by $\gamma_0 w_1 = \gamma_0 w_2 = -Vu$

$$\|\nabla w_1\|_{0;\Omega}^2 + \|\nabla w_2\|_{0;\Omega^c}^2 = \int_{\Omega^c} w_2 \Delta w_2 \, dx + \langle Vu, u\rangle. \tag{5.64}$$

Inserting the identity (5.64) in (5.61) yields the estimate

$$\|u\|_{-1/2;\Gamma}^2 \lesssim \|w_1\|_{0;\Omega}^2 + \|w_2\|_{0;\Omega^c}^2 + \|\Delta w_2\|_{0;\Omega^c}^2 + \int_{\Omega^c} w_2 \Delta w_2 \, dx + \langle Vu, u\rangle,$$

which motivates the following definition of the symmetric compact operator K

$$\langle Ku, u\rangle := \|w_1\|_{0;\Omega}^2 + \|w_2\|_{0;\Omega^c}^2 + \|\Delta w_2\|_{0;\Omega^c}^2 + \int_{\Omega^c} w_2 \Delta w_2 \, dx,$$

i.e.,

$$\langle Ku, v\rangle := 1/2(\langle K(u+v), (u+v)\rangle - \langle K(u-v), (u-v)\rangle).$$

Clearly

$$s(u, v) = \langle Vu, v\rangle + \langle Ku, v\rangle$$

defines a symmetric and $H^{-1/2}(\Gamma)$-elliptic bilinear form, which induces a norm $\|u\| = \sqrt{s(u,u)}$, that is equivalent to $\|\cdot\|_{-1/2}$.

It remains to investigate the estimate (5.58). By the continuity properties of the single layer potential one obtains

$$|k(u - u_{\tilde{\Lambda}}, u_\Lambda - u_{\tilde{\Lambda}})| \lesssim \|u - u_\Lambda\|_{-1;\Gamma}^2 + \|u - u_{\tilde{\Lambda}}\|_{-1;\Gamma}^2.$$

Thus condition (5.58) will be satisfied if $\|u - u_\Lambda\|_{-1;\Gamma}^2$, $\|u - u_{\tilde{\Lambda}}\|_{-1;\Gamma}^2$ are small compared to $\|u - u_\Lambda\|_{-1/2;\Gamma}^2$, $\|u - u_{\tilde{\Lambda}}\|_{-1/2;\Gamma}^2$ which is again a reasonable assumption.

5.6 Conclusions

Exploiting properties of stable multiscale bases, especially norm equivalences for certain ranges of Sobolev norms, we have shown that certain a-posteriori error estimators in terms of wavelet coefficients of residuals are efficient and reliable. Moreover, they give rise to adaptive schemes which are guaranteed to converge for a wide range of elliptic operator equations including those of negative order without assuming the validity of the saturation property beforehand. To our knowledge these are the first results of this type for the latter class of problems.

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