Coorbit Spaces and Banach Frames on Homogeneous Spaces with Applications to Analyzing Functions on Spheres

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Abstract

This paper is concerned with the construction of generalized Banach frames on homogeneous spaces. The major tool is a unitary group representation which is square integrable modulo a certain subgroup. By means of this representation, generalized coorbit spaces can be defined. Moreover, we can construct a specific reproducing kernel which, after a judicious discretization, gives rise to Banach frames for these coorbit spaces. We also discuss nonlinear approximation schemes based on our new Banach frames. As a classical example, we apply our construction to the problem of analyzing and approximating functions on the spheres.

Key Words: Square integrable group representations, time–frequency analysis, frames, homogeneous spaces, coorbit spaces, modulations spaces, nonlinear approximation, spheres.

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1 Introduction

A classical problem in applied mathematics is to analyze and to process a given set of signals. Usually, the first step is to decompose the signal into certain building blocks. A widespread strategy is to use Fourier transform, i.e., to analyze the signal with respect to its components corresponding to different frequencies. Although very successful in many applications, Fourier analysis has the serious disadvantage that the basis functions are not local so that small changes in the signal influence the whole Fourier spectrum. Therefore many attempts have been made to localize the Fourier transform in some natural way. In 1946, Gabor [19] introduced a time–frequency analysis which is often called the *short–time Fourier transform*. The idea is to use a window function g in order to localize the Fourier analysis. In the meantime, the short–time Fourier transform has indeed been established as a powerful tool in signal analysis. Another way to obtain some kind of local analysis would be to use the *wavelet transform*. Then the modulation term in the short–time Fourier transform is replaced by a dilation procedure, and it is possible to work with very localized basis functions. Starting with the pioneering work of Grossmann and Morlet [24], wavelet analysis has become a very important field in applied mathematics with many successful applications in image/signal analysis/compression, numerical analysis, geophysics and in many other fields. Although they may behave quite different in applications, there exists a common thread between Gabor and wavelet transform. Both can be derived from square integrable group representations of a certain group, see, e.g., [25] and Section 2 for details. Both transforms have their advantages and drawbacks, so that the decision which method to use depends on the specific application. For further information and a general overview on both transforms we refer to the excellent textbooks which have appeared quite recently [8, 21, 23, 27, 28, 29, 33].

In any case, when it comes to practical applications, only a *discrete* set of coefficients can be handled. It is therefore necessary to dicretize both transforms to obtain some kind of basis for the function space under consideration. However, constructing some stable basis may be asking to much, nevertheless, it is usually possible to obtain at least a frame. In general, given a Hilbert space H, a system $\{h_m\}_{m\in\mathbb{Z}}$ is called a *frame* if there exist constants A and B, $0 < A \leq B < \infty$ such that

$$A\|F\|_H^2 \leq \sum_{m\in\mathbb{Z}} |\langle F, h_m\rangle|^2 \leq B\|F\|_H^2. \tag{1.1}$$

This setting can also be generalized to Banach spaces, see, e.g., [15, 16] for details. In our case, the frames are obtained by discretizing the underlying group representation in some clever way. A very general machinery for frame constructions has been developed in the pioneering work of Feichtinger and Gröchenig [14, 15, 16, 17]. We shall present a more detailed discussion in Section 4. Once these frames are constructed, they usually also give rise to frames in certain smoothness spaces. These smoothness spaces are again defined by the underlying square integrable group representation, i.e., one collects all functions for which the associated (Gabor are wavelet) transform is contained in some (weighted) L_p–space on the group. These functions spaces are usually called *coorbit spaces* and will be introduced more accurately in Section 3. For the Gabor transform, the coorbit spaces are nothing else but the *modulation spaces*, whereas for the wavelet transform one obtains the *Besov spaces*. We refer to [9, 10, 14, 15, 16, 17, 21, 29, 31] for the definitions and the main properties of modulation and Besov spaces. At this point, the strong analytical properties of wavelets come into play. Indeed, it can be shown that moreover stable wavelet bases for a huge scale of Besov spaces involving those related with L_p–spaces for $p < 1$ can be established, see again [9, 10, 29] for details. These relationships have some very important consequences. In fact, it can be shown that the order of convergence of nonlinear approximation schemes such as best N–term approximation or adaptive wavelet Galerkin methods depends on the regularity of the approximated object in a very specific Besov scale, see, e.g, [5, 7, 9, 10] for details. For the case of the Gabor transform, quite recently results have been derived by Gröchenig and Samarah [22]. They have shown that the approximation order of nonlinear schemes based on local Fourier bases is determined by the regularity in some specific scale of modulation space. Nevertheless, these results are naturally weaker when compared with those for the wavelet case.

In any case, when it comes to practical applications, it is clearly desirable to generalize the theories developed so far to bounded domains and manifolds. This problem has been intensively studied in the last few years. Because of the strong analytical properties of wavelets, one might feel temptered to start with the wavelet transform. However, usually the dilation procedure involved in the wavelet transform does not fit together very well with the boundedness of the domain. Nevertheless, quite recently an almost complete solution to this problem has been given by Antoine and Vandergheynst [2, 3]. Their approach makes heavy use of group theory and can also be used to construct suitable wavelet frames [4]. However, the whole machinery is very complicated. It is fun for the specialists but terrible for the average consumer. In this context, Gabor analysis seems to have a serious advantage. It seems that the generalization of the Gabor transform to manifolds is much simpler than for the wavelet transform. Indeed, quite recently, a first approach for the case of the sphere in $\mathbb{R}^d$ has been presented by Torresani [32].

In summary, the current state of the art suggests the following questions:

- Is it possible to construct a generalized Gabor transform on manifolds and to properly define the associated coorbit spaces?
- Is it possible to generalize the machinery developed by Feichtinger and Gröchenig to this case and to obtain generalized Gabor frames in these coorbit spaces?
- What are the smoothness spaces which determine the order of convergence of the associated best N–term approximation schemes?
- Is it possible to come from abstract general nonsense to concrete applications, e.g., by combining these investigations with Torresani's results in order to obtain Gabor frames on spheres?

In order to execute this program, we proceed in the following way. We start by discussing the group theoretical background in Section 2. Given our manifold $\mathcal{N}$, the first step is clearly to find a locally compact group $\mathcal{G}$ which admits a unitary representation in the Hilbert space $L_2(\mathcal{N})$. To be on safe side, this representation has to be irreducible and square integrable. The first property is usually relatively easy to realize whereas the second one often causes trouble because the group is to 'large'. To obtain a 'smaller' group, *one* natural way would be to extract a closed subgroup $\mathcal{G}_{\mathcal{F}}$ and to restrict the representation to the quotient space $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. However, since $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ has no longer a group structure, one has to ensure that nevertheless all the nice properties of square integrable representations can be saved. Once these relationships are clarified, we are able to define associated coorbit spaces in Section 3. Loosely speaking, these generalized coorbit spaces consist of all function for which the associated Gabor transform is contained in some L_p–space on the quotient manifold $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. According to our program, the next step is to construct Banach frames for these coorbit spaces in Section 4. To this end, we investigate to what extent the general approach of Feichtinger and Gröchenig can be adapted to our setting. The first step is always to define some kind of approximation operator. This operator is usually defined by means of a convolution with the Gabor transform of the analyzing function itself. Since a group structure doesn't longer exist in our setting, a convolution is no longer well–defined. We therefore suggest to replace this convolution by a suitable defined integral transform involving a specific kernel defined by means of the analyzing function, see Subsection 4.2 for details. The next step is to discretize this approximation operator to obtain the desired frames. In Subsection 4.3, we show that under very natural assumptions both, the upper and the lower frame bound, can be established. As outlined above, we also intend to analyze nonlinear approximation schemes based on the new Banach frames. In Section 5, we show that the results of Gröchenig and Samarah on Banach frames carry over to our case without any serious difficulty. Finally, in Section 6, we discuss some applications of our theory, i.e., we treat the problem of analyzing functions on spheres. Our approach is based on the fundamental investigations of Torresani [32]. We show that in the setting of [32] all our assumptions are satisfied so that our theory yields generalized coorbit spaces on spheres and also provides us with suitable Banach frames for these spaces.

Remark 1.1 *i) We want to emphasize that we do not claim to rediscover the whole world of square integrable group representations. It is clear the some of the building blocks used in this paper have already been established before, at least partially. However, we intend to establish the relationships between all these building blocks and to show that they fit together quite nicely.*
ii) The basic idea of this paper has been developed while listening to a talk of K. Gröchenig on "New Results in Time–Frequency Analysis".

2 Group Theoretical Background

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{G}$ be a separable Lie group with (right) Haar measure ν. A *continuous representation* of $\mathcal{G}$ in $\mathcal{H}$ is defined as a mapping

$$U : \mathcal{G} \longrightarrow \mathcal{L}(\mathcal{H}) \tag{2.1}$$

of $\mathcal{G}$ into the space $\mathcal{L}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, such that $U(gg') = U(g)U(g')$ for all $g, g' \in \mathcal{G}$, $U(e) = Id$ and for any $\phi, \psi \in \mathcal{H}$, the function $g \in \mathcal{G} \to \langle \phi, U(g)\psi \rangle_{\mathcal{H}}$ is continuous. The representation U is said to be *square-integrable* if it is irreducible and there exists a nonzero $\psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi \rangle_{\mathcal{H}}|^2 \, d\nu(g) < \infty \, . \tag{2.2}$$

Such a function ψ is called *admissible.* In the sequel, we shall always be concerned with the case that the Hilbert space $\mathcal{H}$ is given as some L_2–space on a manifold $\mathcal{N}$, i.e. $\mathcal{H} = L_2(\mathcal{N})$. As an example, let us consider the *reduced Weyl-Heisenberg group* $\mathcal{G}_{WH}^{red} \cong \mathbb{R}^2 \times S^1$, generated by time and frequency translations on the real line. The group operation is explicitly given by

$$(p, q, \phi)(p', q', \phi') = (p + p', q + q', \phi + \phi' + p'q).$$

The Weyl–Heisenberg group $\mathcal{G}_{WH}^{red}$ admits unitary irreducible representations on $L_2(\mathbb{R})$ which act as follows:

$$U(p, q, \phi)f(x) = \exp(i(\lambda\phi + q(x - \lambda p)))f(x - \lambda p).$$

Because S^1 is compact it is easy to check that U is square integrable for any nonzero $\psi \in \mathcal{H}$. This specific representation can be viewed as the basic building block for the classical Gabor transform, see, e.g., [21] for details. However, there are cases in which square-integrable representations are not available. A simple example is the full Weyl–Heisenberg group $\mathcal{G}_{WH} \cong \mathbb{R}^2 \times \mathbb{R}$. Nevertheless, its coefficients $\langle f, U(q, p, 0)\psi \rangle$ form a square integrable function of $(q, p) \in \mathbb{R}^2$. This example suggests a general strategy. Indeed, the cases where no square-integrable representations are available can very often be handled by restricting U to a convenient quotient $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$. Unless otherwise stated, we shall always consider right coset spaces, i.e.,

$$g_1 \sim g_2 \qquad \text{if and only if} \qquad g_1 = h \circ g_2 \qquad \text{for some} \qquad h \in \mathcal{P}. \tag{2.3}$$

Because U is not directly defined on $\mathcal{G}/\mathcal{P}$, it is necessary to embed $\mathcal{G}/\mathcal{P}$ in $\mathcal{G}$. This can be realized by using the canonical fiber bundle structure of $\mathcal{G}$ with projection $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{P}$. Let $\sigma : \mathcal{G}/\mathcal{P} \to \mathcal{G}$ be a Borel section of this fiber bundle, i.e. $\Pi \circ \sigma(h) = h$ for all $h \in \mathcal{G}/\mathcal{P}$. We introduce $U \circ \sigma$ and some quasi-invariant measure μ on $\mathcal{G}/\mathcal{P}$, which is defined by

$$\int_{\mathcal{G}/\mathcal{P}} \left(\int_{\mathcal{P}} f(h \circ g) d\zeta(h) \right) d\mu([g]) = \int_{\mathcal{G}} f(g) d\nu(g) \quad \text{for all} \quad f \in C_0(\mathcal{G}), \tag{2.4}$$

where ζ denotes the (right) Haar measure on $\mathcal{P}$, see [30, 32] for details.

Then we say that U is *strictly square integrable* mod $(\mathcal{P}, \sigma)$, if there exists $\psi \in L_2(\mathcal{N})$ such that the mapping $V_\psi : L_2(\mathcal{N}) \to L_2(\mathcal{G}/\mathcal{P})$ defined by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi \rangle_{L_2(\mathcal{N})} \tag{2.5}$$

is an isometry. In this case, (ψ, σ) is called a *strictly admissible pair* and ψ a *strictly admissible function* (with respect to σ) [2].

To exploit this concept, the first step is clearly to define an appropriate subgroup of $\mathcal{G}$. We begin with the *adjoint mapping* of $\mathcal{G}$ acting on itself by inner automorphism, i.e. $ad(h)g := hgh^{-1}$, where $g, h \in \mathcal{G}$. This action induces a corresponding action $Ad(h)$ on the Lie algebra $\mathcal{T}_e\mathcal{G}$ of $\mathcal{G}$, $Ad(h)X = hXh^{-1}$ with $X \in \mathcal{T}_e\mathcal{G}$. Finally, the *coadjoint* $Ad(h)^*$ on the dual Lie algebra $\mathcal{T}_e^*\mathcal{G}$ is defined by

$$\langle X, Ad(h)^* F\rangle := \langle Ad(h)X, F\rangle, \quad \text{for } F \in \mathcal{T}_e^*\mathcal{G}.$$

For $\mathcal{F} \in \mathcal{T}_e^*\mathcal{G}$, let

$$\mathcal{G}_{\mathcal{F}} := \{g \in \mathcal{G} : Ad(g)^*\mathcal{F} = \mathcal{F}\} \tag{2.6}$$

denote the stability subgroup of $\mathcal{F}$. Whenever the coadjoint orbit $\mathcal{O}_{\mathcal{F}} \cong \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ can be associated with the representation under consideration, the quotient space $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ is a natural candidate to perform the previous construction.

Assume now that (ψ, σ) is a strictly admissible pair for our setting. Then the isometry V_ψ can be inverted on its image by its adjoint V_ψ^*, which is obviously given by

$$V_\psi^* F(s) := \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(h) U(\sigma(h)^{-1})\psi(s)\, d\mu(h).$$

This provides us with the reconstruction formula

$$f = V_\psi^* V_\psi f = \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \langle f, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} U(\sigma(h)^{-1})\psi\, d\mu(h) \tag{2.7}$$

for $f \in L_2(\mathcal{N})$.

We intend to establish a correspondence principle between $L_2(\mathcal{N})$ and a subspace of $L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ similar to the correspondence principle between $L_2(\mathbb{R}^n)$ and a subspace of the square integrable functions on the reduced Weyl-Heisenberg group. We define a kernel on $\mathcal{G}/\mathcal{G}_{\mathcal{F}} \times \mathcal{G}/\mathcal{G}_{\mathcal{F}}$

$$\begin{aligned} R(h,l) &:= \langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} && (2.8)\\ &= \langle \psi, U(\sigma(h)\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= V_\psi(U(\sigma(h)^{-1})\psi)(l). && (2.9) \end{aligned}$$

Note that $R(h,l) = \overline{R(l,h)}$. Further, we see by (2.9) that $R(h,\cdot) \in L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ for any fixed $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ and by applying Schwarz's inequality in (2.8) that $R \in L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}} \times \mathcal{G}/\mathcal{G}_{\mathcal{F}})$. Now we can prove the following correspondence principle between $L_2(\mathcal{N})$ and the reproducing kernel space

$$\mathcal{M}_2 := \{F \in L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) : \ \langle F, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = F(h)\}. \tag{2.10}$$

Proposition 2.1 *Let U be a strictly square integrable representation of $\mathcal{G}$* mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ *and ψ a strictly admissible function. Let V_ψ and R be defined by* (2.5) *and* (2.8), *respectively.*

i) For every $f \in L_2(\mathcal{N})$, the following equation is satisfied

$$\langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = V_\psi f(h),$$

i.e., $V_\psi f \in \mathcal{M}_2$.

ii) For every $F \in \mathcal{M}_2$ there exists a uniquely determined function $f \in L_2(\mathcal{N})$ such that $F = V_\psi f$.

Consequently, the spaces $L_2(\mathcal{N})$ and $\mathcal{M}_2$ are isometrically isomorph.

Proof *i)* Since $U(\sigma(h)^{-1})\psi \in L_2(\mathcal{N})$ we have by (2.7) that

$$\begin{aligned} V_\psi f(h) &= \langle f, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= \langle f, \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} R(l,h) U(\sigma(l)^{-1})\psi \, d\mu(l)\rangle \\ &= \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \overline{R(h,l)} \langle f, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \, d\mu(l) \\ &= \langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \; . \end{aligned}$$

ii) Let $F \in L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ fulfill

$$F(h) = \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)\overline{R(h,l)} d\mu(l).$$

Then we obtain by the definition of R in (2.8)

$$\begin{aligned} F(h) &= \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l) \overline{\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})}} \, d\mu(l) \\ &= \langle \int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l) U(\sigma(l)^{-1})\psi d\mu(l), U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= V_\psi(V_\psi^* F)(h) \; . \end{aligned}$$

Since V_ψ is an isometry, the mapping $V_\psi V_\psi^*$ is an orthogonal projector onto the image of V_ψ. Thus, there exists $f \in L_2(\mathcal{N})$ such that $F = V_\psi V_\psi^* F = V_\psi f$. The uniqueness of $f \in L_2(\mathcal{N})$ is clear because V_ψ is injective. ■

3 Coorbit Spaces on Homogeneous Spaces

We want to modify the concept of coorbit spaces [17] to functions defined on manifolds. In order to keep comparisons as simple as possible, we adapt the notations given in [13, 14, 15, 16, 17]. Furthermore, to keep the technical difficulties at a reasonable level, we only consider the 'simplest' case, e.g., the weight functions w involved in the usual definition of coorbit spaces is assumed to be $w \equiv 1$. The general case will be studied in a forthcoming paper.

Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ with a strictly admissible function ψ. For the kernel R in (2.8), we will need the basic assumption that

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(h,l)| d\mu(l) \leq C \tag{3.1}$$

with a constant $C < \infty$ independent of $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$.

By H_1' we denote the space of all continuous linear functionals on

$$H_1 := \{ f \in L_2(\mathcal{N}) : \; V_\psi f \in L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \}.$$

As usual, the norm $\|\cdot\|_{H_1}$ on H_1 is defined as

$$\|f\|_{H_1} := \|V_\psi f\|_{L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}.$$

By definition, we have the following continuous embeddings

$$H_1 \hookrightarrow H \hookrightarrow H_1'.$$

Further, we note by (3.1) that $U(\sigma(h)^{-1})\psi \in H_1$ for all $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$. Consequently, the following generalization of the operator V_ψ in (2.5) on H_1' is well defined:

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi \rangle, \tag{3.2}$$

where $f \in H_1'$. For any $f \in H_1'$, we obtain by (3.1) that

$$\begin{aligned} \|V_\psi f\|_{L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} &= \|\langle f, U(\sigma(h)^{-1})\psi \rangle\|_{L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \\ &\leq \|f\|_{H_1'} \operatorname*{ess\,sup}_{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}} \|U(\sigma(h)^{-1})\psi\|_{H_1} \\ &= \|f\|_{H_1'} \operatorname*{ess\,sup}_{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}} \|R\|_{L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} \\ &\leq C\|f\|_{H_1'} \; . \end{aligned} \tag{3.3}$$

Thus, $V_\psi : H_1' \to L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$. For $F \in L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ and $g \in H_1$, we have further that

$$\langle F, V_\psi g \rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l) \overline{V_\psi g(l)} \, d\mu(l)$$

$$= \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)\overline{\langle g, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})}}\, d\mu(l)$$
$$= \langle \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)U(\sigma(l)^{-1})\psi\, d\mu(l), g\rangle_{L_2(\mathcal{N})}.$$

We define the operator $\tilde{V}_\psi : L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \to H_1'$ by

$$\tilde{V}_\psi F := \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)U(\sigma(l)^{-1})\psi \; d\mu(l)\,,$$

where the integral is considered in the weak sense. Then we obtain for $F \in L_\infty(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ that

$$\begin{aligned} V_\psi \tilde{V}_\psi F &= \langle \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)U(\sigma(l)^{-1})\psi\, d\mu(l), U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} F(l)\langle U(\sigma(l)^{-1})\psi, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})}\, d\mu(l) \\ &= \langle F, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}\; . \end{aligned} \tag{3.4}$$

Similar to the coorbit spaces on $\mathbb{R}^n$ we define

$$M_p := \{f \in H_1' :\; V_\psi f \in L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})\}\,, \tag{3.5}$$

with $1 \le p \le \infty$ and norm

$$\|f\|_{M_p} := \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}.$$

It is straightforward that $\|\cdot\|_{M_p}$ defines a seminorm. The property that $\|f\|_{M_p} = 0, i.e., V_\psi f = 0$, implies $f = 0$ follows similarly as in [14] by proving that $\{U(\sigma(h)^{-1})\psi : h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}\}$ is a dense subset of H_1. The basic step for the investigations outlined below is a correspondence principle between these coorbit spaces and certain subspaces on the quotient group $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ which are defined by means of the reproducing kernel R. To this end, we consider the subspaces

$$\mathcal{M}_p := \{F \in L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) :\; \langle F, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = F\} \tag{3.6}$$

of $L_p(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$ with $1 \le p \le \infty$. Then the desired correspondence principle can be formulated as follows:

Proposition 3.1 *Let U be a strictly square integrable representation of $\mathcal{G}$* mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ *and ψ a strictly admissible function. Let V_ψ be defined by* (3.2) *and let R in* (2.8) *fulfill* (3.1).

i) For every $f \in M_p$, the following equation is satisfied

$$\langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} = V_\psi f \ ,$$

i.e., $V_\psi f \in \mathcal{M}_p$.

ii) For every $F \in \mathcal{M}_p$, $1 \le p \le \infty$, there exists a uniquely determined functional $f \in M_p$ such that $F = V_\psi f$.

Consequently, the spaces M_p and $\mathcal{M}_p$, $1 \le p \le \infty$, are isometrically isomorph.

Proof Assertion *i)* follows in the same way as *i)* in Proposition 2.1, where only properties of ψ were used.
ii). For $F \in \mathcal{M}_p$, $1 \le p \le \infty$, we have that

$$\begin{aligned}\|F\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \| \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\, d\mu(l)\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\ &= \operatorname{ess}\sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} | \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\, d\mu(l)|,\end{aligned}$$

and further, by applying Hölder's inequality with $1/p + 1/q = 1$, the fact that $R \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F} \times \mathcal{G}/\mathcal{G}_\mathcal{F})$ and (3.1),

$$\begin{aligned}| \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\, d\mu(l)| &\le \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)||R(h,l)|^{1/p+1/q}\, d\mu(l) \\ &\le (\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)|^p|R(h,l)|\, d\mu(l))^{1/p}(\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |R(h,l)| d\mu(l))^{1/q} \\ &\le c\,\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \ .\end{aligned}$$

Consequently, we have that

$$\|F\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \le c\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \ .$$

Thus, $F \in L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})$ and by (3.4) we obtain that $F = V_\psi(\tilde{V}_\psi F)$, where $\tilde{V}_\psi F \in H_1'$ and since $F \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})$ also $\tilde{V}_\psi F \in M_p$. The uniqueness condition follows by definition of M_p. ∎

Applying Proposition 3.1 *i)* and (3.4) we get for $f \in H_1'$ that

$$V_\psi \tilde{V}_\psi (V_\psi f) = \langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} = V_\psi f \ .$$

Hence, $\tilde{V}_\psi V_\psi$ is the identity in H_1' and we have the reconstruction formula

$$f = \tilde{V}_\psi V_\psi f = \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \langle f, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} U(\sigma(h)^{-1})\psi\, d\mu(h) \ .$$

We finish the section by establishing the relationships of our generalized coorbit spaces to the fundamental spaces $L_2(\mathcal{N})$ and H_1'.

Proposition 3.2 *Under the assumptions outlined above, the following relations are valid:*

i) $M_\infty = H_1'$,

ii) $M_2 = L_2(\mathcal{N})$.

Proof *i)*. For $f \in H_1'$ we have by (3.3) that $\|V_\psi f\|_{L_\infty(\mathcal{G}/\mathcal{G}_\mathcal{F})} \le c||f||_{H_1'}$ which yields the first assertion.
ii). Let $f \in L_2(\mathcal{N})$. Then we obtain by Proposition 2.1 that $V_\psi(f) \in \mathcal{M}_2$. By Proposition 3.1 there exists $g \in M_2$ such that $V_\psi(f) = V_\psi(g)$ which implies by definition of M_2 that $f = g$.

Conversely, let $f \in M_2$. Then we have by Proposition 3.1 that $V_\psi(f) \in \mathcal{M}_2$. By Proposition 2.1 there exists $g \in L_2(\mathcal{N})$ such that $V_\psi(f) = V_\psi(g)$ which implies by definition of M_2 that $f = g$. ■

4 Banach Frames for Coorbit Spaces

Once our generalized coorbit spaces are established, the next step is to derive some atomic decomposition for these spaces, i.e., we want to construct suitable Banach frames. This program is performed in several steps. In the next subsection, we present some preparations and state our main result. The remaining two subsections are devoted to the building blocks which are necessary to prove this result. The major step is the construction of a suitable approximation operator which is defined and analyzed in Subsection 4.2. This approximation operator can then be used to establish the frame bounds in Subsection 4.3.

The results in this section are again inspired by the pioneering work of Feichtinger and Gröchenig, [14, 15, 16, 17].

4.1 Setting and Main Result

Before we can state and prove our main result, some preparations are necessary. Given some neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$, a family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called $\mathcal{U}$*–dense* if $\bigcup_{i\in\mathcal{I}} \mathcal{U}x_i = \mathcal{G}$. A family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *relatively separated,* if for any compact set $\mathcal{Q} \subseteq \mathcal{G}$ there exists a finite partition of the index set $\mathcal{I}$, i.e., $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$, such that $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for all $i, j \in \mathcal{I}_r$ with $i \neq j$.

Let $\mathcal{U}$ be an arbitrary compact neighborhood of the identity in $\mathcal{G}$. By [12], there exists a bounded uniform partition of unity (of size $\mathcal{U}$), i.e., a family of continuous functions $(\varphi_i)_{i\in\mathcal{I}}$ on $\mathcal{G}$ such that

- $0 \le \varphi_i(g) \le 1$ for all $g \in \mathcal{G}$;
- there is an $\mathcal{U}$-dense, relatively separated family $(x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ such that supp $\varphi_i \subseteq \mathcal{U}x_i$;
- $\sum_{i\in\mathcal{I}} \varphi_i(g) \equiv 1$ for all $g \in \mathcal{G}$.

Furthermore, we define the *$\mathcal{U}$–oscillation* with respect to the analyzing wavelet ψ as

$$\mathrm{osc}_{\mathcal{U}}(l,h) := \sup_{u\in\mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})}| \ . \tag{4.1}$$

In the sequel, we shall always assume that $(x_i)_{i\in\mathcal{I}}$ can be chosen such that $\sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \cap \mathcal{U}x_i \neq \emptyset$ implies $x_i \in \sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}})$. Let

$$\mathcal{I}_\sigma := \{ i \in \mathcal{I} : \ \sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \cap \mathcal{U}x_i \neq \emptyset \} \ .$$

Then there exist h_i such that $x_i = \sigma(h_i)$, where $i \in \mathcal{I}_\sigma$. Note that

$$\sum_{i\in\mathcal{I}_\sigma} \varphi_i(\sigma(h)) = 1 \ ,$$

where $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$.

In this setting, we can prove our main theorem.

Theorem 4.1 *Let $\mathcal{G}$ be a separable Lie group with stability subgroup $\mathcal{G}_{\mathcal{F}}$ defined by* (2.6) *and let μ be a quasi–invariant measure on $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$. Further, let U be a strictly square integrable representation of $\mathcal{G}$* mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$ *in $L_2(\mathcal{N})$ with strictly admissible function ψ. Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen so small that*

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \mathrm{osc}_{\mathcal{U}}(l,h)d\mu(l) < 1 \quad \textit{and} \quad \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \mathrm{osc}_{\mathcal{U}}(l,h)d\mu(h) < 1 \, . \tag{4.2}$$

Let $X = (x_i)_{i\in\mathcal{I}}$ be a $\mathcal{U}$–dense and relatively separated family. Furthermore, suppose that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$

$$\mu\{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}} : \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \ge C_{\mathcal{Q}} > 0$$

holds for all $i \in \mathcal{I}_\sigma$. Finally, let us assume that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ our window function ψ fulfills the following inequality

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \sup_{q\in Q} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{L_2(\mathcal{N})}| \, d\mu(l) \le C \tag{4.3}$$

with a constant $C < \infty$ independent of $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$. Then any $f \in M_p$, $1 \le p < \infty$, has an expansion

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi. \tag{4.4}$$

Moreover, the set

$$\psi_i = U(\sigma(h_i)^{-1})\psi, \quad i \in \mathcal{I}_\sigma, \tag{4.5}$$

is a Banach frame for M_p*, i.e., there exist two constants* $0 < A \le B < \infty$ *such that*

$$\frac{1}{B}\|f\|_{M_p} \le \|(\langle f, \psi_i\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \le \frac{1}{A}\|f\|_{M_p}. \tag{4.6}$$

The proof of this theorem is presented in the following two sections.

4.2 Approximation Operators

In this section, we show that under the assumptions of Theorem 4.1 the expansion (4.4) is valid. The basic idea is to construct expansions for the spaces $\mathcal{M}_p$ and then to use Proposition 3.1 *ii)* to derive the desired expansions also for M_p. The major tool is the generalized reproducing kernel $R(h,l)$. Indeed, the definition of $\mathcal{M}_p$ in (3.6) suggests that discretizing $R(h,l)$ may yield a suitable approximation for functions in $\mathcal{M}_p$. We therefore consider the following approximation operator in $\mathcal{M}_p$:

$$\begin{aligned} T_\varphi F(h) &:= \sum_{i\in\mathcal{I}_\sigma} \langle F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} R(h_i, h) \\ &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i(\sigma(l)) d\mu(l) R(h_i, h) \ . \end{aligned}$$

By definition of $\mathcal{M}_p$, we have that

$$\begin{aligned} F(h) &= \langle F, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} = \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\overline{R(h,l)}\, d\mu(l) \\ &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i(\sigma(l)) R(l,h)\, d\mu(l) \end{aligned}$$

and consequently

$$F(h) - T_\varphi F(h) = \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} F(l)\varphi_i(\sigma(l)) \left[R(l,h) - R(h_i,h)\right] d\mu(l) \ .$$

Therefore we obtain

$$|F(h) - T_\varphi F(h)| \le \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)|\varphi_i(\sigma(l))|R(l,h) - R(h_i,h)|\, d\mu(l) \ .$$

Now $\sigma(l) \in \mathcal{U}x_i$ implies that there exists $u_i \in \mathcal{U}$ such that $\sigma(l) = u_i x_i = u_i \sigma(h_i)$. Then $\sigma(h_i)^{-1} = \sigma(l)^{-1} u_i$ and we get by the definition of R that

$$\begin{aligned}
|F(h) - T_\varphi F(h)| &\le \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)|\varphi_i(\sigma(l))|\langle U(\sigma(l)^{-1})\psi, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} - \\
&\qquad \langle U(\sigma(l)^{-1}u_i)\psi, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})}|\, d\mu(l) \\
&= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)|\varphi_i(\sigma(l)) \times \\
&\qquad |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u_i^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})}|\, d\mu(l) \\
&\le \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)| \mathrm{osc}_\mathcal{U}(l,h)\, d\mu(l)\ ,
\end{aligned}$$

where $\mathrm{osc}_\mathcal{U}(l,h)$ is defined by (4.1). Then we conclude that

$$\begin{aligned}
\|F - T_\varphi F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \|(I - T_\varphi)F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \\
&\le \Big(\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \Big(\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |F(l)| \mathrm{osc}_\mathcal{U}(l,h)\, d\mu(l)\Big)^p\, d\mu(h)\Big)^{1/p}\ .
\end{aligned}$$

Now, by applying the generalized Young inequality, see, e.g., [18], p. 185, Theorem 6.18, and recalling the assumptions (4.2), we obtain

$$\|(I - T_\varphi)F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} < \|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\ .$$

Consequently $|||(I - T_\varphi)||| < 1$, i.e., $I - T_\varphi$ is a contraction on $\mathcal{M}_p$ and T_φ is invertible on $\mathcal{M}_p$. Thus we can write

$$F = T_\varphi T_\varphi^{-1} F = \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} R(h_i, h)\ . \tag{4.7}$$

Let $f \in M_p$. Then we have by Proposition 3.1 *ii)* that $F := V_\psi f \in \mathcal{M}_p$ and further by definition of V_ψ that

$$\begin{aligned}
V_\psi f &= \langle f, U(\sigma(h)^{-1})\psi\rangle \\
&= \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} R(h_i, h) \\
&= \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} \langle U(\sigma(h_i)^{-1})\psi, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\
&= \langle \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} U(\sigma(h_i)^{-1})\psi, U(\sigma(h)^{-1})\psi\rangle_{L_2(\mathcal{N})}\ .
\end{aligned}$$

Hence we obtain the following discrete reconstruction formula for $f \in M_p$

$$f = \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} U(\sigma(h_i)^{-1})\psi\ , \tag{4.8}$$

and (4.4) is shown with $c_i := \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})}$.

4.3 Frame Bounds

In this section, we want to prove the second part of Theorem 4.1, i.e., we want to establish (4.6). To this end, it is sufficient to show that there exist two constants $0 < A \le B < \infty$ such that

$$A\|f\|_{M_p} \le \|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{l_p} \le B\|f\|_{M_p} \tag{4.9}$$

holds. Indeed, the coefficients $\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})}$ are given by functionals ξ_i in M_p', i.e.

$$\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} = \langle f, \xi_i\rangle \ ,$$

where $\xi_i = V_\psi^*((T_\varphi^{-1})^*\varphi_i \circ \sigma)$. Now duality arguments [14, 15] yield that

$$\{\psi_i = U(\sigma(h_i)^{-1})\psi : \ i \in \mathcal{I}_\sigma\}$$

is a Banach frame for M_p, $1 \le p < \infty$, i.e., there exist constants $0 < A \le B < \infty$ such that

$$\frac{1}{B}\|f\|_{M_p} \le \|(\langle f, \psi_i\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \le \frac{1}{A}\|f\|_{M_p}$$

and the reconstruction of f from the frame coefficients is

$$f = \sum_{i\in\mathcal{I}\sigma} \langle f, \psi_i\rangle \xi_i \ .$$

In the following lemmata, we show that under the assumptions in Theorem 4.1 both, the upper and the lower bound in (4.9), are valid.

Lemma 4.1 *Suppose that the conditions in Theorem 4.1 are satisfied. For any $f \in M_p$ let the sequence*

$$(c_i)_{i\in\mathcal{I}_\sigma} = (\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})})_{i\in\mathcal{I}_\sigma}$$

be given by (4.8)*. Then there exists a constant $B < \infty$ such that the following inequality holds:*

$$\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \le B\|f\|_{M_p} \ .$$

In particular, we have that $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$.

Proof 1. First we show that for any sequence $(\eta_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$ the inequality

$$\|(\eta_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \le C_\mathcal{U}^{1/p} \|\sum_{i\in\mathcal{I}_\sigma} |\eta_i| 1_{\mathcal{U}x_i} \circ \sigma\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \tag{4.10}$$

holds, where again $x_i = \sigma(h_i)$ and where $1_{\mathcal{U}x_i}$ denotes the characteristic function of $\mathcal{U}x_i$.

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, there exists a splitting $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$ such that $\mathcal{U}x_i \cap \mathcal{U}x_j = \emptyset$ for $i, j \in \mathcal{I}_r$ and $i \neq j$. This results in a decomposition $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma_r}$, where

$$\mathcal{I}_{\sigma_r} = \{i \in \mathcal{I}_r : \ \mathcal{U}x_i \cap \sigma(\mathcal{G}/\mathcal{G}_\mathcal{F}) \neq \emptyset\} \ .$$

Then we obtain (4.10) by

$$\begin{aligned}
\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|^p_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} &= \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}}\left(\sum_{r=1}^{r_0}\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\right)^p d\mu(h)\\
&\geq \sum_{r=1}^{r_0}\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}}\left(\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\right)^p d\mu(h)\\
&= \sum_{r=1}^{r_0}\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}}\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|^p 1_{\mathcal{U}x_i}(\sigma(h))d\mu(h)\\
&\geq C_\mathcal{U}\sum_{i\in\mathcal{I}_\sigma}|\eta_i|^p\ .
\end{aligned}$$

2. Let $F\in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})$. Then the application of (4.10) yields

$$\begin{aligned}
\|(\langle F,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} &\leq \|(\langle |F|,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p}\\
&\leq C_\mathcal{U}^{-1/p}\|\sum_{i\in\mathcal{I}_\sigma}\langle |F|,\varphi_i\circ\sigma\rangle 1_{\mathcal{U}x_i}\circ\sigma\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\ .
\end{aligned}$$

Further, we see for an arbitrary fixed $h\in\mathcal{G}/\mathcal{G}_\mathcal{F}$ that

$$\sum_{i\in\mathcal{I}_\sigma}\langle |F|,\varphi_i\circ\sigma\rangle 1_{\mathcal{U}x_i}(\sigma(h))=\sum_{i\in\mathcal{I}_h}\langle |F|,\varphi_i\circ\sigma\rangle\ ,$$

where $\mathcal{I}_h:=\{i\in\mathcal{I}_\sigma:\ x_i\in\mathcal{U}^{-1}\sigma(h)\}$ and further that

$$\begin{aligned}
\sum_{i\in\mathcal{I}_h}\langle |F|,\varphi_i\circ\sigma\rangle &= \sum_{i\in\mathcal{I}_h}\langle |F|,\varphi_i(\sigma(\cdot))\rangle\\
&\leq \langle |F|,1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(\cdot)\sigma(h)^{-1})\rangle\ .
\end{aligned}$$

Since

$$\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}}1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(l)\sigma(h)^{-1})\,d\mu(l) = \mu\{l\in\mathcal{G}/\mathcal{G}_\mathcal{F}:\ \sigma(l)\in\mathcal{U}\mathcal{U}^{-1}\sigma(h)\}\leq c\ ,$$

for all $h\in\mathcal{G}/\mathcal{G}_\mathcal{F}$ we obtain by the generalized Young inequality, compare again with the appendix, that

$$\begin{aligned}
\|(\langle F,\varphi_i\circ\sigma\rangle)_{i\in I_\sigma}\|_{\ell_p} &\leq C_\mathcal{U}^{-1/p}\|\langle |F|,1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(\cdot)\sigma(h)^{-1})\rangle\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\
&\leq C_\mathcal{U}^{-1/p}c\|F\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\ .
\end{aligned}$$

Finally, we conclude by using $F=T_\varphi^{-1}V_\psi f\in\mathcal{M}_p$ in the above inequality that

$$\begin{aligned}
\|(\langle T_\varphi^{-1}V_\psi f,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} &\leq c\,\|T_\varphi^{-1}V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\
&\leq c\,|||T_\varphi^{-1}|||\ \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}\\
&\leq c\,|||T_\varphi^{-1}|||\ \|f\|_{M_p}\ . \qquad \blacksquare
\end{aligned}$$

The next step is to derive the lower bound in (4.9).

Lemma 4.2 *Suppose that the conditions in Theorem 4.1 are satisfied. Then there exists a constant $A > 0$ such that for any sequence $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_p$, $1 \le p \le \infty$, the following inequality holds:*

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_p} \le \frac{1}{A}\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \ . \tag{4.11}$$

In particular, we have by (4.8) that

$$\|f\|_{M_p} \le \frac{1}{A}\|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \ .$$

Proof By definition of the norm in M_p and (2.9) we have

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_p} = \|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})} \ .$$

By the Riesz–Thorin Interpolation Theorem, see, e.g., [18] Chapter 6 and the appendix for details, it suffices to prove the inequality (4.11) for $p = 1$ and $p = \infty$. For $p = 1$, we obtain

$$\begin{aligned} by(3.1)\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_1} &= \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)|\, d\mu(h) \\ &\le \sum_{i\in\mathcal{I}_\sigma} |c_i| \sup_{i\in\mathcal{I}_\sigma} \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} |R(h_i,h)|\, d\mu(h) \\ &\le C\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_1} \ . \end{aligned}$$

For $p = \infty$ it follows that

$$\begin{aligned} \|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_\infty} &= \sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} |\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)| \\ &\le \sup_{i\in\mathcal{I}_\sigma} |c_i| \sup_{h\in\mathcal{G}/\mathcal{G}_\mathcal{F}} \sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)| \ . \end{aligned} \tag{4.12}$$

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, we have for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ that $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma_r}$ and $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for $i,j \in \mathcal{I}_{\sigma_r}$ and $i \ne j$. Hence we obtain

$$\sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)| = \sum_{r=1}^{r_0} \sum_{i\in\mathcal{I}_{\sigma_r}} |R(h_i,h)| \ .$$

For all $l \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ with the property that $\sigma(l) \in \mathcal{Q}\sigma(h_i)$, we have that $\sigma(h_i)^{-1} \in \sigma(l)^{-1}\mathcal{Q}$ and hence

$$\begin{aligned}\sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{L_2(\mathcal{N})}| &\geq |\langle U(\sigma(h)^{-1})\psi, U(\sigma(h_i)^{-1})\psi\rangle_{L_2(\mathcal{N})}| \\ &= |R(h,h_i)| = |R(h_i,h)| \,.\end{aligned}$$

Let $\mathcal{B}_i := \{l \in \mathcal{G}/\mathcal{G}_{\mathcal{F}} : \ \sigma(l) \in \mathcal{Q}\sigma(h_i)\}$. Then the above inequality implies

$$\int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{L_2(\mathcal{N})}| \, d\mu(l) \geq |R(h_i,h)|\mu(\mathcal{B}_i) \ .$$

Now we have that for $i,j \in \mathcal{I}_{\sigma_r}$ and $i \neq j$ the sets $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint. Consequently, we obtain

$$\begin{aligned}&\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{L_2(\mathcal{N})}| \, d\mu(l) \ \geq \\ &\geq \sum_{i\in\mathcal{I}_{\sigma_r}} \int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{L_2(\mathcal{N})}| \, d\mu(l) \\ &\geq \sum_{i\in\mathcal{I}_{\sigma_r}} |R(h_i,h)|\mu(\mathcal{B}_i) \\ &\geq C_{\mathcal{Q}} \sum_{i\in\mathcal{I}_{\sigma_r}} |R(h_i,h)|\end{aligned}$$

and further by (4.3) for all $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$

$$\sum_{i\in\mathcal{I}_{\sigma_r}} |R(h_i,h)| \leq \frac{C}{C_{\mathcal{Q}}} \quad , \quad \sum_{i\in\mathcal{I}_{\sigma}} |R(h_i,h)| \leq \frac{r_0 C}{C_{\mathcal{Q}}} \ .$$

Together with (4.12) this yields

$$\| \sum_{i\in\mathcal{I}_{\sigma}} c_i U(\sigma(h_i)^{-1}\psi)\|_{M_\infty} \leq \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_\infty} \frac{r_0 C}{C_{\mathcal{Q}}} \ .$$

■

5 Nonlinear Approximation with Banach Frames

Once our Banach frames are established, they can clearly be used to decompose, to approximate and to analyze certain functions on $\mathcal{N}$. Then it is clearly desirable to determine the quality of certain approximation schemes based on our frames, i.e., the approximation order comes into play. In this section, we discuss *nonlinear approximation schemes* based on our Banach frames. Especially, we are interested in the quality of the *best N–term approximation*. The setting can be described as follows.

Let $\{\psi_i = U(\sigma(h_i)^{-1})\psi : \; i \in \mathcal{I}_\sigma\}$ denote the Banach frame constructed in the previous section, i.e., we have for any $f \in M_p$ that

$$f = \sum_{i\in\mathcal{I}_\sigma} \langle f, \xi_i\rangle \psi_i \quad , \quad \langle f, \xi_i\rangle = c_i := \langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle_{L_2(\mathcal{G}/\mathcal{G}_\mathcal{F})} \tag{5.1}$$

and

$$\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_p} \sim \|f\|_{M_p} \; . \tag{5.2}$$

We want to approximate our functions $f \in M_p$ by elements from the nonlinear manifolds Σ_n, $n \in \mathbb{N}$, which consist of all functions $S \in M_p$ whose expansions with respect to our frame have at most n nonzero coefficients, i.e.,

$$\Sigma_n := \{S \in M_p : \; S = \sum_{i\in J} a_i \psi_i, \; J \subseteq \mathcal{I}_\sigma, \; \mathrm{card} J \leq n\} \; .$$

Then we are interested in the asymptotic behavior of the error

$$E_n(f)_{M_p} := \inf_{S\in\Sigma_n} \|f - S\|_{M_p} \; .$$

Usually, the order of approximation which can be achieved depends on the regularity of the approximated function as measured in some associated smoothness space. For instance, for nonlinear wavelet approximation, the order of convergence is determined by the regularity as measured in a specific scale of Besov spaces. For nonlinear approximation based on Gabor frames, it has been shown in [22] that the 'right' smoothness spaces are given by a specific scale of modulation spaces. It turns out that the results from [22] carry over to our case without any difficulty. The basic ingredient in the proof of the theorem is the following lemma which has been shown in [22], see also [11].

Lemma 5.1 *Let $a = (a_i)_{i=1}^\infty$ be a decreasing sequence of positive numbers. For $p, q > 0$ set $\alpha := 1/p - 1/q$ and $E_{n,q}(a) := (\sum_{i=n}^\infty a_i^q)^{1/q}$. Then for $0 < p < q \leq \infty$ we have*

$$2^{-1/p}\|a\|_{\ell_p} \leq \left(\sum_{n=1}^\infty (n^\alpha E_{n,q}(a))^p \frac{1}{n}\right)^{1/p} \leq c\,\|a\|_{\ell_p}$$

with a constant $c > 0$ depending only on p.

Now one can prove the following theorem, see also [22].

Theorem 5.1 *Let $\{\psi_i : \; i \in \mathcal{I}_\sigma\}$ be a Banach frame for M_p, $1 \leq p \leq \infty$, given by Theorem 4.1. If $1 \leq p < q$, $\alpha := 1/p - 1/q$ and $f \in M_p$, then*

$$\left(\sum_{n=1}^\infty \frac{1}{n} \left(n^\alpha E_n(f)_{M_q}\right)^p\right)^{1/p} \leq c\|f\|_{M_p}$$

for a constant $c < \infty$.

Proof Let π permutate the sequence $(|\langle f, \xi_i \rangle|)_{i\in\mathcal{I}_\sigma}$ in (5.1) in a decreasing order, i.e. $|\langle f, \xi_{\pi(1)}\rangle| \geq |\langle f, \xi_{\pi(2)}\rangle| \geq \dots$. Then we obtain that

$$E_n(f)_{M_q} \leq \| \sum_{i=n+1}^{\infty} \langle f, \xi_{\pi(i)}\rangle \psi_{\pi(i)} \|_{M_q}$$

and by (5.2) further that

$$E_n(f)_{M_q} \leq c \left(\sum_{i=n+1}^{\infty} |\langle f, \xi_{\pi(i)}\rangle|^q \right)^{1/q} = c\, E_{n+1,q}(|\langle f, \xi_{\pi(i)}\rangle|) \leq c\, E_{n,q}(|\langle f, \xi_{\pi(i)}\rangle|) \ .$$

Now we finish by applying Lemma 5.1 and (5.2)

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(n^\alpha E_n(f)_{M_q} \right)^p \right)^{1/p} &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(n^\alpha\, c\, E_{n,q} \right)^p \right)^{1/p} \\ &\leq c \,\| (|\langle f, \xi_{\pi(i)}\rangle|) \|_{\ell_p} \\ &\leq c \, \|f\|_{M_p} \ . \end{aligned}$$

■

6 Application to the Sphere

In this section, we want to explain how the machinery developed in the previous sections can be applied to very specific manifolds, namely to the spheres S^{n-1} contained in $\mathbb{R}^n$. The aim is to derive a generalized windowed Fourier transform on the spheres and to construct the associated Gabor frames. We therefore explain how the basic steps outlined above can be realized for this specific setting. First of all, in Subsection 6.1, we construct a suitable group acting on the Hilbert space $L_2(S^{n-1})$. Here we follow the lines of B. Torresani [32]. Then, in Subsection 6.2, we introduce and discuss the associated coorbit spaces. In case of the windowed Fourier transform these spaces can be interpreted as generalized modulation spaces. The basic technical step is to establish a generalized Young inequality, i.e., we have to verify (3.1). Subsection 6.3 is devoted to the frame construction. We therefore have to verify that all the assumptions in Theorem 4.1 can be established.

Although some parts of the theory are presented for the general setting, we shall mainly confine the discussion to the simplest case, that is, to the sphere S^1 contained in $\mathbb{R}^2$. The reason for proceeding this way is to keep the technical difficulties at a reasonable level. The general case will be discussed in a forthcoming paper.

6.1 Basic Setting

In this subsection, we want to establish a suitable group representation for the Hilbert space $\mathcal{H} = L_2(S^{n-1})$. To this end, we shall mainly follow the lines of fundamental approach derived by B. Torresani [32]. We are interested in building a version of the windowed Fourier transform on the sphere. Since the usual windowed Fourier transform is generated with translations and modulations, we need similar transformations on the sphere. A good candidate to start with is the Euclidean group $E(n)$. Let $SO(n)$ denote the special orthogonal group of rotations in $\mathbb{R}^n$, then

$$\mathcal{G} := E(n) = SO(n) \ltimes \mathbb{R}^n$$

with group operation

$$(R, p) \circ (\tilde{R}, \tilde{p}) = (R\tilde{R}, R\tilde{p} + p), \quad (R, p)^{-1} = (R^{-1}, -R^{-1}p). \tag{6.1}$$

The group $\mathcal{G}$ is a separable Lie group with Haar measure ν. As a natural analogue to the Schrödinger representation of the Weyl-Heisenberg group on $L_2(\mathbb{R}^n)$, we consider the continuous unitary representation U of $\mathcal{G}$ on $L_2(S^{n-1})$ defined by

$$(U(R, p))f(s) := e^{i<s,p>} f(R^{-1}s) \;, \tag{6.2}$$

where $s \in S^{n-1}$. Note that U can be derived in a more sophisticated way by Makeys induction from some subgroup $\mathcal{P}$ of $\mathcal{G}$ with $\mathcal{G}/\mathcal{P} \cong S^{n-1}$, see, e.g., [32] for details. Unfortunately, there does not exist any function $\psi \in L_2(S^{n-1})$ satisfying

$$\int_{\mathcal{G}} |\langle \psi, U(g^{-1})\psi \rangle_{L_2(S^{n-1})}|^2 d\nu(g) < \infty \;,$$

so that the representation U in $L_2(S^{n-1})$ is not square integrable. However, the way out clearly consists in considering representations modulo a subgroup of $\mathcal{G}$ as explained in Section 2.

As already stated above, we shall mainly restrict ourselves to the case $\mathcal{H} = L_2(S^1)$ in the sequel. In this case, $R \in SO(2)$ and $s \in S^1$ are given explicitly by

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad s = \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix} .$$

Hence, we have by this parametrization $L_2(S^1) \cong L_2([-\pi, \pi])$. This leads to

$$U(\theta, p_1, p_2)\psi(\gamma) = e^{i(p_1 \sin\gamma + p_2 \cos\gamma)} \psi(\gamma - \theta) \;. \tag{6.3}$$

To overcome the integrability problem we have to choose an appropriate subgroup. A natural candidate is given by the stability group $\mathcal{G}_{\mathcal{F}} \cong \{(0, 0, p_2) \in \mathcal{G}\}$. As explained in the previous sections, the whole construction depends on the choice of the section σ of the principal bundle $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\mathcal{F}}$. In the following, we will primarily consider the flat section defined by $\sigma(\theta, p_1) = (\theta, p_1, 0)$. We have to verify that U is strictly square integrable mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$. To this end, we have to show that there exists a function $\psi \in L_2(S^1)$ such that the associated wavelet transform

$$\begin{aligned} V_\psi g(h) &= \langle g, U(\sigma(h)^{-1})\psi\rangle_{L_2(S^1)} \\ &= \langle g, U((\theta, p_1, 0)^{-1})\psi\rangle_{L_2([-\pi,\pi])} \\ &= \int_{-\pi}^{\pi} e^{ip_1 \sin\gamma}\overline{\psi}(\gamma) g(\gamma - \theta) d\gamma \end{aligned} \tag{6.4}$$

is an isometry. The next lemma can also be found in [32].

Lemma 6.1 *Assume that the function* $\psi \in L_2([-\pi, \pi])$ *is such that* $\operatorname{supp}\psi \subset [-\pi/2, \pi/2]$ *and*

$$2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} d\gamma = 1 \ . \tag{6.5}$$

Then the map

$$L_2(S^1) \ni g \mapsto V_\psi g \in L_2(\mathcal{G}/\mathcal{G}_F) \ ,$$

where $V_\psi g$ *is defined by* (6.4) *is an isometry.*

Proof Assume that $g \in L_2([-\pi, \pi])$ and $\psi \in L_2([-\pi, \pi])$. Then we can write

$$\begin{aligned} V_\psi g(\theta, p) &= \langle g, U(\sigma(\theta, p)^{-1})\psi\rangle_{L_2(S^1)} = \langle U(\sigma(\theta, p))g, \psi\rangle_{L_2(S^1)} \\ &= \int_{-\pi/2}^{\pi/2} e^{ip\sin\gamma} g(\gamma - \theta)\bar{\psi}(\gamma) d\gamma \ . \end{aligned}$$

By using the substitution $\sin\gamma = t$ we obtain

$$\begin{aligned} \int_{\mathcal{G}/\mathcal{G}_F} |V_\psi g(\theta, p)|^2 d\mu(\theta, p) &= \int_{\mathbb{R}} \int_{-\pi}^{\pi} \Big| \int_{-\pi/2}^{\pi/2} e^{ip\sin\gamma} g(\gamma - \theta)\bar{\psi}(\gamma) d\gamma \Big|^2 d\theta dp \\ &= \int_{\mathbb{R}} \int_{-\pi}^{\pi} \Big| \int_{-1}^{1} e^{ipt} \frac{g(\arcsin t - \theta)\bar{\psi}(\arcsin t)}{\sqrt{1-t^2}} dt \Big|^2 d\theta dp \end{aligned}$$

and further by Parseval's equality

$$\int_{\mathcal{G}/\mathcal{G}_F} |V_\psi g(\theta, p)|^2 d\mu(\theta, p) = 2\pi \int_{-\pi}^{\pi} \int_{-1}^{1} \Big| \frac{g(\arcsin t - \theta)\bar{\psi}(\arcsin t)}{\sqrt{1-t^2}} \Big|^2 dt d\theta$$

$$= 2\pi \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{|g(\gamma-\theta)|^2 |\psi(\gamma)|^2}{\cos\gamma} \, d\gamma d\theta$$

$$= \|g\|^2_{L_2(S^1)} \, 2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} \, d\gamma \; .$$

■

As a consequence, the wavelet transform can be inverted by using the adjoint V_ψ^*. Of course the approach works also if

$$0 < c_\psi := 2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} \, d\gamma \; < \infty.$$

Then the inverse of the wavelet transform is given by $V_\psi^*/\sqrt{c_\psi}$.

6.2 Modulation Spaces on the Sphere S^1

To construct properly defined modulations spaces, it is clearly necessary to ensure the correspondence principle in Proposition 3.1. Therefore we have to establish the basic property (3.1). Hence, we have to verify that $R(l,\cdot) \in L_1(\mathcal{G}/\mathcal{G}_\mathcal{F})$ for every $l \in \mathcal{G}/\mathcal{G}_\mathcal{F}$ with a norm that can be bounded independently of h. We shall always work with an admissible wavelet ψ in the sense of Lemma 6.1, i.e., we assume that supp$\psi \subset [-\pi/2, \pi.2]$ and that condition (6.5) is satisfied. The group law (6.1) combined with the Euler angle parameterization yields for $h = (\theta_h, p_h, 0), l = (\theta_l, p_l, 0) \in \mathcal{G}/\mathcal{G}_\mathcal{F}$

$$\sigma(h)\sigma(l)^{-1} = (\theta_h - \theta_l, p_h - p_l \cos(\theta_h - \theta_l), p_l \sin(\theta_h - \theta_l)) \; .$$

We therefore obtain

$$\begin{aligned} R(l,h) &= \int_{-\pi/2}^{\pi/2} e^{i(\sin\gamma(-p_l\cos\theta + p_h) + \cos\gamma(p_l \sin\theta))} \psi(\gamma-\theta)\overline{\psi}(\gamma) \, d\gamma \\ &= \int_{-\pi/2}^{\pi/2} e^{i(p_h \sin\gamma - p_l \sin(\gamma-\theta))} \psi(\gamma-\theta)\overline{\psi}(\gamma) \, d\gamma, \end{aligned}$$

where $\theta := \theta_h - \theta_l$. By substituting $t = \sin\gamma$ one has

$$R(l,h) = \int_{-1}^{1} e^{-ip_l \sin(\arcsin t - \theta)} e^{ip_h t} \psi(\arcsin t - \theta)\overline{\psi}(\arcsin t) \frac{dt}{\sqrt{1-t^2}} \; .$$

Furthermore, by defining

$$F_{\theta,p_l}(t) := e^{-ip_l \sin(\arcsin t-\theta)}\psi(\arcsin t-\theta)\overline{\psi}(\arcsin t)/\sqrt{1-t^2}$$

and recalling the fact that $\mathrm{supp}\psi \subset [-\pi/2, \pi/2]$ we may write

$$R(l,h) = \hat{F}_{\theta,p_l}(-p_h)\ . \tag{6.6}$$

The quasi–invariant measure $d\mu(h)$ of the quotient space $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$ is given by $dp_h d\theta_h$, hence we have

$$\int\limits_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(l,h)|\, d\mu(h) = \int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|\, dp_h\, d\theta_h.$$

Interpreting $\int |\hat{F}_{\theta,p_l}(p_h)|\, dp_h$ as the inverse Fourier transform at point 0 and regarding that the outer integration is over a finite interval, we see that property (3.1) is equivalent to

$$|\hat{F}_{\theta,p_l}(\cdot)|^{\vee}(0) < C, \tag{6.7}$$

with some constant C independent of p_l and θ_l.

We have checked numerically that for one of the typical admissible functions suggested by Torresani [32] condition (6.7) is satisfied. We have chosen function ψ by

$$\psi(x) = \cos^6 x \cdot \chi_{[-\pi/2,\pi/2]}(x)\ ,$$

which is admissible in the sense of Lemma 6.1. In Figure 1 we have displayed two typical

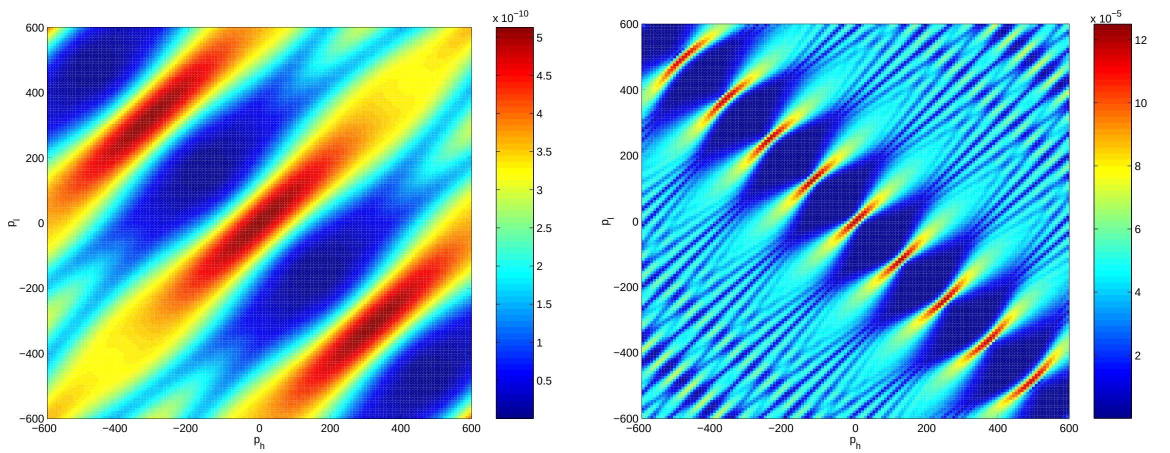

Figure 1: Left: $|\hat{F}_{\theta,p_l}(-p_h)|$ for $\theta = -2.7416$, right: $|\hat{F}_{\theta,p_l}(-p_h)|$ for $\theta = 2.0584$

plots of $\hat{F}_{\theta,p_l}(-p_h)$ for $\theta = -2.7416$ and $\theta = 2.0584$. Numerival experiments were done for θ on the whole grid $-\pi/2 : \pi/16 : \pi/2$. These figures indicate that for fixed θ the expression

$$\int |\hat{F}_{\theta,p_l}(-p_h)| dp_h$$

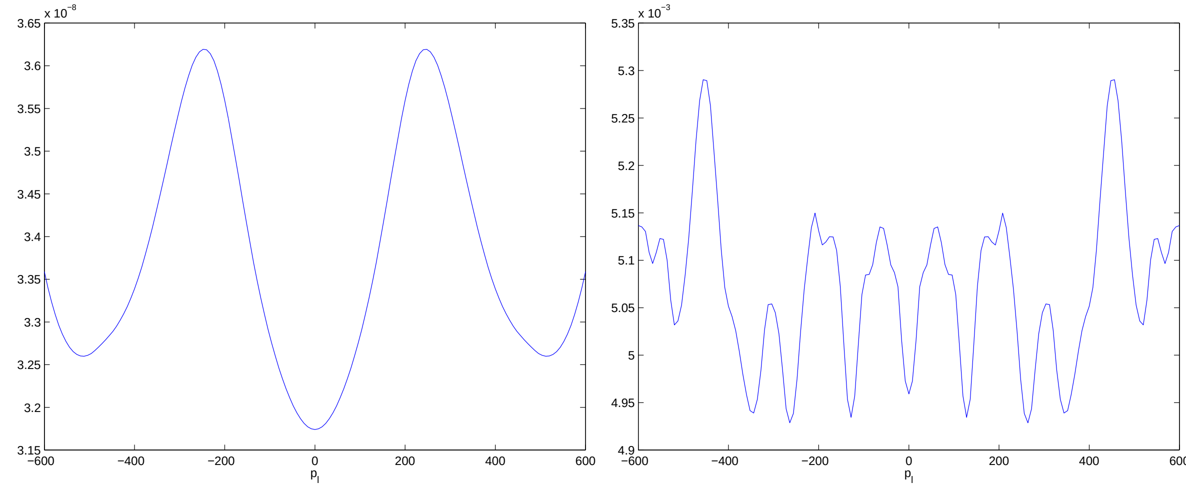

Figure 2: $\int |\hat{F}_{\theta,p_l}(-p_h)|dp_h$ $\theta = -2.7416$, $\theta = 2.0584$

is bounded independently of p_l. This is confined by Figure 2 which shows the approximated values of $\int |\hat{F}_{\theta,p_l}(-p_h)|dp_h$ as functions of p_l. Finally, in Figure 3 we have displayed the maximal value of $\int |\hat{F}_{\theta,p_l}(-p_h)|dp_h$ with respect to p_l as a function of θ. From this figure, we observe that condition (6.7) is satisfied.

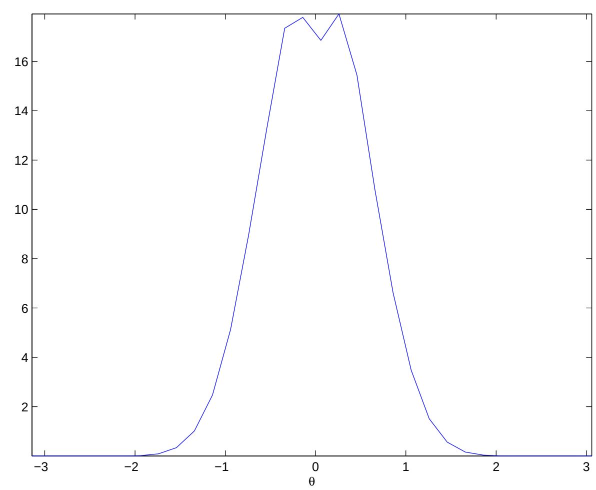

Figure 3: $\max_{p_l} \int |\hat{F}_{\theta,p_l}(-p_h)|dp_h$ as a function of θ

6.3 Banach Frames on the Sphere S^1

In this subsection, we want to derive some atomic decomposition for the new modulation spaces. To this end, we have to check that all assumptions in Theorem 4.1 can be satisfied. Therefore we have to define some neighborhood $\mathcal{U}$ and a related $\mathcal{U}$–dense family X which is relatively separated.

Let $\mathcal{U}$ be given by $\mathcal{U} := (-\pi/N, \pi/N) \times (-\pi/M, \pi/M) \times (-\pi/M, \pi/M)$ and $X := (x_{n,m})_{(n,m)\in\mathcal{I}}$ by $x_{n,m} = (\theta_n, p_m, q_m)$. One basic premise we have to verify is that the $\mathcal{U}$–oscillation (4.1) fulfills (4.2). For $u = (\theta_u, p_u, q_u) \in \mathcal{U}$ we start by evaluating

$$\sigma(h)\sigma(l)^{-1}u = (\theta + \theta_u, p_h - p_l\cos\theta + p_u\cos\theta + q_u\sin\theta, p_l\sin\theta - p_u\sin\theta + q_u\cos\theta)\ ,$$

where $\theta := \theta_h - \theta_l$. By (6.3) and since supp $\psi \in [-\pi/2, \pi/2]$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} (U(\sigma(h)\sigma(l)^{-1})\psi(\gamma)\,\bar{\psi}(\gamma) - U(\sigma(h)\sigma(l)^{-1}u)\psi(\gamma)\,\bar{\psi}(\gamma))\,d\gamma \\
= &\int_{-\pi/2}^{\pi/2} (U(\theta, p_h - p_l\cos\theta, p_l\sin\theta)\psi(\gamma)\,\bar{\psi}(\gamma) - U(\theta+\theta_u, \\
&\quad p_h - p_l\cos\theta + p_u\cos\theta + q_u\sin\theta, p_l\sin\theta - p_u\sin\theta + q_u\cos\theta)\psi(\gamma)\,\bar{\psi}(\gamma))\,d\gamma \\
= &\int_{-\pi/2}^{\pi/2} e^{i(p_h\sin\gamma + p_l\sin(\theta-\gamma))}\left[\psi(\gamma-\theta) - e^{i(p_u\sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\,\psi(\gamma-\theta-\theta_u)\right]\bar{\psi}(\gamma)\,d\gamma \\
= &\int_{-\pi/2}^{\pi/2} e^{i(p_h\sin\gamma + p_l\sin(\theta-\gamma))}\{[\psi(\gamma-\theta) - \psi(\gamma-\theta-\theta_u)]\,\bar{\psi}(\gamma) + \\
&\quad \left[1 - e^{i(p_u\sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\right]\psi(\gamma-\theta-\theta_u)\bar{\psi}(\gamma)\}\,d\gamma.
\end{aligned}$$

Now we can estimate $\mathrm{osc}_{\mathcal{U}}(l,h)$ by

$$\begin{aligned}
\mathrm{osc}_{\mathcal{U}}(l,h) \quad\leq\quad &\sup_{u\in\mathcal{U}}\left|\int_{-\pi/2}^{\pi/2} e^{i(p_h\sin\gamma+p_l\sin(\theta-\gamma))}\left[\psi(\gamma-\theta)-\psi(\gamma-\theta-\theta_u)\right]\bar{\psi}(\gamma)d\gamma\right| + \\
&\sup_{u\in\mathcal{U}}\left|\int_{-\pi/2}^{\pi/2} e^{i(p_h\sin\gamma+p_l\sin(\theta-\gamma))}\left[1-e^{i(p_u\sin(\gamma-\theta)+q_u\cos(\gamma-\theta))}\right]\times\right. \\
&\left.\psi(\gamma-\theta-\theta_u)\bar{\psi}(\gamma)d\gamma\right|\ .
\end{aligned}$$

We have to verify that $\mathrm{osc}_{\mathcal{U}}(l,h)$ fulfills the conditions (4.2). We restrict our attention to the condition

$$I := \int_{\mathcal{G}/\mathcal{G}_F} \mathrm{osc}_{\mathcal{U}}(l,h)\,d\mu(h) < 1.$$

The other condition follows in a similar way. By our estimate of $\mathrm{osc}_{\mathcal{U}}(l,h)$, we have that

$$I \leq \int_{-\pi}^{\pi} (I_1 + I_2)\, d\theta_h\,, \tag{6.8}$$

where

$$I_1 := \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \Big| \int_{-\pi/2}^{\pi/2} e^{i(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \left[\psi(\gamma-\theta) - \psi(\gamma-\theta-\theta_u)\right] \bar{\psi}(\gamma) d\gamma \Big|\, dp_h,$$

and

$$I_2 := \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \Big| \int_{-\pi/2}^{\pi/2} e^{i(p_h \sin\gamma + p_l \sin(\theta-\gamma))} \left[1 - e^{i(p_u \sin(\gamma-\theta) + q_u \cos(\gamma-\theta))}\right] \psi(\gamma-\theta-\theta_u) \bar{\psi}(\gamma)\, d\gamma \Big|\, dp_h.$$

Substituting $t = \sin\gamma$ in I_1, we get

$$I_1 = \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \Big| \int_{-1}^{1} e^{ip_h t}\, e^{ip_l \sin(\theta - \arcsin t)} \left[\psi(\arcsin t - \theta) - \psi(\arcsin t - \theta - \theta_u)\right] \frac{\bar{\psi}(\arcsin t)}{\sqrt{1-t^2}}\, dt \Big|\, dp_h$$

Introducing the functions

$$g(t) := \begin{cases} \frac{e^{ip_l \sin(\theta - \arcsin t)}\, \bar{\psi}(\arcsin t)^{1/2}}{\sqrt{1-t^2}} & \text{for } t \in [-1,1], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_{\theta_u}(t) := \begin{cases} \left[\psi(\arcsin t - \theta) - \psi(\arcsin t - \theta - \theta_u)\right]\, \bar{\psi}(\arcsin t)^{1/2} & \text{for } t \in [-1,1], \\ 0 & \text{otherwise,} \end{cases}$$

the above expression can be written as

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \Big| \int_{\mathbb{R}} w_{\theta_u}(t) g(t) e^{ip_h t}\, dt \Big|\, dp_h \\ &= \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \left| ((\hat{w}_{\theta_u} * \hat{g})(-p_h) \right| dp_h \\ &\leq \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} \int_{\mathbb{R}} |\hat{w}_{\theta_u}(v)| |\hat{g}(p_h - v)|\, dv\, dp_h. \end{aligned} \tag{6.9}$$

We choose ψ sufficiently smooth, e.g., $\psi(t) = \cos^6(t)$, so that $w_{\theta_u}^{(r)}(t)$ is a continuous function for some $r \geq 2$ and $\hat{g} \in L_1$. Note that $w_{\theta_u}^{(r)}(t)$ has compact support. Then $\lim_{\theta_u \to 0} w_{\theta_u}^{(r)}(t) = 0$ and we obtain by dominated convergence that

$$\lim_{\theta_u \to 0} ||w_{\theta_u}^{(r)}||_{L_1} = 0.$$

The Fourier transform maps L_1 continuously onto a dense subalgebra of C_0. Here C_0 denotes the Banach space of continuous functions which tend to zero at $\pm\infty$ with norm

$$||f||_\infty := \max\{|f(t)| : t \in \mathbb{R}\}.$$

Thus

$$\lim_{\theta_u \to 0} ||(w_u^{(r)})\check{}\,||_\infty = 0. \tag{6.10}$$

Further, we have that

$$\hat{w}_{\theta_u}(v) = (-iv)^{-r}\,(w_{\theta_u}^{(r)})\check{}\,(v),$$

which by (6.10) implies

$$|\hat{w}_{\theta_u}(v)| \le (1+|v|)^{-r}\,C(\theta_u), \tag{6.11}$$

where $C(\theta_u)$ is a continuous function with $\lim_{\theta_u\to 0} C(\theta_u) = 0$. Inserting (6.11) into (6.9), we get

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}} \sup_{u\in\mathcal{U}} C(\theta_u) \int_{\mathbb{R}} (1+|v|)^{-r} |\hat{g}(p_l - v)|\, dv\, dp_h \\ &= ||\hat{g}||_{L_1} \sup_{|\theta_u|\le\pi/N} C(\theta_u) \int_{\mathbb{R}} (1+|v|)^{-r}\, dv \\ &\le C \sup_{|\theta_u|\le\pi/N} C(\theta_u). \end{aligned}$$

This expression becomes arbitrary small for sufficiently large N. The term I_2 can be treated in a similar way. Now (4.2) follows by (6.8).

Finally it is easy to check that

$$\mu\{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}} \ :\ \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \ge c_{\mathcal{Q}}$$

for all $i \in I_\sigma$ as follows: Let $\mathcal{Q}$ be of the standard form $\mathcal{Q} = [-\pi/N, \pi/N]\times[-\pi/M,\pi/M]\times[-\pi/M,\pi/M]$ and let $\sigma(h_i) = (\theta_i, p_i, 0)$. For $l = (\gamma, q_1, q_2) \in \mathcal{Q}$ we obtain

$$\begin{aligned} (\gamma, q_1, q_2)\circ\sigma(h_i) &= (\gamma, q_1, q_2)\circ(\theta_i, p_i, 0) \\ &= (\gamma+\theta_i, q_1 + \cos(\gamma)p_i, q_2 - \sin(\gamma)p_i). \end{aligned}$$

The term on the right–hand side can be interpreted as some $\sigma(h)$, $h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}$ if $q_2 - \sin(\gamma)p_i = 0$, i.e.,

$$\sin(\gamma) = \frac{q_2}{p_i} \quad \text{if } p_i \ne 0, \quad q_2 = 0 \quad \text{if } p_i = 0.$$

For fixed $p_i \ne 0$, the above equation can be satisfied if $q_2 \in [-\epsilon, \epsilon]$ and $\gamma \in [-\delta, \delta]$ for some sufficiently small parameters ϵ and δ. Then we obtain

$$(\gamma, q_1, q_2)\circ\sigma(h_i) = (\gamma+\theta_i, q_1 + (p_i^2 - q_2^2)^{1/2}, 0).$$

For $\gamma \in [-\delta,\delta]$, $q_2 \in [-\epsilon,\epsilon]$ and $q_1 \in [-\pi/M, \pi/M]$ this set has obviously a positive measure.

The remaining condition (4.3) can be checked numerically by performing similar calculations as in Subsection 6.2.

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