Adaptive Wavelet Methods for Saddle Point Problems

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1. Introduction

- Numerical treatment of saddle point problems

X, M Hilbert spaces, $\langle \cdot, \cdot \rangle_X \quad \langle \cdot, \cdot \rangle_M$

$$a(\cdot,\cdot) : X \times X \longrightarrow \mathbb{R}, \quad b(\cdot,\cdot) : X \times M \longrightarrow \mathbb{R}$$

$$|a(v,w)| \lesssim \|v\|_X \|w\|_X, \quad a(v,v) \sim \|v\|_X^2,$$

$$|b(v,q)| \lesssim \|v\|_X \|q\|_M$$

$$\begin{aligned} a(u,v) \quad + \quad b(v,p) &= \langle f, v\rangle_X \quad \forall\, v \in X, \\ b(u,q) \qquad\qquad\quad &= \langle g, q\rangle_M \quad \forall\, p \in M, \end{aligned}$$

$$\inf_{p \in M} \sup_{v \in X} \frac{b(v,p)}{\|v\|_X \|p\|_M} > \beta \quad \Longrightarrow$$

unique solution $U = (u,q) \in X \times M$

- Example: Stokes problem

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } \Omega \subset \mathbb{R}^d, \quad u|_{\partial\Omega} = 0, \\ \nabla \cdot u &= g, \end{aligned}$$

$$
\begin{aligned}
a(u,v) + b(v,p) &= \langle f, v\rangle \quad \forall\ v \in H_0^1(\Omega)^d,\\
b(u,q) &= \langle g, q\rangle \quad \forall\ q \in L_{2,0}(\Omega)
\end{aligned}
$$

$$
a(u,v) := \langle \nabla u, \nabla v\rangle = \sum_{i,l=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_l}(x)\frac{\partial v_i}{\partial x_l}(x)dx,
$$

$$
b(v,q) := -\langle \nabla \cdot v, q\rangle = -\sum_{i=1}^{d} \int_\Omega q(x)\frac{\partial v_i}{\partial x_i}(x)dx.
$$

- block operator equation

$$
\begin{aligned}
&B : X \to M', \quad b(v,p) = \langle Bv, p\rangle_M\\
&A : X \to X', \quad a(v,w) = \langle Av, w\rangle_X
\end{aligned}
$$

$$
\mathcal{L}U := \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} =: F,
$$

$$
\begin{aligned}
c_{\mathcal{L}} \left(\|v\|_X^2 + \|p\|_M^2\right)^{1/2} &\le \|\mathcal{L}\begin{pmatrix} v \\ p \end{pmatrix}\|_{X'\times M'}\\
&\le C_{\mathcal{L}} \left(\|v\|_X^2 + \|p\|_M^2\right)^{1/2}
\end{aligned}
$$

- Galerkin approach

$$\langle \mathcal{L} u_V, v \rangle = \langle F, v \rangle, \quad v \in V$$

Wavelets: Wavelet Galerkin approach

2. Wavelets

specific bases for X and M,

$$\Psi_X = \{\psi_{X,\lambda} : \lambda \in \mathcal{J}_X\}, \quad \Psi_M = \{\psi_{M,\lambda} : \lambda \in \mathcal{J}_M\}$$

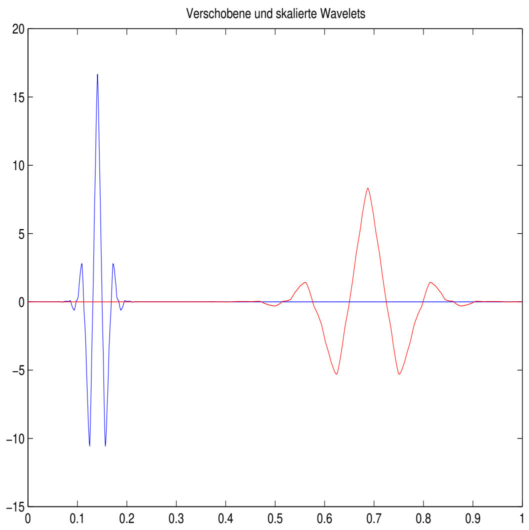
$$\lambda = (j,k), \quad |\lambda| = j$$

$$u = \mathbf{d}^T \Psi_X := \sum_{\lambda \in \mathcal{J}_X} d_\lambda \psi_{X,\lambda}$$

(NE) $$c_X \|\mathbf{D}_X \mathbf{d}\|_{\ell_2(\mathcal{J}_X)} \le \|\mathbf{d}^T \Psi_X\|_X \le C_X \|\mathbf{D}_X \mathbf{d}\|_{\ell_2(\mathcal{J}_X)}$$
$$c_M \|\mathbf{D}_M \mathbf{c}\|_{\ell_2(\mathcal{J}_M)} \le \|\mathbf{c}^T \Psi_M\|_M \le C_M \|\mathbf{D}_M \mathbf{c}\|_{\ell_2(\mathcal{J}_M)}$$

(L) $\mathrm{supp}\psi_\lambda \subset Q_\lambda, \quad \mathrm{diam} Q_\lambda \lesssim 2^{-|\lambda|}$

(CP) $|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|(m+d/2)} \|v\|_{W^m(L_\infty(\mathrm{supp}\psi_\lambda))}$



3. Discretization

scaled wavelet representations

$$\mathbf{A} := \mathbf{D}_X^{-1}\langle \Psi_X, A\Psi_X\rangle_X \mathbf{D}_X^{-1},$$
$$\mathbf{B} := \mathbf{D}_M^{-1}\langle \Psi_M, B\Psi_X\rangle_M \mathbf{D}_X^{-1},$$
$$\mathbf{f} := \mathbf{D}_X^{-1}\langle \Psi_X, f\rangle_X,$$
$$\mathbf{g} := \mathbf{D}_M^{-1}\langle \Psi_M, g\rangle_M, \ \mathbf{F} := (\mathbf{f}^T, \mathbf{g}^T)^T$$
$$\mathbf{u} = \mathbf{D}_X \mathbf{d}, \ \mathbf{q} = \mathbf{D}_M \mathbf{c} \quad \Longrightarrow$$

$$\mathbf{L}\begin{pmatrix}\mathbf{u}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix}$$

(NE) $\Longrightarrow$

$$\mathbf{L} : \ell_2(\mathcal{J}) := \ell_2(\mathcal{J}_X) \times \ell_2(\mathcal{J}_M) \to \ell_2(\mathcal{J}),$$

is an isomorphism, i.e.,

$$c_L \|\mathbf{V}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{L}\mathbf{V}\|_{\ell_2(\mathcal{J})} \le C_L \|\mathbf{V}\|_{\ell_2(\mathcal{J})},$$
$$\mathbf{V} := (\mathbf{u}^T, \mathbf{q}^T)^T \in \ell_2(\mathcal{J}),$$
$$\|\mathbf{V}\|^2_{\ell_2(\mathcal{J})} = \|\mathbf{u}\|^2_{\ell_2(\mathcal{J}_X)} + \|\mathbf{q}\|^2_{\ell_2(\mathcal{J}_M)}$$

The original problem has been transformed to an equivalent ℓ_2–problem!

2. Adaptivity

Concept of an adaptive strategy (updating)

- choose index set
- compute Galerkin approximation
- estimate the current error
- adapt: add indices to reduce the error

Goal: Adaptive wavelet algorithm

+ convergent

+ optimal, i.e., error rate comparable to best N-*term approximation*, i.e.,

$$u \rightsquigarrow u_n = \sum_{\lambda \in \Gamma_n} c_\lambda \psi_{X,\lambda}$$

$\Gamma_n \mathrel{\widehat{=}} n$ biggest wavelet coefficients

State of the art:

convergent scheme for elliptic problems

> S. D., W. Dahmen, R. Hochmuth, R. Schneider, Stable Multiscale Bases and Local Error Estimation for Elliptic Problems, *Appl. Numer. Math.* **23(1)** (1997), 21–48.

asymptotically optimal scheme

> A. Cohen, W. Dahmen, and R. DeVore, Adaptive Wavelet Methods for Elliptic Operator Equations — Convergence Rates, *Math. Comp.* **70** (2001), 27–75.

convergent scheme for saddle point problems

> S. D., R. Hochmuth, and K. Urban, Adaptive Wavelet methods for saddle point problems, *M2AN* **34** (2000), 1003–1022.

Question: *Optimal* adaptive scheme for saddle point problems?

2.1 An Adaptive Uzawa-Strategy

UZAWA: *Given any* $\mathbf{q}_0 \in \ell_2(\mathcal{J}_M)$*, compute for* $i = 1, 2, \ldots$

$$\begin{aligned} \mathbf{A}\mathbf{u}_i &= \mathbf{f} - \mathbf{B}^T \mathbf{q}_{i-1}, \\ \mathbf{q}_i &= \mathbf{q}_{i-1} + \omega(\mathbf{B}\mathbf{u}_i - \mathbf{g}) \end{aligned}$$

[D./Hochmuth/Urban]: **UZAWA** converges when $q < 1$

$$\begin{aligned} q : &= \|\mathbf{id} - \omega \mathbf{S}\|_{\ell_2(\mathcal{J}_M)} \\ \mathbf{S} &:= \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T : \ell_2(\mathcal{J}_M) \to \ell_2(\mathcal{J}_M) \end{aligned}$$

We need

- Approximations of $\mathbf{B}\mathbf{u}, \mathbf{B}^T\mathbf{q}$, $\mathbf{u}, \mathbf{q}$ finitely supported

  $\mathbf{APPLY}\, [\eta, \mathbf{C}, \mathbf{v}] \to (\mathbf{w}, \Lambda)$, $\mathbf{w}$ *with finite support* $\Lambda \subset \mathcal{J}$ *such that*

  $$\|\mathbf{C}\mathbf{v} - \mathbf{w}\|_{\ell_2(\mathcal{J})} \leq \eta.$$

- Adaptive solver for $\mathbf{A}\hat{\mathbf{u}} = \mathbf{h}$

ADALG $[\mathbf{A}, \mathbf{v}, \mathbf{h}, \varepsilon] \to (\bar{\mathbf{u}}, \Lambda)$
Given data $\mathbf{h} \in \ell_2(\mathcal{J}_X)$ *and approximation* $\mathbf{v}$ *to the solution* $\hat{\mathbf{u}}$ *of* $\mathbf{A}\hat{\mathbf{u}} = \mathbf{h}$ *such that*

$$\|\hat{\mathbf{u}} - \mathbf{v}\|_{\ell_2(\mathcal{J}_X)} \leq \eta, \quad \varepsilon < \eta.$$

Then **ADALG** *produces* $\bar{\mathbf{u}}$ *with support* Λ *such that*

$$\|\hat{\mathbf{u}} - \bar{\mathbf{u}}\|_{\ell_2(\mathcal{J}_X)} \leq \varepsilon.$$

- discard unsignificant coefficients

NCOARSE $[\eta, \mathbf{v}] \to (\bar{\mathbf{v}}, \Lambda)$ $\quad$ $\mathbf{v}$ finitely supported
$\longrightarrow$ $\bar{\mathbf{v}}$ with smallest possible support Λ such that

$$\|\mathbf{v} - \mathbf{w}\|_{\ell_2(\mathcal{J})} \leq \eta.$$

ADV $[\bar{\mathbf{u}}, \bar{\mathbf{q}}, \delta] \to (\tilde{\mathbf{u}}, \tilde{\mathbf{q}}, \Lambda_u, \Lambda_q)$
Given $\bar{\mathbf{u}}, \bar{\mathbf{q}}$ *such that*

$$\|\bar{\mathbf{u}} - \mathbf{u}\|_{\ell_2(\mathcal{J}_X)} \leq \delta, \quad \|\bar{\mathbf{q}} - \mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \leq \delta,$$

ADV $[\bar{\mathbf{u}}, \bar{\mathbf{q}}, \delta]$ *produces new approximations* $\tilde{\mathbf{u}}, \tilde{\mathbf{q}}$ *as follows*

(i) *Set* $i = 1$, $\bar{\mathbf{q}}_0 := \bar{\mathbf{q}}$, $\bar{\mathbf{u}}_0 := \bar{\mathbf{u}}$.

(ii) *If* $i \leq K$ *go to (iii); else*
NCOARSE $[2\delta/5, \bar{\mathbf{q}}_{i-1}] \to (\tilde{\mathbf{q}}, \Lambda_q)$;
set $\tilde{\mathbf{u}} := \bar{\mathbf{u}}_{i-1}$, $\Lambda_u := \operatorname{supp} \bar{\mathbf{u}}_{i-1}$.

(iii) **NCOARSE** $[\gamma_i q^i \delta, \mathbf{f}] \to (\mathbf{f}_i, \Lambda_i^f)$; $\sum_{i=1}^{\infty} \gamma_i < \infty$

(iv) **ADALG** $[\mathbf{A}, \bar{\mathbf{u}}_{i-1}, \mathbf{f}_i - \mathbf{B}^T \bar{\mathbf{q}}_{i-1}, \gamma_i q^i \delta] \to (\bar{\mathbf{u}}_i, \Lambda_i^X)$.

(v) **NCOARSE** $[\gamma_i q^i \delta, \mathbf{g}] \to (\mathbf{g}_i, \Lambda_i^g)$;

APPLY $[\gamma_i q^i \delta, \mathbf{B}, \bar{\mathbf{u}}_i] \to (\mathbf{p}_i, \Lambda_i^p)$; *set*

$$\bar{\mathbf{q}}_i = \bar{\mathbf{q}}_{i-1} + \omega(\mathbf{p}_i - \mathbf{g}_i);$$

set $i + 1 \to i$ *and go to (ii).*

$$\|\bar{\mathbf{u}}-\mathbf{u}\|_{\ell_2(\mathcal{J}_X)} \leq \delta, \quad \|\bar{\mathbf{q}}-\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \leq \delta \qquad \Longrightarrow$$

$$\|\tilde{\mathbf{u}}-\mathbf{u}\|_{\ell_2(\mathcal{J}_X)} \leq \delta/2, \quad \|\tilde{\mathbf{q}}-\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \leq \delta/2 \quad \Longrightarrow$$

Adaptive Uzawa–Algorithm:

UZAWA $[\mathbf{A},\mathbf{B},\mathbf{f},\mathbf{g},\varepsilon] \to (\mathbf{u}(\varepsilon),\mathbf{q}(\varepsilon))$:
ε *target accuracy,* $\Lambda_0^M = \Lambda_0^X = \emptyset$ $\mathbf{q}_0 = \bar{\mathbf{q}}_0 = \mathbf{0}$, $\bar{\mathbf{u}} = \mathbf{0}$, $\delta_0 := c_L^{-1}\left(\|\mathbf{f}\|^2_{\ell_2(\mathcal{J}_X)} + \|\mathbf{q}\|^2_{\ell_2(\mathcal{J}_M)}\right)$, $J = 0$.

(i) **ADV** $[\bar{\mathbf{u}},\bar{\mathbf{q}},\delta_J] \to (\tilde{\mathbf{u}},\tilde{\mathbf{q}},\Lambda_u,\Lambda_q)$;

(ii) *Set* $\delta_{J+1} := \delta_J/2$.
If $\delta_{J+1} \leq \varepsilon$, *stop and accept* $\mathbf{u}(\varepsilon),\mathbf{q}(\varepsilon)$ *as solution.*
Else, set $\bar{\mathbf{u}} = \tilde{\mathbf{u}}$, $\bar{\mathbf{q}} = \tilde{\mathbf{q}}$, $J+1 \to J$ *and go to (i).*

Properties: Suitable (optimal) building block $\Longrightarrow$

- **UZAWA** converges
- optimal convergence rate
- (LBB)–condition not needed! $\Longrightarrow$
  Adaptivity stabilizes!
  (Quite recently: similar result by Bänsch et.al.)
- $\mathbf{A}$ elliptic on Ker$\mathbf{B}$ okay!

2.2 Building Blocks

- **APPLY:** Approximation of $\mathbf{C}\mathbf{v}$!

(L), (CP) $\Longrightarrow$ $\mathbf{A}, \mathbf{B}$ *compressible*:

$$
\begin{aligned}
\mathbf{A}_J : \quad &= \quad (a'_{\lambda,\lambda'})_{\lambda,\lambda' \in \mathcal{J}_X} \\
\tilde{a}_{\lambda,\lambda'} \quad &:= \quad \begin{cases} a_{\lambda,\lambda'}, & ||\lambda| - |\lambda'|| \leq J/d, \\ 0, & \text{else} \end{cases} \\
a'_{\lambda,\lambda'} \quad &:= \quad \begin{cases} \tilde{a}_{\lambda,\lambda'}, & 2^{\min(|\lambda|,|\lambda'|)} \mathrm{dist}(Q_\lambda, Q_{\lambda'}) \leq \\ & \qquad 2^{J/d - ||\lambda| - |\lambda'||} \gamma(||\lambda| - |\lambda'||) \\ 0, & \text{else} \end{cases} \\
& \qquad \sum_n \gamma(n)^d < \infty
\end{aligned}
$$

has at most 2^J nonzero entries per row and column and

$$
\|\mathbf{A} - \mathbf{A}_J\|_{\ell_2} \quad \lesssim \quad \nu_J 2^{-Js}
$$

Fast matrix–vector multiplication: **FMVM**

$$(\mathbf{A} \circledast \mathbf{v})_J :=$$

$$\mathbf{A}_J \mathbf{v}_{[0]} + \mathbf{A}_{J-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \mathbf{A}_0(\mathbf{v}_{[J]} - \mathbf{v}_{[J-1]})$$

$\mathbf{v}_{[j]}$:= best N–term approximation for $N = 2^j$

$\mathbf{v}^*$:= nonincreasing rearrangement

$\mathbf{v}_0^*$	$(j,k)_0$	$\mathbf{v}_{[0]}$
$\mathbf{v}_1^*$	$(j,k)_1$	$\mathbf{v}_{[1]} - \mathbf{v}_{[0]}$
$\mathbf{v}_2^*$ $\mathbf{v}_3^*$	$(j,k)_2$ $(j,k)_3$	$\mathbf{v}_{[2]} - \mathbf{v}_{[1]}$
$\mathbf{v}_4^*$ $\mathbf{v}_5^*$ $\mathbf{v}_6^*$ $\mathbf{v}_7^*$	$(j,k)_4$ $(j,k)_5$ $(j,k)_6$ $(j,k)_7$	$\mathbf{v}_{[3]} - \mathbf{v}_{[2]}$
$\cdots$	$\cdots$	

$$\|\mathbf{A}\mathbf{v} - (\mathbf{A} \circledast \mathbf{v})_J\|_{\ell_2(\mathcal{J})} \lesssim 2^{-sJ}$$

$$\# \operatorname{supp}(\mathbf{A} \circledast \mathbf{v})_J \lesssim 2^J$$

$$\mathbf{A}\mathbf{v} \rightsquigarrow \mathbf{A} \circledast \mathbf{v}, \quad 2^{-sJ} < \eta \ !$$

- **NCOARSE**

discard unsignificant coefficients, find smallest k such that

$$\sum_{l=1}^{k} |\mathbf{v}^*|^2 > \|\mathbf{v}\|^2_{\ell_2(\mathcal{J})} - \eta^2$$

- **ADALG**

Adaptive solver for

$$\mathbf{A}\hat{\mathbf{u}} = \mathbf{h}$$

General strategy: Given Λ_j, find (small) $\Lambda_{j+1} \supset \Lambda_j$ s.t.

$$\|\hat{\mathbf{u}} - \bar{\mathbf{u}}_{j+1}\| \le \theta \|\hat{\mathbf{u}} - \bar{\mathbf{u}}_j\|, \;\; \theta \in (0,1).$$

Key idea: Choose Λ_{j+1} s.t. for fixed $\alpha \in (0,1)$

$$\begin{aligned} \|\mathbf{r}_j|_{\Lambda_{j+1}}\|_{\ell_2(\mathcal{J}_X)} &\ge \alpha \|\mathbf{r}_j\|_{\ell_2(\mathcal{J}_X)} \\ \mathbf{r}_j &= \mathbf{h} - \mathbf{A}\bar{\mathbf{u}}_j \end{aligned}$$

Note: residual has infinitely many coefficients.

Approximation of the residual: use **FMVM**!

$$\mathbf{r}_j \rightsquigarrow \tilde{\mathbf{r}}_j = \mathbf{h} - \mathbf{A} \circledast \bar{\mathbf{u}}_j$$

ADALG

i) Target accuracy: ε, $\Lambda_0 = \emptyset$, $\mathbf{r}_0 = \mathbf{h}$, $\varepsilon_0 := \|\mathbf{h}\|_{\ell_2(\mathcal{J}_X)}$.

ii) For $j = 0, 1, 2, \ldots,$ determine $(\Lambda_{j+1}, \bar{\mathbf{u}}_{j+1})$ from $(\Lambda_j, \bar{\mathbf{u}}_j)$ as follows

- $\varepsilon_{j+1} := 2^{-(j+1)}\|\mathbf{h}\|_{\ell_2(\mathcal{J}_X)}, \Lambda_{j,0} := \Lambda_j, \bar{\mathbf{u}}_{j,0} := \bar{\mathbf{u}}_j$;
- For $k = 1, 2, \ldots, K$
  - Apply **Galerkin-solver**. Determine approximative solution $\bar{\mathbf{u}}_{j,k-1}$ on index set $\Lambda_{j,k-1}$
  - Determine coefficients of the approximated residual $\tilde{\mathbf{r}}_{j,k-1} = \mathbf{h} - \mathbf{A} \circledast \bar{\mathbf{u}}_{j,k-1}$
  - If $\|\tilde{\mathbf{r}}_{j,k-1}\| > c(\mathbf{A})\varepsilon_{j+1}$: apply **GROW**, find $\Lambda_{j,k}$ smallest set s.t.
$$\|\tilde{\mathbf{r}}_{j,k-1}|_{\Lambda_{j,k}}\|_{\ell_2(\mathcal{J}_X)} \geq \frac{1}{2}\|\tilde{\mathbf{r}}_{j,k-1}\|_{\ell_2(\mathcal{J}_X)},$$
    else return
- Apply **NCOARSE**, $(\Lambda_{j,k}, \bar{\mathbf{u}}_{j,k}) \longrightarrow (\Lambda_{j+1}, \bar{\mathbf{u}}_{j+1})$

if $\varepsilon_{j+1} \leq \varepsilon$ stop

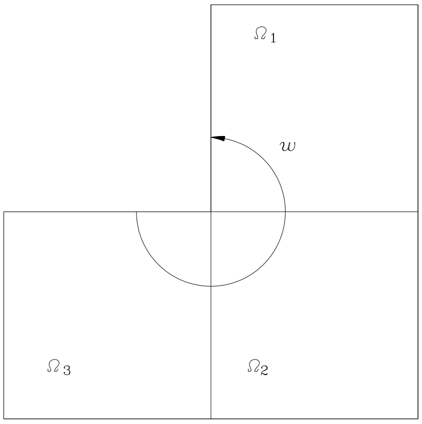
- **[D./Dahmen/Hochmuth/Schneider] ADALG** converges

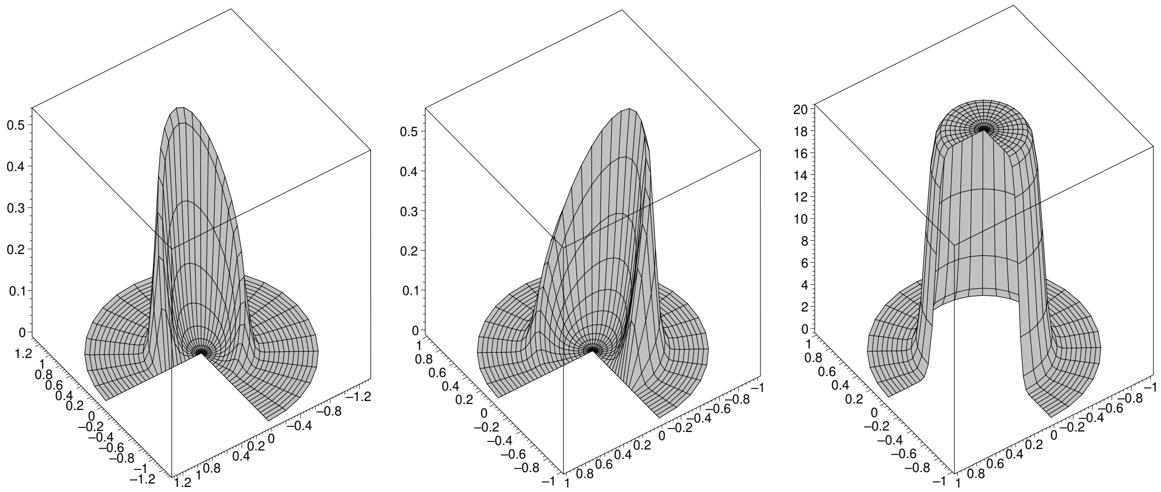
- **[Cohen/Dahmen/DeVore] ADALG** converges with optimal order!

5. Numerical Results

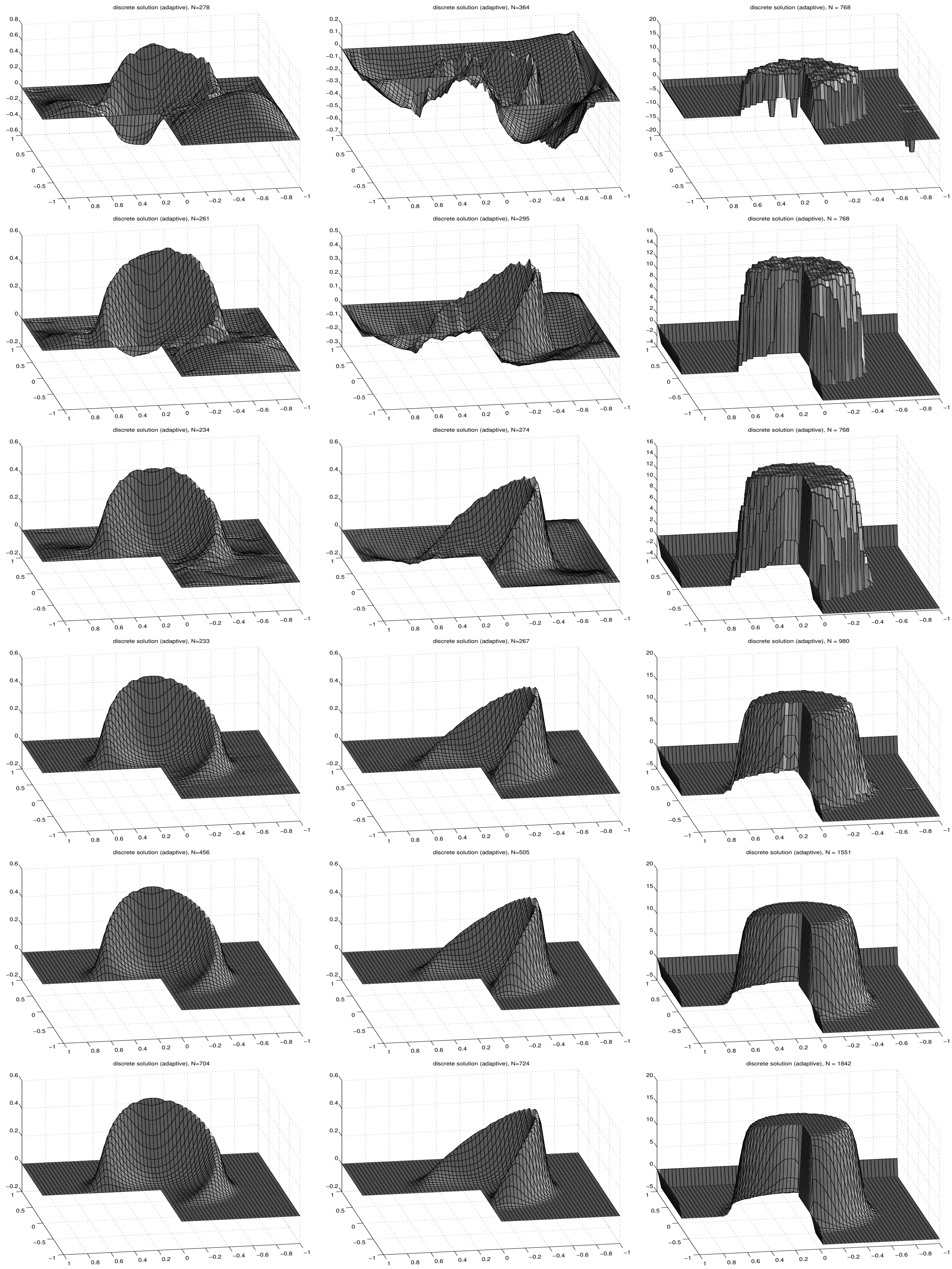
Stokes problem

$$
\begin{aligned}
-\triangle u + \nabla p &= f \quad \text{in } \Omega \subset \mathbb{R}^d, \qquad u|_{\partial\Omega} = 0,\\
\nabla \cdot u &= g,
\end{aligned}
$$

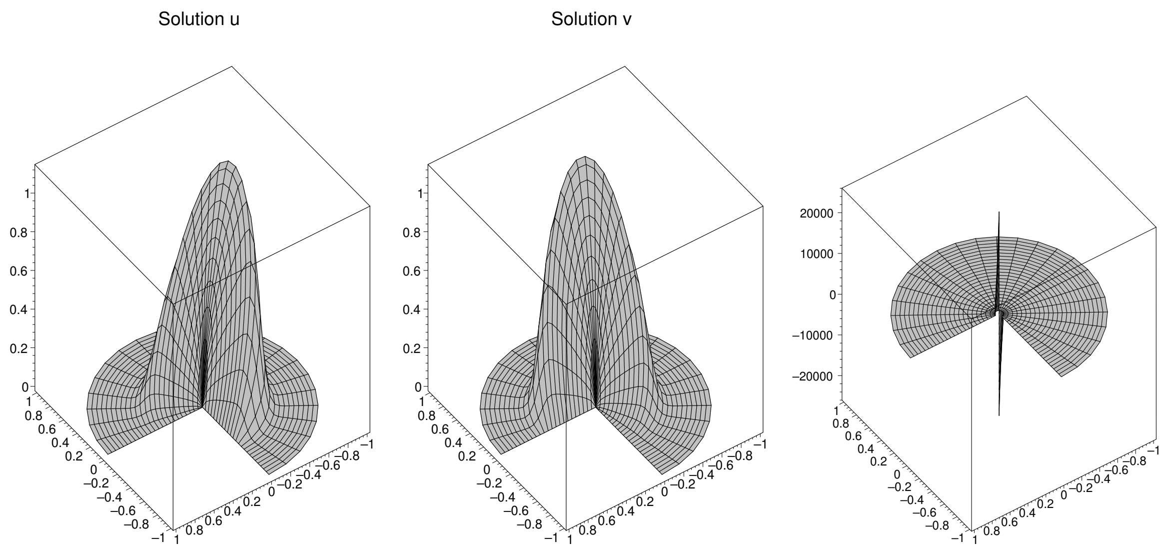




Exact solution, velocity components (left and middle) and pressure (right)



First/second velocity component (left/middle), pressure (right) reentrant corner $\rightsquigarrow$ singular solutions!



Exact solutions, velocity components (left and middle) and pressure (right).

Adaptivity pays, if the Besov regularity of the exact solution is higher than its Sobolev regularity!

Sobolev regularity $\longleftrightarrow$ uniform refinement $\longleftrightarrow$ non-adaptive schemes

Besov regularity $\longleftrightarrow$ best N–term approximation $\longleftrightarrow$ adaptive schemes

for singular solutions:
Besov regularity $\gg$ Sobolev regularity $\Longrightarrow$
Adaptivity helps!

Summary

- adaptive schemes for saddle point problems
- convergent adaptive Uzawa algorithm based on wavelets
- building blocks
- numerical experiments