Adaptive Frame Methods for Elliptic Operator Equations

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(joint work with M. Fornasier, T. Raasch, R. Stevenson, and M. Werner)

Outline

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Numerical Treatment of Operator Equations:

- $$\mathcal{L}u = f$$
  $$\mathcal{L}: \; H \longrightarrow H', \quad H \text{ Sobolev space}, \quad H' \text{ dual space}$$
  $$(H, L_2, H') \qquad \text{Gelfand triple}$$

- elliptic

  $$\|\mathcal{L}u\|_{H'} \sim \|u\|_{H}, \quad u \in H$$

  example:

  $$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz} \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

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- wavelet Galerkin approach

$$\ldots S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots,$$

$$S_j = \mathrm{span}\{\psi_\lambda,\ \lambda \in \Lambda_j \subset \mathcal{J}\}$$

$$\langle \mathcal{L} u_j, v \rangle = \langle f, v \rangle \quad \text{for all} \quad v \in S_j$$

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Wavelets...

- multiresolution analysis $\{V_j\}_{j\geq 0}$:

$$V_0 \subset V_1 \subset V_2 \subset \dots \qquad \overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\Omega)$$

$$V_j = \overline{\text{span}\{\varphi_{j,k},\ k \in I_j\}}$$

-

$$V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\Omega) = \oplus_{j=0}^{\infty} W_j$$

$$W_j = \overline{\text{span}\{\psi_{j,k},\ k \in J_j\}}$$

$$\lambda = (j,k), \quad |\lambda| = j, \quad \mathcal{J} = \bigcup_{j=0}^{\infty} (\{j\} \times J_j)$$

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Further Properties...

- (NE) $$\left\| \sum_{\lambda \in \mathcal{J}} d_\lambda \psi_\lambda \right\|_{H^s} \sim \left(\sum_{\lambda \in \mathcal{J}} 2^{2|\lambda| s} |d_\lambda|^2 \right)^{1/2} \qquad s_1 < s < s_2$$

- (L) $$\mathrm{supp}\,\psi_\lambda \subset Q_\lambda, \quad |Q_\lambda| \lesssim 2^{-|\lambda| d}$$

- (CP) $$|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|(m+d/2)} \|v\|_{W^m(L_\infty(\mathrm{supp}\,\psi_\lambda))}$$

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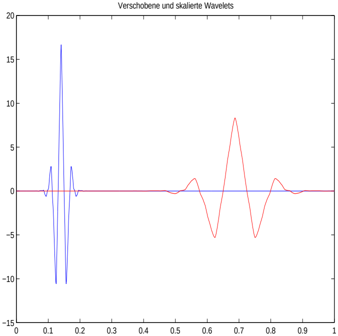
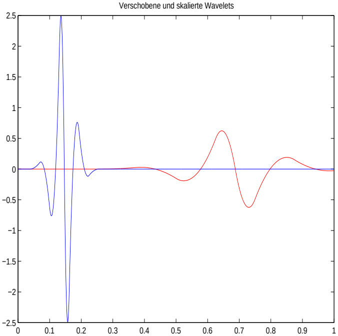
Some Examples:

$$\Omega = \mathbb{R}:$$
$$\psi_{j+1,k}(x) = 2^{j/2}\psi(2^j x - k), \quad \varphi_{j,k}(x) = \varphi(2^j x - k)$$





Wavelets in Numerical Analysis:

- advantages:
  - uniformly bounded condition numbers
  - efficient compression techniques
  - adaptive strategies, guaranteed to converge with optimal order
- disadvantages:
  - construction of wavelets on domains is complicated

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Motivation

Wavelets

Frames
- Hilbert/Banach Frames
- Gelfand Frames
- Wavelet Gelfand Frames on Domains

Adaptive Numerical Frame Schemes
- Series Representation
- Numerical Realization
- The Steepest Descent Approach
- Domain Decomposition

Numerical Experiments
- Steepest Descent
- Additive Schwarz

Frames in a Hilbert Space:

- $\mathcal{F} = \{f_n\}_{n \in \mathcal{N}}$ is called a frame, if

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{n \in \mathcal{N}} |\langle f, f_n \rangle_{\mathcal{H}}|^2 \leq B\|f\|_{\mathcal{H}}^2 \qquad f = \sum_{n \in \mathcal{N}} c_n f_n$$

- the operators

$$F : \mathcal{H} \to \ell_2(\mathcal{N}), \quad f \mapsto (\langle f, f_n \rangle_{\mathcal{H}})_{n \in \mathcal{N}},$$

$$F^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \mathbf{c} \mapsto \sum_{n \in \mathcal{N}} c_n f_n,$$

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- frame operator

$$\mathcal{S}(f) := F^* F(f) = \sum_{n \in \mathcal{N}} \langle f, f_n \rangle f_n$$

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Dual Frames:

Theorem

Let $\{f_n\}_{n\in\mathcal{N}}$ be a frame. Then

- *$\mathcal{S}$ is invertible and $B^{-1}I \le \mathcal{S}^{-1} \le A^{-1}I$;*
- *$\tilde{\mathcal{F}} = \{\tilde{f}_n\}_{n\in\mathcal{N}}$, $\tilde{f}_n := \mathcal{S}^{-1} f_n$ is also a frame, the dual frame;*
- *every $f \in \mathcal{H}$ has an expansion*

$$f = \sum_{n\in\mathcal{N}} \langle f, \mathcal{S}^{-1} f_n\rangle_{\mathcal{H}} f_n = \sum_{n\in\mathcal{N}} \langle f, f_n\rangle_{\mathcal{H}} \mathcal{S}^{-1} f_n.$$

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Banach Frames:

$\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ in $\mathcal{B}'$ is a Banach frame for $\mathcal{B}$ with sequence space $\mathcal{B}_d$ if:

- $Ff = \left(\langle f, f_n\rangle_{\mathcal{B}\times\mathcal{B}'}\right)_{n\in\mathcal{N}}$ is bounded from $\mathcal{B}$ into $\mathcal{B}_d$;
- norm equivalence:

$$\|f\|_{\mathcal{B}} \sim \left\| \left(\langle f, f_n\rangle_{\mathcal{B}\times\mathcal{B}'}\right)_{n\in\mathcal{N}} \right\|_{\mathcal{B}_d};$$

- there exists a bounded synthesis operator R, such that

$$R(\langle f, f_n\rangle_{\mathcal{B}\times\mathcal{B}'})_{n\in\mathcal{N}}) = f.$$

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are bounded operators. $\Longrightarrow$

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Theorem

If $\mathcal{F}$ is a Gelfand frame for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, then $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are Banach frames for $\mathcal{B}$ and $\mathcal{B}'$, respectively.

Wavelet Gelfand Frames on Domains:

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- $\mathcal{B} := H_0^t(\Omega)$, $t \geq 0$, $\quad \mathcal{H} := L_2(\Omega)$, $\quad \mathcal{B}' := H^{-t}(\Omega)$
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Aggregated Wavelet Gelfand Frames:

- $\Psi^{\square} = \{\psi_\lambda^{\square}\}_{\lambda \in \mathcal{J}}$ Wavelet Gelfand frame for $(H_0^t(\square), L_2(\square), H^{-t}(\square))$, $t \geq 0$, $\square := (0,1)^d$
  e.g., tensor products of boundary adapted wavelets as in

  W. Dahmen, R. Schneider, Wavelets with complementary boundary conditions – Functions spaces on the cube, *Result. Math.* **34** (1998), 255–293.

- overlapping partition $\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\Omega_i}$
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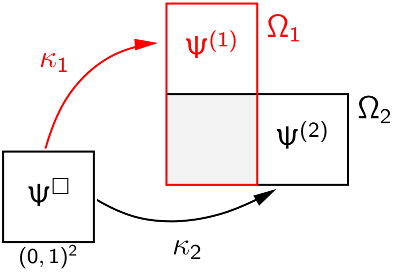

- define

$$\Psi := \{\psi_{i,\lambda}\}_{1 \le i \le n, \lambda \in \mathcal{J}},$$

where

$$\psi_{i,\lambda}(x) := \frac{\psi_\lambda^{\square}(\kappa_i^{-1}(x))}{|\det D\kappa_i(\kappa_i^{-1}(x))|^{1/2}}, \quad 1 \le i \le n, \quad \lambda \in \mathcal{J},$$

- if $\Psi^{\square}$ has sufficient order of vanishing moments $\Longrightarrow$ Ψ is Wavelet–Gelfand–Frame for $(H_0^t(\Omega), L_2(\Omega), H^{-t}(\Omega))$

$\rightsquigarrow$ Easy to implement!

$\rightsquigarrow$ High smoothness possible!

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Adaptive Numerical Frame Schemes:

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$$(\mathcal{B}, \mathcal{H}, \mathcal{B}') = (H, L_2, H') \quad \text{Gelfand triple}$$

Idea: use frame discretization! We have seen that

- construction of wavelet frames

is simple, and we hope that

- adaptive strategies can still be derived
- their (optimal) convergence can be shown

R. Stevenson, Adaptive solution of operator equations using wavelet frames, *SIAM J. Numer. Anal.* **41** (2003), 1074–1100.

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$$\mathcal{L}u = f, \quad \mathcal{L} : \mathcal{B} \longrightarrow \mathcal{B}', \tag{1}$$

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Idea: use frame discretization! We have seen that

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is simple, and we hope that

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Series Representation:

- fix a Gelfand frame $\mathcal{F} = \{f_n\}_{n \in \mathcal{N}}$ for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with $(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d')$
- assume that there exist isomorphisms $D_{\mathcal{B}} : \mathcal{B}_d \to \ell_2(\mathcal{N})$, $D_{\mathcal{B}}^* : \ell_2(\mathcal{N}) \to \mathcal{B}_d'$ (alway satisfied for wavelet Gelfand frames)

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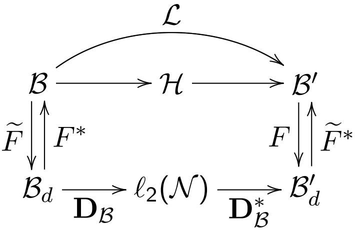

Lemma

The operator

$$\mathbf{G} := (D_{\mathcal{B}}^*)^{-1} F \mathcal{L} F^* D_{\mathcal{B}}^{-1}$$

is bounded from $\ell_2(\mathcal{N})$ *to* $\ell_2(\mathcal{N})$, $\mathbf{G} = \mathbf{G}^*$, *and boundedly invertible on its range* $\mathrm{ran}(\mathbf{G}) = \mathrm{ran}((D_{\mathcal{B}}^*)^{-1} F)$.

It works...

Theorem

The solution u of

$$\mathcal{L}u = f$$

can be computed by

$$u = F^* D_{\mathcal{B}}^{-1} \mathbf{Pu}$$

where $\mathbf{u}$ solves

$$\mathbf{Pu} = \left(\alpha \sum_{n=0}^{\infty} (id - \alpha \mathbf{G})^n \right) \mathbf{f}, \quad \mathbf{f} := (D_{\mathcal{B}}^*)^{-1} F f$$

with $0 < \alpha < 2/\lambda_{\max}$ and $\lambda_{\max} = \|\mathbf{G}\|$. Here $\mathbf{P} : \ell_2(\mathcal{N}) \to ran(\mathbf{G})$ is the orthogonal projection onto $ran(\mathbf{G})$.

Numerical Realization:

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} - \alpha(\mathbf{G}\mathbf{u}^{(i)} - \mathbf{f}), \qquad \mathbf{u}^{(0)} = \mathbf{0}$$

- $\mathbf{RHS}[\varepsilon, \mathbf{g}] \to \mathbf{g}_\varepsilon$: determines for $\mathbf{g} \in \ell_2(\mathcal{N})$ a finitely supported $\mathbf{g}_\varepsilon \in \ell_2(\mathcal{N})$ such that

$$\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2(\mathcal{N})} \leq \varepsilon;$$

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ALGORITHM $\mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}] \to \mathbf{u}_\varepsilon$:

Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that
$3\rho^K < \theta,\ \rho = \|\mathrm{id} - \alpha \mathbf{G}_{|\mathrm{ran}(\mathbf{G})}\|$.
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Properties of $\mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}]$:

Theorem

$\mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}]$ *produces* $\mathbf{u}_\varepsilon$ *with*

$$\|u - F^* D_{\mathcal{B}}^{-1} \mathbf{u}_\varepsilon\|_{\mathcal{B}} \leq \|F^*\| \|D_{\mathcal{B}}^{-1}\| \varepsilon.$$

Theorem

If the best N–term approximation of u satisfies $\mathcal{O}(N^{-s})$, then

- $\|u - F^* D_{\mathcal{B}}^{-1} \mathbf{u}_\varepsilon\|_{\mathcal{B}} = \mathcal{O}((\#\text{supp } \mathbf{u}_\varepsilon)^{-s})$
- *$\#$flops needed to compute $\mathbf{u}_\varepsilon$* $= \mathcal{O}(\#\text{supp } \mathbf{u}_\varepsilon)$

numerical realization of **RHS**, **APPLY**, **COARSE**:

A. Cohen, W. Dahmen, and R. DeVore, Adaptive wavelet methods for elliptic operator equations — Convergence rates, *Math. Comp.* **70** (2001), 27–75.

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How to Realize $\mathbf{APPLY}$?

- $\mathbf{APPLY}[\varepsilon, \mathbf{G}, \mathbf{v}] \rightarrow \mathbf{w}_\varepsilon,$ s.t. $\|\mathbf{G}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$
- find significant columns of $\mathbf{G}$ and compress them (corresponding to significance)!

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- (CP) $\Longrightarrow$ it works!

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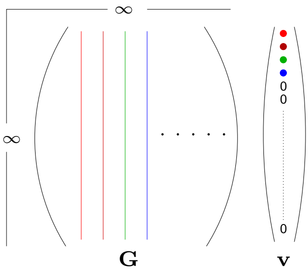

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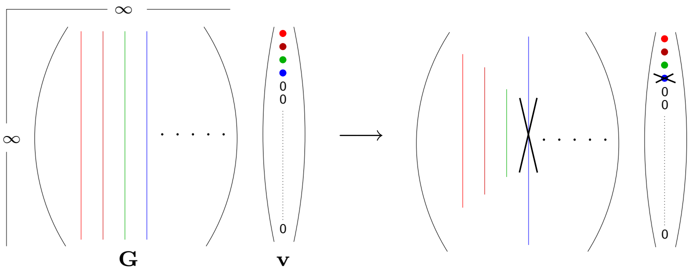

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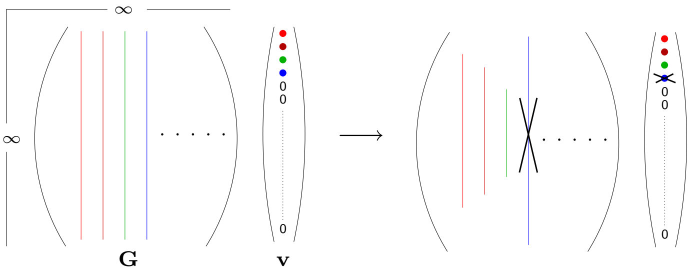

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$\mathbf{N} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$ is s^*–compressible if

$$\|\mathbf{N} - \mathbf{N}_J\| \lesssim \alpha_J 2^{-Js}, \qquad 0 < s < s^*$$

$\alpha_J 2^J$ non–zero entries per row/column, $\{\alpha_k\}_{k\in\mathbb{Z}}$ summable

Theorem

$\mathbf{G}$ *s^*–compressible,* $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, $s \in (0, s^*)$,
$\tau = (1/2 + s)^{-1}$, $\mathbf{f}$ *is s^*–optimal, i.e.,* $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon, \mathbf{f}]$

$$\#\text{supp } \mathbf{f}_\varepsilon, \textit{ number of arithmetic operations } \lesssim \varepsilon^{-1/s} |\mathbf{f}|^{1/s}_{\ell^w_\tau(\mathcal{N})}.$$

$\mathbf{P}$ *is bounded on* $\ell^w_{\tilde{\tau}}(\mathcal{N})$, $\tilde{s} \in (s, s^*)$. *Then, for* K *in* $\mathbf{SOLVE}$ *sufficiently large* $\mathbf{u}_\epsilon := \mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}]$ *satisfies*

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The Steepest Descent Approach:

Richardson iteration revisited:

- Richardson requires computation of optimal relaxation parameter $\alpha = 2.0/\left(\|\mathbf{G}_{|\mathrm{ran}\mathbf{G}}\| + \|\mathbf{G}_{|\mathrm{ran}\mathbf{G}}^{-1}\|^{-1}\right)$
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choose: $\mathbf{w}^{(0)} \in \mathrm{ran}(\mathbf{G})$

iterate:

$$\mathbf{r}^{(\mathbf{n})} := \mathbf{f} - \mathbf{G}\mathbf{w}^{(\mathbf{n})},$$

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Advantages:

- descent parameter individually computed for each step
- asymptotically: Richardson ⇔ steepest descent
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The Algorithm...

ALGORITHM $\quad \mathbf{STEEP\_SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}] \rightarrow \mathbf{w}_\varepsilon$:

Fix constants etc...

inner loop:

for $i := 1$ to K

$\bar{\mathbf{w}}_i := \mathbf{w}_{i-1}$

$\tilde{\mathbf{r}}_i := \mathbf{RHS}[\eta, \mathbf{f}] - \mathbf{APPLY}[\eta, \mathbf{G}, \bar{\mathbf{w}}_i]$

$\mathbf{z}_i := \mathbf{APPLY}[\eta, \mathbf{G}, \tilde{\mathbf{r}}_i]$

$\bar{\mathbf{w}}_i := \bar{\mathbf{w}}_i + \frac{\langle \tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_i \rangle}{\langle \mathbf{z}_i, \tilde{\mathbf{r}}_i \rangle} \tilde{\mathbf{r}}_i$

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The Algorithm...

ALGORITHM $\mathbf{STEEP\_SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}] \rightarrow \mathbf{w}_\varepsilon$:

Fix constants etc...

inner loop:

for $i := 1$ to K

$\bar{\mathbf{w}}_i := \mathbf{w}_{i-1}$

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Properties of **STEEP_SOLVE**$[\varepsilon, \mathbf{G}, \mathbf{f}]$:

Theorem

STEEP_SOLVE$[\varepsilon, \mathbf{G}, \mathbf{f}]$ *produces* $\mathbf{w}_\varepsilon$ *with*

$$\|u - F^* D_{\mathcal{B}}^{-1} \mathbf{w}_\varepsilon\|_{\mathcal{B}} \lesssim \varepsilon$$

Theorem

If the best N–term approximation of u satisfies $\mathcal{O}(N^{-s})$, then

- $\|u - F^* D_{\mathcal{B}}^{-1} \mathbf{w}_\varepsilon\|_{\mathcal{B}} = \mathcal{O}((\#\text{supp } \mathbf{u}_\varepsilon)^{-s})$
- *$\#$flops needed to compute* $\mathbf{w}_\varepsilon = \mathcal{O}(\#\text{supp } \mathbf{u}_\varepsilon)$

Domain Decomposition:

drawbacks of steepest descent:

- sometimes not efficient enough
- $\kappa(\mathbf{G}) \gg 0 \longrightarrow$ many iteration steps

possible alternative:

- domain decomposition methods !
- overlapping partition
- parallelization

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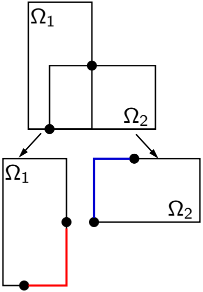
Domain Decomposition [H.A. Schwarz, 1870]

- decompose the problem into coupled subproblems in smaller subdomains

Schwarz iteration:



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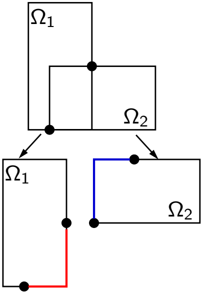

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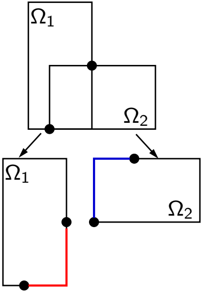

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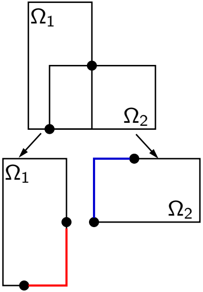

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Disretization: Additive Schwarz Iteration:

weak formulation of the first subproblem:

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$$u_1^{k+1} := u^k + w$$

- Gelfand frame $\Psi = \Psi^{(1)} \cup \Psi^{(2)}$
- $\mathbf{G}_i$ discretization of $a(\cdot,\cdot)$ on Ω_i w.r.t. $\Psi^{(i)}$
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for $k = 1, \dots,$

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- scheme corresponds to a Richardson iteration for the preconditioned system

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Adaptive Version:

ALGORITHM $\mathbf{ADDSCHWARZSOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}] \rightarrow \mathbf{u}_\varepsilon$:

Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that
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Outline

Numerical Experiments: Steepest Descent:

Poisson equation on an L–shaped domain Ω

$$-\triangle u \quad = \quad f \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

overlapping partition with two rectangles

$\Omega = (-1,1) \times (-1,0) \cup (-1,0) \times (-1,1)$

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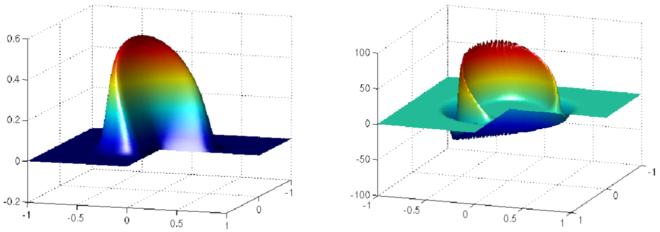

Figure: Exact solution and right–hand side.

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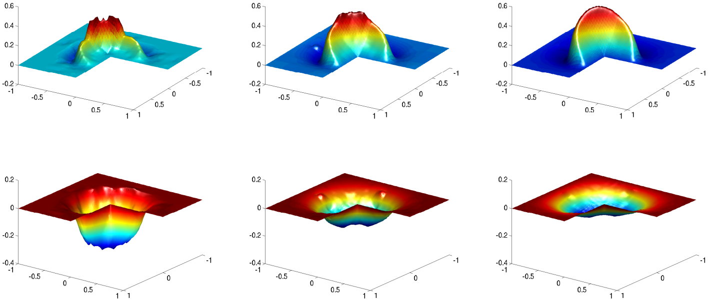
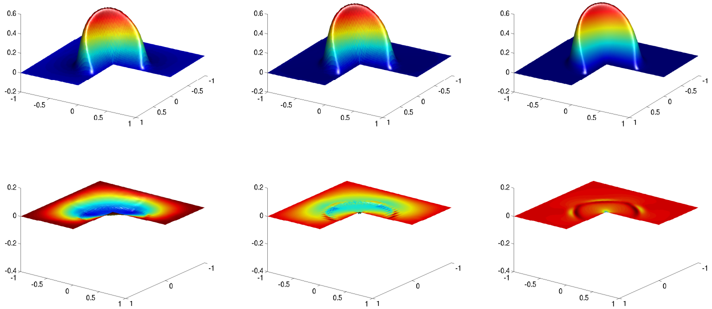
The First Iterations...

$\#\text{supp } \mathbf{u}_\varepsilon = 287$ $\#\text{supp } \mathbf{u}_\varepsilon = 694$ $\#\text{supp } \mathbf{u}_\varepsilon = 809$



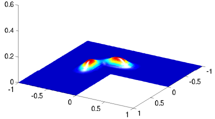
The Next Iterations...

$\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 952$ | $\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 1139$ | $\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 4617$

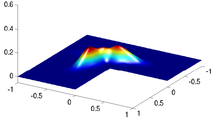


Numerical Experiments: Additive Schwarz:

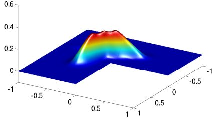
#supp $\mathbf{u}_\varepsilon = 2$



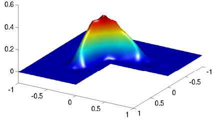
#supp $\mathbf{u}_\varepsilon = 8$



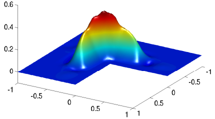
#supp $\mathbf{u}_\varepsilon = 15$



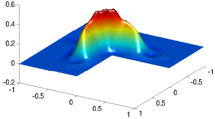
#supp $\mathbf{u}_\varepsilon = 27$



#supp $\mathbf{u}_\varepsilon = 43$

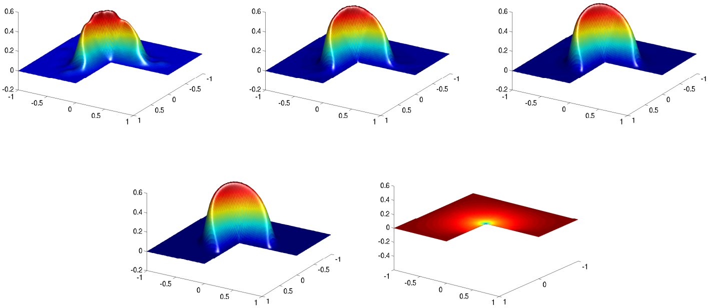


#supp $\mathbf{u}_\varepsilon = 62$

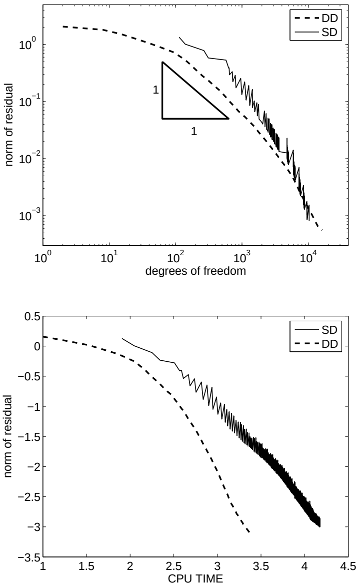


The Next Iterations...

$\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 160$ $\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 362$ $\#\mathrm{supp}\ \mathbf{u}_\varepsilon = 645$



Additive Schwarz vs. Steepest Descent:



Summary:

- Use frames instead of bases
- Adaptive numerical frame schemes
- Wavelet Gelfand frames
- Richardson Iteration
- Steepest Descent
- Domain Decomposition
- Numerical experiments

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