Invariant Manifolds for Products of Random Diffeomorphisms

Stephan Dahlke

Abstract

This paper is concerned with the construction of invariant families of submanifolds for products of random diffeomorphisms on a compact Riemannian manifold. These submanifolds can be obtained for almost arbitrary parameters disjoint from the Lyapunov spectrum of the resulting cocycle. Local measurable families are constructed and the globalization problem is discussed. We present a globalization result for generalized stable and unstable manifolds.

Key Words: invariant manifolds, random diffeomorphisms, Lyapunov exponents, ergodic theory

AMS Subject classification: primary 34C35, 58F11, secondary 28D05, 60J05

Author's affiliation and address:
Institut für Geometrie und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
phone: (0241) 80-6592
e-mail: dahlke@igpm.rwth-aachen.de

1 Introduction

One important aspect of the study of a dynamical system is the description of its invariant sets. In accordance with this philosophy, it is the aim of this paper to construct invariant manifolds for specific dynamical systems arising from random diffeomorphisms. Roughly speaking, we present a generalization of results developed by Pesin (1976), (1977a), (1977b). In the fundamental first paper, Pesin (1976) proved the existence of stable manifolds for diffeomorphisms on compact Riemannian manifolds. The basic tool for the construction is the multiplicative ergodic theorem proved by Oseledec (1968). One part of this theorem states the existence of appropriate subbundles of the tangent bundle of the manifold under consideration (Oseledec spaces). The stable manifolds are obtained by "pulling down" appropriate parts of these subbundles to the manifold. Therefore, the resulting invariant families of submanifolds can be interpreted as the nonlinear analoga of the Oseledec spaces.

During the last few years, it has turned out that the multiplicative ergodic theorem remains valid in much more general situations, see e.g. Boxler (1989) and Carverhill (1985) for details. Therefore, it seems plausible that Pesin's results also permit further generalizations. This program is carried out in this paper. The generalizations are concerned with the following points.

- Instead of one diffeomorphism we study the dynamic of products of random diffeomorphisms.
- We construct (generalized) stable and unstable manifolds with respect to almost arbitrary parameters disjoint from the Lyapunov spectrum.
- We construct nonlinear analoga to each of the Oseledec spaces itself (Oseledec manifolds).
- We consider the globalization problem, i.e., we try to establish the connections between the generalized stable manifolds of points from different orbits.

The multiplicative ergodic theorem is used to obtain an appropriate splitting of the tangent spaces. In principle, it is possible to obtain similar results for systems having another spectral decomposition. For instance, we could study systems to which the dynamical spectral theory of Sacker and Sell (1978) applies. However, in our setting, we would have to restrict ourselves to systems on compact probability spaces to apply this spectral theory which is very restrictive. Since this assumption is not needed for the spectral theory of Oseledec, the multiplicative ergodic theorem seems to be more suitable for what we have in mind.

This paper is organized as follows. In Section 2, we briefly recall the concept of random diffeomorphisms and state a version of the multiplicative ergodic theorem which is appropriate for our purpose. Section 3 is devoted to invariant families of local manifolds. In Section 3.1, we construct measurable families of local stable manifolds. The idea is to reduce the problem of finding an invariant family of submanifolds to the problem of constructing an invariant manifold for a mapping in a Banach space of sections of an appropriate fibre bundle having a hyperbolic fixed point. Furthermore, we show that for special parameters our construction yields strongly stable manifolds having a specific dynamical characterization. In Section 3.2 we construct unstable and Oseledec manifolds by using a similar technique. Section 3.3 is devoted to the proof of some technical lemmata. In Section 4 we study the globalization problem. It is a well-known fact that the strongly stable manifolds give rise to a global foliation. For the generalized stable manifolds also considered here this is not necessarily true, but we can show that at least along the smaller strongly stable manifolds the generalized stable manifolds of different orbits are tangent to the same spaces. A similar result holds for the unstable manifolds. We have stated this result in the stochastic setting, but to our knowledge a result of this type is new even for the deterministic case (which is clearly a special case of our construction). The proof of this result is based on several technical lemmata which are proved in Section 4.3 and on a specific construction principle for invariant manifolds in Banach spaces which is explained in Section 4.1.

The investigations in this paper where at least partially inspired by the work of Fathi *et al.* (1983). They give a new proof of Pesin's stable manifold theorem by means of the reduction idea sketched above. The results presented here are also closely related to the work of Carverhill (1985) and Boxler (1989) on stochastic flows. By using a quite different method, they show the existence of strongly stable manifolds and center manifolds, respectively. In Section 3, we will study the relations of their work to our approach. Furthermore, there are relations to the work of Pugh and Shub (1989). Using methods similar to ours, they also give a new proof of Pesin's stable manifold theorem and use it to show the existence of ergodic attractors and the absolute continuity of the stable foliation. Moreover, they give sharp differentiability estimates and generalize Pesin's results to arbitrary parameters, e.g., they also study point two of our program. However, their work is restricted to the deterministic case, and they have no results concerning the globalization problem for arbitrary parameters.

2 The Setting

In this section, we briefly recall the concept of random diffeomorphisms and multiplicative ergodic theory. Let M be a compact Riemannian manifold equipped with the Levi-Civita connection, and let $\mathrm{Diff}^2(M)$ denote the set of C^2-diffeomorphisms of M. Furthermore, let $(\Omega, \mathcal{A}, P)$ be a probability space and $\{\xi_n\}_{n\in\mathbf{Z}}$ be a stationary and ergodic sequence of random variables with values in a measurable space $(Y, \mathcal{Y})$. Without loss of generality, we may assume that $\Omega = Y^{\mathbf{Z}}$. On $Y^{\mathbf{Z}}$, we define the shift

$$\vartheta_n(\omega)(\cdot) := \omega(\cdot + n), \quad n \in \mathbf{Z}.$$

ϑ is a measure preserving map and one has $\xi_n = \xi_0 \circ \vartheta_n$. For some measurable mapping $\Upsilon : Y \longrightarrow \mathrm{Diff}^2(M)$ we set

$$\varphi(n, \omega, \cdot) := \begin{cases} \Upsilon_n(\omega) \circ \Upsilon_{n-1}(\omega) \circ \ldots \circ \Upsilon_1(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ \Upsilon_{n+1}^{-1}(\omega) \circ \Upsilon_{n+2}^{-1}(\omega) \circ \ldots \circ \Upsilon_{-1}^{-1}(\omega) \circ \Upsilon_0^{-1}(\omega) & : \quad n < 0, \end{cases} \tag{2.1}$$

where $\Upsilon_n(\omega) := \Upsilon(\xi_n(\omega)) = \Upsilon(\xi_0 \circ \vartheta_n(\omega))$. It is easy to check that φ defines a **cocycle** with respect to ϑ_n, i.e.,

$$\varphi(n+m, \omega, \cdot) = \varphi(n, \vartheta_m(\omega), \cdot) \circ \varphi(m, \omega, \cdot) \quad \text{for all } m, n \in \mathbf{Z}. \tag{2.2}$$

This cocycle gives rise to a **linear skew-product flow** Θ_n defined by

$$\begin{aligned} \Theta_n : \ \Omega \times M \ &\longrightarrow \ \Omega \times M \\ (\omega, x) \ &\longmapsto \ (\vartheta_n(\omega), \varphi(n, \omega, x)) \ . \end{aligned} \tag{2.3}$$

The compactness of M implies that φ possesses an **invariant measure** ν on $\Omega \times M$, i.e.,

$$\Theta_n(\nu) = \nu \ \text{ for all } n \in \mathbf{Z} \text{ and } \pi_\Omega(\nu) = P,$$

where π_Ω denotes the canonical projection onto the first factor. For details, the reader is referred e.g. to Crauel (1990). One way to describe the dynamic of a cocycle φ is given by the famous multiplicative ergodic theorem which says that almost everywhere with respect to ν the differential of φ possesses a well-defined asymptotic behaviour. Furthermore, it ensures the existence of appropriate invariant subbundles of the tangent bundle TM of M.

Theorem 2.1 *(Multiplicative ergodic theorem of Oseledec)*
Let φ be a cocycle on M with ergodic invariant measure ν on $\Omega \times M$. Suppose that

$$\int_{\Omega \times M} \left[\log^+ \|T\varphi(1, \omega, x)\| + \log^+ \|(T\varphi(1, \omega, x))^{-1}\|\right] d\nu(\omega, x) < \infty. \tag{2.4}$$

Then there exists a Θ_n-invariant set $\Gamma \subset \Omega \times M$ with $\nu(\Gamma) = 1$ such that for some real numbers $\{\lambda_i\}_{i=1,\ldots,r}$ with $\lambda_1 < \ldots < \lambda_r$ and some integers $\{d_i\}_{i=1,\ldots,r}$ satisfying $\sum_{i=1}^r d_i = d = dim(M)$ the following holds.

i) There exists a measurable splitting

$$T_x M = E_1(\omega, x) \oplus \ldots \oplus E_r(\omega, x), \quad \dim E_i(\omega, x) = d_i,$$

on Γ which is invariant with respect to Θ_n in the sense that

$$E_i(\Theta_n(\omega, x)) = T\varphi(n, \omega, x) E_i(\omega, x) \ .$$

ii) For all ξ in $E_i(\omega, x) \backslash \{0\}$ one has

$$\lambda^\pm(\omega, x, \xi) := \lim_{n \to \pm\infty} \frac{1}{n} \log \|T\varphi(n, \omega, x)\xi\| = \lambda_i \ .$$

The numbers $\{\lambda_i\}_{i=1,\ldots,r}$ are called the **Lyapunov exponents**, and the set of pairs $\{\lambda_i, d_i\}_{i=1,\ldots,r}$ is called the **Lyapunov spectrum** of the cocycle φ. For further information concerning random dynamical systems and multiplicative ergodic theory, the reader is referred e.g. to Arnold and Crauel (1991).

3 Local Invariant Manifolds

3.1 Local Stable Manifolds

In our setting, a **local stable manifold** with respect to a parameter $\mu \in \mathbf{R}$ consists of an invariant family of immersed C^1−submanifolds tangent to the spaces $V^s_\mu(\omega, x) := \oplus_{\lambda_i<\mu} E_i(\omega, x)$. Observe that it is not necessary to assume that $\mu < 0$. (Stable manifolds with respect to parameters $\mu > 0$ are sometimes called **center stable manifolds** or **pseudo stable manifolds**, see e.g. Abraham and Robbin (1967) or Pugh and Shub (1989)). The main result of this section shows the existence of such families under certain integrability conditions on the functions

$$G(\omega) := \sup_{x\in M} \|T\varphi(1, \omega, x)\|, \quad H(\omega) := \sup_{x\in M} \|T^2\varphi(1, \omega, x)\| . \tag{3.1}$$

To simplify the terminology, we will sometimes not distinguish between invariant manifolds and invariant families of submanifolds. As stated above, we proceed by transforming the problem to a global situation in an appropriate Banach space. To do that , we start by lifting the cocycle locally to the tangent spaces by means of the exponential mapping. These local lifts give rise to a mapping in a Banach space of sections of an appropriate fibre bundle. It can be shown that this mapping can be approximated in a certain sense by the differential of the cocycle. By using specific measurable norms, the zero section of the bundle is converted into a hyperbolic fixed point and we can apply well-known results concerning invariant manifolds of these points. Each invariant manifold with respect to this fixed point corresponds to a family of local stable manifolds for the original cocycle. This technique was first used by Hirsch and Pugh (1970) to construct invariant manifolds on hyperbolic sets. Further generalizations were given by Fathi *et al.* (1983).

The construction depends on some parameters that will be fixed once and for all. We choose an arbitrary small number $a > 0$ and construct stable manifolds for parameters μ disjoint from all intervalls $[\lambda_i - a, \lambda_i + a]$. For that purpose, we have to fix a Riemannian metric on M and to construct a suitable C^∞ function g associated with this metric. Furthermore, we have to fix a parameter ϵ which controls the exponential growth rate of some specific functions. In principle, the resulting manifolds can depend on the choice of a, μ and g. For $\mu < 0$, it can be shown that this dependence is ameliorated in a certain sense, see Corollary 3.1 below.

Theorem 3.1 *(Local stable manifolds)*
Let $a \in \mathbf{R}^+$, $\mu \in \mathbf{R}$ be some fixed numbers such that $\mu \notin [\lambda_i - a, \lambda_i + a]$ for all λ_i and consider the associated splitting

$$V^s_\mu(\omega, x) := \bigoplus_{\lambda_i<\mu} E_i(\omega, x), \quad V^u_\mu(\omega, x) := \bigoplus_{\lambda_i>\mu} E_i(\omega, x) . \tag{3.2}$$

Suppose that

$$E \log^+(G(\omega)) < \infty, \quad E \log^+(H(\omega)) < \infty. \tag{3.3}$$

Then there exists an invariant set $\Lambda \subseteq \Gamma$ such that $\nu(\Lambda) = 1$, a measurable function $\alpha : \Lambda \times \mathbf{N} \longrightarrow (0,\infty)$ and a family $\{W^s_\mu(\omega,x)|(\omega,x) \in \Lambda\}$ of immersed C^1-submanifolds such that

i) $\varphi(m,\omega,\cdot)(W^s_\mu(\omega,x) \bigcap B(x,\alpha(\omega,x,n))) \subset W^s_\mu(\Theta_m(\omega,x))$ *for all* $0 \leq m \leq n$,

ii) $x \in W^s_\mu(\omega,x)$ *and* $T_x W^s_\mu(\omega,x) = V^s_\mu(\omega,x)$.

Remark 3.1

i) Clearly, $B(x,\alpha(\omega,x,n))$ denotes the ball of radius $\alpha(\omega,x,n)$ at x.

ii) The first statement clarifies the meaning of invariance in our setting, whereas the second one expresses the fact that the manifolds have indeed the desired tangentiality.

iii) The integrability conditions are needed to prove Lemma 3.2. Without this lemma, one can still prove the existence of invariant manifolds, but their size can no longer be controlled.

Proof of Theorem 3.1:
First of all, we define a new metric on the tangent spaces of M which depends on ω and is adapted to the Oseledec spaces. We set

$$\begin{aligned} |\xi|_{(\omega,x)} &:= \sum_{i=0}^{\infty} e^{(-(\lambda_i+a)n)}\|T\varphi(n,\omega,x)\xi\| + \sum_{n=1}^{\infty} e^{((\lambda_i-a)n)}\|T\varphi(-n,\omega,x)\xi\| \quad \text{for } \xi \in E_i(\omega,x), \\ |\xi|_{(\omega,x)} &:= \max_i |\xi_i|_{(\omega,x)} \qquad \text{for } \xi = \sum_{i=1}^{r}\xi_i,\ \xi_i \in E_i(\omega,x) \ . \end{aligned} \tag{3.4}$$

A metric of this form is sometimes called a **Lyapunov metric**. For later use, we have to clarify the relations between this new metric and the original Riemannian metric. Furthermore, we have to study the behaviour of the differential of the cocycle with respect to the Lyapunov metric. Concerning the first point, we have the following lemma which will be proved in Section 3.3.

Lemma 3.1 *For all $\epsilon > 0$ there exists a measurable function $C(\omega,x,\epsilon)$ satisfying*

$$C(\Theta_n(\omega,x),\epsilon) \leq C(\omega,x,\epsilon)e^{(\epsilon|n|)} \quad \textit{for all } n \in \mathbf{Z} \tag{3.5}$$

and

$$r^{-1}\|\xi\| \leq |\xi|_{(\omega,x)} \leq C(\omega,x,\epsilon)\|\xi\|. \tag{3.6}$$

The norm of the differential can be estimated as follows.

$$\begin{aligned} e^{(\lambda_i+a)}|\xi|_{(\omega,x)} &\geq |T\varphi(1,\omega,x)\xi|_{\Theta(\omega,x)} \geq e^{(\lambda_i-a)}|\xi|_{(\omega,x)} \qquad \text{for } \xi \in E_i(\omega,x), \\ e^{(-(\lambda_i-a))}|\xi|_{\Theta(\omega,x)} &\geq |(T\varphi(1,\omega,x))^{-1}\xi|_{(\omega,x)} \geq e^{(-(\lambda_i+a))}|\xi|_{\Theta(\omega,x)} \text{ for } \xi \in E_i(\Theta(\omega,x)) \ . \end{aligned} \tag{3.7}$$

We will only prove the first inequality in detail. By using the definition of Θ_n in (2.3) and the cocycle property (2.2) of φ we obtain

$$
\begin{aligned}
|T\varphi(1,\omega,x)\xi|_{\Theta(\omega,x)} &= \sum_{n=0}^{\infty} e^{(-(\lambda_i+a)n)} \|T\varphi(n,\Theta(\omega,x))T\varphi(1,\omega,x)\xi\| \\
&\quad + \sum_{n=1}^{\infty} e^{((\lambda_i-a)n)} \|T\varphi(-n,\Theta(\omega,x))T\varphi(1,\omega,x)\xi\| \\
&= \sum_{n=0}^{\infty} e^{(-(\lambda_i+a)n)} \|T_x(\varphi(n,\vartheta(\omega),\cdot)\circ\varphi(1,\omega,\cdot))\xi\| \\
&\quad + \sum_{n=1}^{\infty} e^{((\lambda_i-a)n)} \|T_x(\varphi(-n,\vartheta(\omega),\cdot)\circ\varphi(1,\omega,\cdot))\xi\| \\
&= \sum_{n=0}^{\infty} e^{(-(\lambda_i+a)n)} \|T\varphi(n+1,\omega,x)\xi\| \\
&\quad + \sum_{n=1}^{\infty} e^{((\lambda_i-a)n)} \|T\varphi(-(n-1),\omega,x)\xi\| \\
&= e^{(\lambda_i+a)} \sum_{n=0}^{\infty} e^{(-(\lambda_i+a)(n+1))} \|T\varphi(n+1,\omega,x)\xi\| \\
&\quad + e^{(\lambda_i-a)} \sum_{n=1}^{\infty} e^{((\lambda_i-a)(n-1))} \|T\varphi(-(n-1),\omega,x)\xi\| \\
&\le e^{(\lambda_i+a)} |\xi|_{(\omega,x)}.
\end{aligned}
$$

The remaining estimates can be proved analogously.

Next claim is to lift the underlying cocycle locally to the tangent spaces by means of the exponential mapping, i.e., we define for a sufficiently small neighbourhood $U(\omega,x)$ of 0_x in T_xM

$$
\begin{aligned}
f_{(\omega,x)}:\ T_xM \supset U_{(\omega,x)} &\longrightarrow T_{\varphi(1,\omega,x)}M \\
\xi &\longmapsto \mathrm{Exp}^{-1}_{\varphi(1,\omega,x)} \circ \varphi(1,\omega,\cdot) \circ \mathrm{Exp}_x(\xi)\ .
\end{aligned}
\tag{3.8}
$$

These mappings have to be modified in such a way that they are defined on the whole tangent plane. This is performed by composing them in an appropriate way with the differential of φ. More precisely, one has the following lemma which will also be proved in Section 3.3.

Lemma 3.2 *For all $\epsilon,\zeta > 0$ there exists on a Θ_n−invariant set $\Lambda, \Lambda \subset \Gamma, \nu(\Lambda) = 1$ a function $D(\omega,x,\epsilon)$ such that the mapping*

$$
\begin{aligned}
F_{(\omega,x)}:\ T_xM &\longrightarrow T_{\varphi(1,\omega,x)}M \\
\xi &\longmapsto T\varphi(1,\omega,x)\xi + (f_{(\omega,x)} - T\varphi(1,\omega,x))(\xi)\cdot g\Big(\frac{\tilde r^2\|\xi\|^2}{2r^2D(\omega,x,\epsilon)^2}\Big)
\end{aligned}
\tag{3.9}
$$

is well-defined and satisfies

a) $Lip_{|\ |}(F_{(\omega,x)} - T\varphi(1,\omega,x)) < \zeta$,

b) $F_{(\omega,x)}(\xi) = f_{(\omega,x)}(\xi)$ *for* $|\xi|_{(\omega,x)} \le D(\omega,x,\epsilon)$,

c) $D(\Theta_n(\omega,x),\epsilon) \ge D(\omega,x,\epsilon)e^{(-\epsilon|n|)}$ *for all* $n \in \mathbf{Z}$,

d) $Lip^{\frac{1}{2}}_{|\ |}DF_{(\omega,x)} < 1$,

where g *denotes a suitable* $C^\infty-$*function and* $\tilde{r}$ *is a constant depending only on the geometry of* M.

Now we want to collect all the mappings $F_{(\omega,x)}$ to one global function in an appropriate Banach space. To this end, let us consider the measurable fibre bundle

$$(\Omega \times M, \Omega \times TM, \pi) \quad \text{with} \quad \pi(\omega,\xi) := (\omega, \pi_M\xi), \tag{3.10}$$

where π_M denotes the canonical projection from TM onto M. According to Lemma 3.2 we will henceforth consider the subbundle

$$(\Lambda, \pi^{-1}(\Lambda), \pi).$$

The Banach space we want to deal with consists of bounded sections of $(\Lambda, \pi^{-1}(\Lambda), \pi)$. Each section S is of the form

$$S(\omega,x) = (\omega, \sigma(\omega,x)),$$

so that we may define the Banach space B by

$$B := \{S \mid |S| = \sup_{(\omega,x)\in\Lambda} |\sigma(\omega,x)|_{(\omega,x)} < \infty\}. \tag{3.11}$$

Theorem 2.1 immediately implies that B possesses a splitting

$$B = B_1 \oplus B_2 \oplus \ldots \oplus B_r,$$

where

$$B_i := \{S \in B | \sigma(\omega,x) \in E_i(\omega,x) \text{ for all } (\omega,x) \in \Lambda\}.$$

We set

$$B^s_\mu := \bigoplus_{\lambda_i<\mu} B_i, \quad B^u_\mu := \bigoplus_{\lambda_i>\mu} B_i \ . \tag{3.12}$$

The mapping $F_{(\omega,x)}$ as well as the differential of φ admit a certain continuation to the space B. Indeed, we may define

$$\begin{aligned} \mathcal{F}: \quad & B \longrightarrow B && (3.13)\\ & (\mathcal{F}S)(\omega,x) = (\omega, F_{\Theta_{-1}(\omega,x)}\sigma(\Theta_{-1}(\omega,x))), \\ \mathcal{T}: \quad & B \longrightarrow B && (3.14)\\ & (\mathcal{T}S)(\omega,x) = (\omega, T\varphi(1,\Theta_{-1}(\omega,x))\sigma(\Theta_{-1}(\omega,x))). \end{aligned}$$

Now we are able to apply existence theorems concerning invariant manifolds in Banach spaces to $\mathcal{F}$ and $\mathcal{T}$. More precisely, we want to use the following version of the stable manifold theorem proved by Irwin (1972).

Theorem 3.2 *Let T be an isomorphism of a Banach space E with invariant subspaces E_1 and E_2 such that $E = E_1 \oplus E_2$, we define $T_i = T|_{E_i}$. Suppose that $\|T_1\| < \|T_2^{-1}\|^{-1}$ and let κ be such that $\|T_1\| < \kappa < \|T_2^{-1}\|^{-1}$. Let $f:\ E \to E$ be a (global) Lipschitz map such that $f(0) = 0$ and $Lip(f - T) = l < \min(\kappa - \|T_1\|, \|T_2^{-1}\|^{-1} - \kappa)$. Then the set*

$$\mathcal{W}^s_\kappa := \{x \in E | \sup_{n\geq 0} \|\kappa^{-n} f^n(x)\| < \infty\}$$

is the graph of a Lipschitz map $g:\ E_1 \to E_2$, with $Lip(g) < 1$. Moreover, $Lip(f|\mathcal{W}^s_\kappa) \leq \|T_1\| + l$, which implies that $\kappa^{-n} f^n(x) \to 0$ for $n \to \infty$ if $x \in \mathcal{W}^s_\kappa$.

To apply Theorem 3.2 it is sufficient to check

a) $\mathcal{T}$ is an isomorphism,

b) $|\mathcal{T}|_{B^s_\mu}| \leq e^{(\lambda_k + a)} < e^\mu < e^{(\lambda_{k+1} - a)} \leq |\mathcal{T}^{-1}|_{B^u_\mu}|^{-1}$ where $\lambda_k := \max_{\lambda_i < \mu} \lambda_i$,

c) $\mathcal{F} - \mathcal{T}$ is a Lipschitz map with $\mathrm{Lip}(\mathcal{F} - \mathcal{T}) \leq \zeta$, ζ sufficiently small.

For then, the conditions of Theorem 3.2 are satisfied with $\kappa = e^\mu$. Eq. (3.7) yields

$$\begin{aligned} |\mathcal{T}|_{B_i}| &= \sup_{S\in B_i, |S|=1} \sup_{(\omega,x)} |T\varphi(1, \Theta_{-1}(\omega, x))\sigma(\Theta_{-1}(\omega, x))|_{(\omega,x)} \\ &\leq \sup_{S\in B_i, |S|=1} \sup_{(\omega,x)} (e^{(\lambda_i + a)} |\sigma(\Theta_{-1}(\omega, x))|_{\Theta_{-1}(\omega,x)}) \\ &\leq e^{(\lambda_i + a)}, \end{aligned}$$

which implies that $\mathcal{T}$ is bounded. The boundedness of the inverse $(\mathcal{T}^{-1}S)(\omega, x) = (\omega, (T\varphi(1, \omega, x))^{-1}\sigma(\Theta(\omega, x)))$ can be proved analogously, so that a) is shown. Statement b) follows from the estimates

$$|\mathcal{T}|_{B^s_\mu}| \leq \max_{\lambda_i < \mu} |\mathcal{T}|_{B_i}| \leq \max_{\lambda_i < \mu} e^{(\lambda_i + a)} \leq e^{(\lambda_k + a)}, \quad |\mathcal{T}^{-1}|_{B^u_\mu}| \leq \max_{\lambda_i > \mu} e^{(-(\lambda_i - a))} \leq e^{(-(\lambda_{k+1} - a))}.$$

Finally, c) is a consequence of Lemma 3.2 a):

$$|(\mathcal{F} - \mathcal{T})S - (\mathcal{F} - \mathcal{T})\tilde{S}| \leq \zeta \sup_{(\omega,x)} |\sigma(\Theta_{-1}(\omega, x)) - \tilde{\sigma}(\Theta_{-1}(\omega, x))|_{\Theta_{-1}(\omega,x)} \leq \zeta |S - \tilde{S}|.$$

Therefore, we can use Theorem 3.2 to deduce that the set

$$\mathcal{W}^s_\mu := \{S \in B | \sup_{n\geq 0} |e^{(-\mu n)} \mathcal{F}^n S| < \infty\}$$

is the graph of a function $\mathcal{G}:\ B^s_\mu \to B^u_\mu$ satisfying $\mathcal{G}(0) = 0$ and Lip $\mathcal{G} < 1$.

It can be checked that the global mapping $\mathcal{G}$ gives rise to local mappings $\mathcal{G}(\omega, x)$ of the tangent spaces into themselves. The desired local manifolds are determined by $\mathcal{G}(\omega, x)$, i.e., they can be obtained by simply applying the exponential map to the graph of $\mathcal{G}(\omega, x)$. To carry out this construction, we want to use the following lemma which will be proved in Section 4.1.

Lemma 3.3 *$\mathcal{G}$ induces a family of mappings*

$$\mathcal{G}(\omega,x):\ V^s_\mu(\omega,x)\longrightarrow V^u_\mu(\omega,x)$$

such that

$$\mathcal{G}(S)(\omega,x)=(\omega,\mathcal{G}(\omega,x)\sigma(\omega,x)).$$

Using the functions $\mathcal{G}(\omega,x)$ we define

$$W^s_\mu(\omega,x):=\mathrm{Exp}_x\{\xi\in\text{graph }\mathcal{G}(\omega,x)|\ |\xi|_{(\omega,x)}\le D(\omega,x,\epsilon)\}\ . \tag{3.15}$$

We have to verify that these objects satisfy the conditions i) and ii). Let us start by proving i). To this end, it is sufficient to construct the measurable function α in such a way that for all $m\le n$

$$z\in W^s_\mu(\omega,x)\cap B(x,\alpha(\omega,x,n))\Rightarrow|F_{\Theta_{m-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}\circ\mathrm{Exp}_x^{-1}(z)|_{\Theta_m(\omega,x)}\le D(\Theta_m(\omega,x),\epsilon). \tag{3.16}$$

This can be seen as follows. For $z\in W^s_\mu(\omega,x),\ z=\mathrm{Exp}_x(\xi)$ we define

$$\delta^\xi_{(\omega,x)}(\tilde\omega,\tilde x):=\begin{cases}(\tilde\omega,\xi) & \text{if } (\tilde\omega,\tilde x)=(\omega,x),\\ (\tilde\omega,0) & \text{otherwise .}\end{cases} \tag{3.17}$$

Obviously, $\delta^\xi_{(\omega,x)}\in$ graph $\mathcal{G}$, and since graph $\mathcal{G}$ is invariant with respect to $\mathcal{F}$ it follows that $\mathcal{F}^m\delta^\xi_{(\omega,x)}\in$ graph $\mathcal{G}$. Therefore, since

$$\mathcal{F}^m\delta^\xi_{(\omega,x)}=\delta^{F_{\Theta_{m-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)}_{\Theta_m(\omega,x)}\ , \tag{3.18}$$

we obtain

$$F_{\Theta_{m-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)\in\text{graph }\mathcal{G}(\Theta_m(\omega,x)).$$

Hence, by using Eq. (3.16), Lemma 3.2 b), Eq. (3.8) and the cocycle property (2.2), we get

$$\begin{aligned}D(\Theta_m(\omega,x),\epsilon)&\ge|F_{\Theta_{m-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)|_{\Theta_m(\omega,x)}\\
&=|f_{\Theta_{m-1}(\omega,x)}\circ\ldots\circ f_{(\omega,x)}(\xi)|_{\Theta_m(\omega,x)}\\
&=|\mathrm{Exp}^{-1}_{\varphi(1,\vartheta_{m-1}(\omega),\varphi(m-1,\omega,x))}\circ\varphi(1,\vartheta_{m-1}(\omega),\cdot)\circ\mathrm{Exp}_{\varphi(m-1,\omega,x)}\\
&\quad\circ\mathrm{Exp}^{-1}_{\varphi(1,\vartheta_{m-2}(\omega),\varphi(m-2,\omega,x))}\circ\ldots\circ\varphi(1,\omega,\cdot)\circ\mathrm{Exp}_x(\xi)|_{\Theta_m(\omega,x)}\\
&=|\mathrm{Exp}^{-1}_{\varphi(m,\omega,x)}\circ\varphi(1,\vartheta_{m-1}(\omega),\cdot)\circ\varphi(1,\vartheta_{m-2}(\omega),\cdot)\circ\ldots\circ\varphi(1,\omega,\cdot)\circ\mathrm{Exp}_x(\xi)|_{\Theta_m(\omega,x)}\\
&=|\mathrm{Exp}^{-1}_{\varphi(m,\omega,x)}\circ\varphi(m,\omega,\cdot)(z)|_{\Theta_m(\omega,x)},\end{aligned}$$

which implies that

$$\varphi(m,\omega,z)\in W^s_\mu(\Theta_m(\omega,x)).$$

Let us define the function $\alpha(\omega, x, n)$ as follows:

$$\alpha(\omega, x, n) := \begin{cases} C(\omega, x, \epsilon)^{-1} D(\omega, x, \epsilon) e^{(-\epsilon n)} & \text{if } e^{(\lambda_k + a)} + \zeta \leq 1, \\ (e^{(\lambda_k + a)} + \zeta)^{-n} C(\omega, x, \epsilon)^{-1} D(\omega, x, \epsilon) e^{(-\epsilon n)} & \text{otherwise .} \end{cases} \tag{3.19}$$

We will only discuss the second case in detail, the first case can be studied analogously. Suppose that $z \in W^s_\mu(\omega, x)$, $d(x, z) \leq \alpha(\omega, x, n)$. Then, by employing Theorem 3.2, Lemma 3.1 and Lemma 3.2 we obtain

$$\begin{aligned} |F_{\Theta_{m-1}(\omega,x)} \circ \ldots \circ F_{(\omega,x)}(\xi)|_{\Theta_m(\omega,x)} &= |\mathcal{F}^m \delta^\xi_{(\omega,x)}| \\ &\leq (e^{(\lambda_k+a)} + \zeta)^m |\delta^\xi_{(\omega,x)}| \\ &= (e^{(\lambda_k+a)} + \zeta)^m |\xi|_{(\omega,x)} \\ &\leq (e^{(\lambda_k+a)} + \zeta)^m C(\omega, x, \epsilon) d(x, z) \\ &\leq (e^{(\lambda_k+a)} + \zeta)^n C(\omega, x, \epsilon) \alpha(\omega, x, n) \\ &\leq D(\omega, x, \epsilon) e^{(-\epsilon n)} \\ &\leq D(\Theta_m(\omega, x), \epsilon), \end{aligned}$$

and i) is proved. It remains to prove ii). To this end, we want to use the following theorem.

Theorem 3.3 *Suppose that the conditions of Theorem 3.2 are satisfied. Furthermore, suppose that f is a C^1-map and that $Lip(f - T)$ is sufficiently small. Then g is also a C^1- map, and if $Df(0) = T$, then $Dg(0) = 0$.*

For $\kappa \leq 1$, the proof of Theorem 3.3 was also given by Irwin (1972). However, it is easy to prove the theorem in its full generality by using the graph transform method described in Section 4, see e.g. Dahlke (1988) for details. To apply Theorem 3.3, we introduce the mapping

$$\begin{aligned} D\mathcal{F}(S): \quad & B \longrightarrow B \\ & D\mathcal{F}(S)(\tilde{S})(\omega, x) = (\omega, D_{\sigma(\Theta_{-1}(\omega,x))} F_{\Theta_{-1}(\omega,x)} \tilde{\sigma}(\Theta_{-1}(\omega, x))). \end{aligned}$$

Lemma 3.2 d) implies that

$$|\mathcal{F}(S + \tilde{S}) - \mathcal{F}(S) - D\mathcal{F}(S)\tilde{S}| \leq \sup_{(\omega,x)} |\tilde{\sigma}(\Theta_{-1}(\omega, x))|^{\frac{1}{2}}_{\Theta_{-1}(\omega,x)} |\tilde{\sigma}(\Theta_{-1}(\omega, x))|_{\Theta_{-1}(\omega,x)} \leq |\tilde{S}|^{\frac{3}{2}},$$

which shows that $D\mathcal{F}(S)$ is indeed the derivative of $\mathcal{F}$. Furthermore, since $T_0 \mathrm{Exp}_x = \mathrm{Id}$, we obtain

$$\begin{aligned} D\mathcal{F}(0)S(\omega, x) &= (\omega, D_0 F_{\Theta_{-1}(\omega,x)} \sigma(\Theta_{-1}(\omega, x))) \\ &= (\omega, D_0(\mathrm{Exp}_x^{-1} \circ \varphi(1, \vartheta_{-1}(\omega), \cdot) \circ \mathrm{Exp}_{\varphi(-1,\omega,x)}) \sigma(\Theta_{-1}(\omega, x))) \\ &= (\omega, T\varphi(1, \Theta_{-1}(\omega, x)) \sigma(\Theta_{-1}(\omega, x))), \end{aligned}$$

which implies that $D\mathcal{F}(0) = \mathcal{T}$. Therefore, it follows from Theorem 3.3 that $\mathcal{G}$ is C^1 and $D\mathcal{G} = 0$, and hence, by employing the continuous linear operators

$$\begin{array}{rlcl} i_{(\omega,x)}: & V_\mu^s(\omega,x) & \longrightarrow & B_\mu^s \\ & \xi & \longmapsto & \delta^\xi_{(\omega,x)} \end{array} \qquad \begin{array}{rlcl} \pi_{(\omega,x)}: & B_\mu^u & \longrightarrow & V_\mu^u(\omega,x) \\ & S & \longmapsto & \sigma(\omega,x) \end{array}$$

we obtain that each local map $\mathcal{G}(\omega,x)$ is C^1 and satisfies $D_0\mathcal{G}(\omega,x) = 0$, i.e., $\mathcal{G}(\omega,x)$ is tangent to V_μ^s. □

Remark 3.2

i) The proof of Theorem 3.1 shows that, in principle, similar results can be obtained for dynamical systems on non-compact manifolds. However, in this case, one has to assume the existence of an invariant measure. Furthermore, the radius of injectivity of the manifold under consideration has to be different from zero.

ii) To be on safe side and to obtain a C^1- family of submanifolds, we have assumed that $\Upsilon(\omega) \in \mathrm{Diff}^2(M)$. As one would expect, the smoothness of the submanifolds $W_\mu^s(\omega,x)$ increases with the smoothness of the cocycle φ. In fact, it can be checked that if $\Upsilon(\omega) \in \mathrm{Diff}^r(M)$, then the $W_\mu^s(\omega,x)$ are C^{r-1}. In the deterministic case, sharper results are available. For instance, it can be shown that for a $C^{r+\beta}-$ diffeomorphism the strongly stable manifolds are also $C^{r+\beta}$, provided that $r+\beta > 1$, see Pugh and Shub (1989) for details. Under some additional conditions, a similar result holds for the generalized stable manifolds.

iii) The set $\mathcal{W}_\mu^s$ is also invariant with respect to $\mathcal{F}^{-1}$. This is a consequence of its dynamical characterization according to Theorem 3.2. Using this property, it can be checked that the family $\{W_\mu^s(\omega,x)|(\omega,x) \in \Lambda\}$ is also backward invariant in the sense stated in part i) of Theorem 3.1.

According to Theorem 2.1, the distribution formed by the subspaces $V_\mu^s(\omega,x)$ is measurable, i.e., it is given by a measurable mapping into the corresponding Grassmann bundle over M. The local stable manifolds $W_\mu^s(\omega,x)$ can be interpreted as the nonlinear analoga of the spaces $V_\mu^s(\omega,x)$. Consequently, they are measurable in a similar sense.

Theorem 3.4 *Let $C^0(V_\mu^s, V_\mu^u)$ denote the measurable bundle over Λ whose fiber with respect to (ω,x) consists of the continuous functions from $V_\mu^s(\omega,x)$ into $V_\mu^u(\omega,x)$, equipped with the topology of uniform convergence on compact sets. Then the mapping*

$$\begin{array}{rcl} \Lambda & \longrightarrow & C^0(V_\mu^s, V_\mu^u) \\ (\omega,x) & \longmapsto & \mathcal{G}(\omega,x) \end{array}$$

provides a measurable section of this bundle.

Proof: Let $\mathcal{B}(M)$ denote the Borel $\sigma-$algebra of M. It can be checked that the subset

$$B_m := \{S \in B |\ S \text{ is } \mathcal{A}\otimes\mathcal{B}(M), \mathcal{A}\otimes\mathcal{B}(TM) - \text{measurable}\}$$

forms a closed subspace of B, see e.g. Dahlke (1989) for details. Analogously to B, B_m can be decomposed as

$$B_m = B^s_{m,\mu} \bigoplus B^u_{m,\mu},$$

where

$$\begin{aligned} B^s_{m,\mu} &:= \{S \in B_m | \, \sigma(\omega,x) \in V^s_\mu(\omega,x)\}, \\ B^u_{m,\mu} &:= \{S \in B_m | \, \sigma(\omega,x) \in V^u_\mu(\omega,x)\}. \end{aligned}$$

Therefore, the construction of Theorem 3.1 can also be performed for the space B_m, which implies that the set

$$\mathcal{W}^s_{m,\mu} := \{S \in B_m | \sup_{n\geq 0} |e^{(-\mu n)}\mathcal{F}^n S| < \infty\}$$

is the graph of a Lipschitz map $\mathcal{G}_m : \; B^s_{m,\mu} \to B^u_{m,\mu}$. Once again, $\mathcal{G}_m$ induces local maps $\mathcal{G}_m(\omega,x)$, and it can be checked that they coincide with the mappings $\mathcal{G}(\omega,x)$, see again Dahlke (1989) for details. We want to use the following theorem.

Theorem 3.5 *Let $(X,\mathcal{A})$ be a measurable space and M,N separable and metrizable $C^\infty-$ manifolds. Let $C^0(M,N)$ denote the space of continuous functions from M into N, equipped with the topology of uniform convergence on compact sets. Furthermore, let $map(X,M)$ and $map(X,N)$ denote the space of all mappings from X into M and N, respectively. To an arbitrary function $f : \; X \to C^0(M,N)$ we associate the mapping*

$$\begin{aligned} J : \; map(X,M) &\to map(X,N) \\ J(\theta)(x) &= f(x)\theta(x). \end{aligned}$$

If J maps measurable functions into measurable functions, then f is measurable.

For the case that X is a polish space with Borel $\sigma-$algebra this theorem was proved by Fathi *et al.* (1983). However, it can be checked that this assumption is in fact not necessary, see Dahlke (1989).
Consider the sets

$$\begin{aligned} V^s &:= \bigcup_{(\omega,x)\in\Lambda} (\{\omega\} \times V^s_\mu(\omega,x)) \subset \Omega \times TM, \\ V^u &:= \bigcup_{(\omega,x)\in\Lambda} (\{\omega\} \times V^u_\mu(\omega,x)) \subset \Omega \times TM. \end{aligned}$$

Since the spaces $V^s_\mu(\omega,x)$ and $V^u_\mu(\omega,x)$ depend measurable on (ω,x), it follows that the bundles V^s and V^u are measurable trivial, i.e., there exist measurable mappings

$$\begin{aligned} T^s : \; V^s &\longrightarrow \Lambda \times \mathbf{R}^k, \quad k = \dim V^s_\mu(\omega,x), \\ T^u : \; V^u &\longrightarrow \Lambda \times \mathbf{R}^{d-k}. \end{aligned}$$

Using the first trivialization, the space $B^s_{m,\mu}$ can be identified with the set $\mathcal{M}(\Lambda, \mathbf{R}^k)$ of measurable functions from Λ into $\mathbf{R}^k$. The space $B^u_{m,\mu}$ can be treated similarly. Therefore, $\mathcal{G}_m$ induces mappings

$$\begin{array}{rcl} G: \quad \Lambda & \longrightarrow & C^0(\mathbf{R}^k, \mathbf{R}^{d-k}) \\ (\omega, x) & \longmapsto & \mathcal{G}(\omega, x). \end{array}$$

However, since J defined by

$$\begin{array}{rcl} J: \mathcal{M}(\Lambda, \mathbf{R}^k) & \longrightarrow & \mathcal{M}(\Lambda, \mathbf{R}^{d-k}) \\ J(\theta)(\omega, x) & \longmapsto & \mathcal{G}(\omega, x)(\theta(\omega, x)) \end{array}$$

is well-defined, i.e., maps measurable functions into measurable functions, Theorem 3.5 implies that the mapping $(\omega, x) \mapsto \mathcal{G}(\omega, x)$ is measurable. $\square$

Remark 3.3 We have not used the space B_m in the proof of Theorem 3.1 since it was sometimes convenient to use the functions $\delta^{\xi}_{(\omega,x)}$. However, without further assumptions on the space $(\Omega, \mathcal{A})$ it is not a priori clear that these sections are measurable.

If we choose $\mu < 0$, then a little bit more can be said about the structure of the resulting stable manifolds, for then, the points in $W^s_\mu(\omega, x)$ possess a dynamical characterization. By using a quite different approach, a result of this type was proved before by Carverhill (1985).

Corollary 3.1 *(Strongly stable manifolds)*
Let $\mu < 0$ and ϵ sufficiently small. Then there exist measurable functions $\beta(\omega, x), \gamma(\omega, x)$ such that

$$\begin{aligned} & z \in W^s_\mu(\omega, x) \cap B(x, \beta(\omega, x)) \quad \textit{if and only if} \\ & z \in B(x, \beta(\omega, x)) \textit{ and } d(\varphi(n, \omega, x), \varphi(n, \omega, z)) \leq \gamma(\omega, x) e^{((\mu-\epsilon)n)} \textit{ for all } n \in \mathbf{N}. \end{aligned}$$

Proof: It is sufficient to show

a) $z \in W^s_\mu(\omega, x) \Longrightarrow |\mathrm{Exp}^{-1}_{\varphi(n,\omega,x)}(\varphi(n, \omega, z))|_{\Theta_n(\omega,x)} \leq (e^{(\lambda_k + a)} + \zeta)^n |\mathrm{Exp}^{-1}_x(z)|_{(\omega,x)}$,

b) $z \in W^s_\mu(\omega, x) \Longleftarrow |\mathrm{Exp}^{-1}_{\varphi(n,\omega,x)}(\varphi(n, \omega, z))|_{\Theta_n(\omega,x)} \leq e^{(\mu n)} D(\omega, x, \epsilon)$.

This can be seen as follows. Suppose that a) and b) hold. By means of Lemma 3.1 these implications can be easily transformed to the original Riemannian metric. We obtain

ã) $z \in W^s_\mu(\omega, x) \Longrightarrow d(\varphi(n, \omega, x), \varphi(n, \omega, z)) \leq rC(\omega, x, \epsilon)(e^{(\lambda_k + a)} + \zeta)^n d(x, z)$,

$\tilde{\text{b}}$) $z \in W^s_\mu(\omega, x) \Longleftarrow d(\varphi(n, \omega, x), \varphi(n, \omega, z)) \leq D(\omega, x, \epsilon) C(\omega, x, \epsilon)^{-1} e^{((\mu-\epsilon)n)}$.

Therefore, if we define

$$\beta(\omega,x) := \frac{D(\omega,x,\epsilon)}{rC(\omega,x,\epsilon)^2}, \qquad \gamma(\omega,x) := \frac{D(\omega,x,\epsilon)}{C(\omega,x,\epsilon)},$$

then ã) yields for $z \in W^s_\mu(\omega,x) \cap B(x,\beta(\omega,x))$ and ϵ small enough since $e^{(\lambda_k+a)} + \zeta < e^\mu$

$$\begin{aligned}
d(\varphi(n,\omega,x),\varphi(n,\omega,z)) &\leq rC(\omega,x,\epsilon)(e^{(\lambda_k+a)}+\zeta)^n d(x,z) \\
&\leq rC(\omega,x,\epsilon)e^{((\mu-\epsilon)n)}\frac{D(\omega,x,\epsilon)}{rC(\omega,x,\epsilon)^2} \\
&\leq \frac{D(\omega,x,\epsilon)}{C(\omega,x,\epsilon)}e^{((\mu-\epsilon)n)} \\
&= \gamma(\omega,x)e^{((\mu-\epsilon)n)},
\end{aligned}$$

and the other implication is a direct consequence of $\tilde{\text{b}}$).

Let us carry on by proving a). Since $\mu < 0$ we may choose ϵ small enough such that

$$e^{(\lambda_k+a)} + \zeta < e^\mu \leq e^{-\epsilon} \ .$$

Suppose that $z \in W^s_\mu(\omega,x),\ z = \mathrm{Exp}_x(\xi)$. Then Theorem 3.2 and Lemma 3.2 imply

$$|F_{\Theta_{n-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)|_{\Theta_n(\omega,x)} \leq (e^{(\lambda_k+a)}+\zeta)^n|\xi|_{(\omega,x)} \leq e^{(-\epsilon n)}D(\omega,x,\epsilon) \leq D(\Theta_n(\omega,x),\epsilon).$$

Therefore, we obtain

$$\begin{aligned}
(e^{(\lambda_k+a)}+\zeta)^n|\mathrm{Exp}_x^{-1}(z)|_{(\omega,x)} &\geq |F_{\Theta_{n-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)|_{\Theta_n(\omega,x)} \\
&= |f_{\Theta_{n-1}(\omega,x)}\circ\ldots\circ f_{(\omega,x)}(\xi)|_{\Theta_n(\omega,x)} \\
&= |\mathrm{Exp}^{-1}_{\varphi(n,\omega,x)}(\varphi(n,\omega,z))|_{\Theta_n(\omega,x)},
\end{aligned}$$

and a) is shown. It remains to prove b). We have to show that $\xi \in \text{graph}\ \mathcal{G}(\omega,x)$, i.e., $\delta^\xi_{(\omega,x)} \in \text{graph}\ \mathcal{G}$. To do that, we want to use the dynamical characterization of the set $\mathcal{W}^s_\mu$ according to Theorem 3.2. By employing once again the fact that $e^{(\mu n)}D(\omega,x,\epsilon) \leq e^{(-\epsilon n)}D(\omega,x,\epsilon) \leq D(\Theta_n(\omega,x),\epsilon)$ we obtain

$$|F_{\Theta_{n-1}(\omega,x)}\circ\ldots\circ F_{(\omega,x)}(\xi)|_{\Theta_n(\omega,x)} = |\mathrm{Exp}^{-1}_{\varphi(n,\omega,x)}\varphi(n,\omega,z)|_{\Theta_n(\omega,x)}$$

and therefore

$$|e^{(-\mu n)}\mathcal{F}^n\delta^\xi_{(\omega,x)}| = e^{(-\mu n)}|\mathrm{Exp}^{-1}_{\varphi(n,\omega,x)}\varphi(n,\omega,z)|_{\Theta_n(\omega,x)} \leq D(\omega,x,\epsilon) < \infty.$$

The Corollary is proved. □

From the construction of the spaces $V^s_\mu(\omega,x)$, it is clear that $V^s_{\mu_1}(\omega,x) \subset V^s_{\mu_2}(\omega,x)$ if $\mu_1 < \mu_2$. Since we have interpreted the manifolds $W^s_\mu(\omega,x)$ as the nonlinear analoga of the spaces $V^s_\mu(\omega,x)$, they should behave in a similar way. Indeed, one has the following

Corollary 3.2 *Let the numbers $\mu_1, \dots, \mu_{r-1}$ be chosen such that*

$$\lambda_1 + a < \mu_1 < \lambda_2 - a < \lambda_2 + a < \ldots < \mu_{r-1} < \lambda_r - a \ .$$

Then

$$W^s_{\mu_1}(\omega, x) \subset W^s_{\mu_2}(\omega, x) \subset \ldots \subset W^s_{\mu_{r-1}}(\omega, x) \ .$$

Proof: The dynamical characterization of the set $\mathcal{W}^s_\mu$ implies that it is sufficient to show that the conditions of the Theorems 3.2 and 3.3 are satisfied simultaneously for all $\mu_i, \ i = 1, \dots, r-1$. However, this can be easily checked by using the definition of the Lyapunov metric in Eq. (3.4) and the fact that the constant ζ may be chosen arbitrary small. $\square$

3.2 Unstable and Oseledec Manifolds

From our point of view, an invariant family of C^1-submanifolds $W^u_\mu(\omega, x)$ is called a family of **unstable manifolds** if each $W^u_\mu(\omega, x)$ is tangent to $V^u_\mu(\omega, x)$. In principle, the existence of such a family can be shown by mimicking the proof of Theorem 3.1 with $\varphi(-1, \omega, x)$ instead of $\varphi(1, \omega, x)$. However, this could e.g. produce a different function $D(\omega, x, \epsilon)$ which would be very unconvenient in the following. As we will see, it is necessary for our purpose to fix all the functions and parameters that are used once and for all. Therefore, we will prove the existence of the unstable families by employing the setting of Theorem 3.1.

Theorem 3.6 *Suppose that the conditions of Theorem 3.1 are satisfied. Then there exists a set $\Lambda \subseteq \Gamma$ such that $\nu(\Lambda) = 1$, a measurable function $\delta : \ \Lambda \times \mathbf{N} \to (0, \infty)$ and a family $\{W^u_\mu(\omega, x) | (\omega, x) \in \Lambda\}$ of immersed C^1-submanifolds such that*

i) $\varphi(-m, \omega, \cdot)(W^u_\mu(\omega, x) \cap B(x, \delta(\omega, x, n))) \subset W^u_\mu(\Theta_{-m}(\omega, x))$ *for* $m \leq n$,

ii) $x \in W^u_\mu(\omega, x)$ *and* $T_x W^u_\mu(\omega, x) = V^u_\mu(\omega, x)$.

Proof: We want to apply the Theorems 3.2 and 3.3 to $\mathcal{F}^{-1}$ and $\mathcal{T}^{-1}$. It can be checked that

$$\mathrm{Lip}\mathcal{F}^{-1} \leq (|\mathcal{T}^{-1}|^{-1} - \mathrm{Lip}(\mathcal{T} - \mathcal{F}))^{-1} \tag{3.20}$$

and therefore

$$\mathrm{Lip}(\mathcal{F}^{-1} - \mathcal{T}^{-1}) \leq |\mathcal{T}^{-1}| \mathrm{Lip}(\mathcal{T} - \mathcal{F})(|\mathcal{T}^{-1}|^{-1} - \mathrm{Lip}(\mathcal{T} - \mathcal{F}))^{-1} \ .$$

Hence, we are able to fulfil the conditions of Theorem 3.2 by choosing the parameter ζ in Lemma 3.2 sufficiently small. Once again, Lemma 3.3 implies the existence of local functions $\mathcal{H}(\omega, x)$ which determine the unstable manifolds. By construction, one has

$$\mathcal{F}^{-1}(S)(\omega, x) = (\omega, F^{-1}_{(\omega,x)} \sigma(\Theta(\omega, x))).$$

Therefore, it remains to show that there exists a function $L(\omega,x,\epsilon)$ satisfying

$$L(\Theta_n(\omega,x),\epsilon) \geq L(\omega,x)e^{(-\epsilon|n|)}, \quad n\in \mathbf{Z},$$

and

$$F^{-1}_{\Theta_{-1}(\omega,x)}(\xi) = f^{-1}_{\Theta_{-1}(\omega,x)}(\xi) \ \text{ for } |\xi|_{(\omega,x)} \leq L(\omega,x,\epsilon),$$

for then, after setting

$$W^u_\mu(\omega,x) := \mathrm{Exp}_x\{\xi \in \text{graph } \mathcal{H}(\omega,x)|\ |\xi|_{(\omega,x)} < L(\omega,x,\epsilon)\}$$

the function δ can be obtained by following the lines of the construction of the function α in Theorem 3.1. The function L can be generated from the function D. By construction

$$|F^{-1}_{\Theta_{-1}(\omega,x)}(\xi)|_{\Theta_{-1}(\omega,x)} \leq D(\Theta_{-1}(\omega,x),\epsilon)$$

implies

$$F^{-1}_{\Theta_{-1}(\omega,x)}(\xi) = f^{-1}_{\Theta_{-1}(\omega,x)}(\xi)\ .$$

Furthermore, one has

$$\begin{aligned}|F^{-1}_{\Theta_{-1}(\omega,x)}(\xi)|_{\Theta_{-1}(\omega,x)} &\leq \mathrm{Lip} F^{-1}_{\Theta_{-1}(\omega,x)}|\xi|_{(\omega,x)}\\ &\leq [|T\varphi(1,\Theta_{-1}(\omega,x))^{-1}|^{-1} - \mathrm{Lip}(F_{\Theta_{-1}(\omega,x)} - T\varphi(1,\Theta_{-1}(\omega,x)))]^{-1}|\xi|_{(\omega,x)}\\ &\leq (e^{(\lambda_1+a)}-\zeta)^{-1}|\xi|_{(\omega,x)}\ ,\end{aligned}$$

compare with (3.20). Therefore, by setting

$$L(\omega,x,\epsilon) := D(\omega,x,\epsilon)e^{(-\epsilon)}(e^{(\lambda_1+a)}-\zeta)$$

it follows for $|\xi|_{(\omega,x)} \leq L(\omega,x,\epsilon)$

$$|F^{-1}_{\Theta_{-1}(\omega,x)}(\xi)|_{\Theta_{-1}(\omega,x)} \leq (e^{(\lambda_1+a)}-\zeta)^{-1}|\xi|_{(\omega,x)} \leq D(\omega,x,\epsilon)e^{-\epsilon} \leq D(\Theta_{-1}(\omega,x),\epsilon).$$

Finally, since $\mathcal{F}$ is Lipschitz-close to $\mathcal{T}$, $\mathcal{F}^{-1}$ is C^1 which implies that $W^u_\mu(\omega,x)$ is C^1. $\square$

So far, we have constructed stable and unstable manifolds associated with appropriate parameters μ. Our construction shows that this parameter can in fact be chosen almost arbitrary, so that it seems natural to try to intersect the stable and unstable manifolds with respect to different parameters to obtain invariant manifolds tangent to the Oseledec spaces $E_i(\omega,x)$ themselves. The following theorem shows that these so-called **Oseledec manifolds** really exist.

Theorem 3.7 *Suppose that the conditions of Theorem 3.1 are satisfied and that the number $\mu_1 < \mu_2$ are disjoint from all intervals $[\lambda_i - a, \lambda_i + a]$. We set*

$$V^s_{\mu_1}(\omega,x) := \bigoplus_{\lambda_i<\mu_1} E_i(\omega,x),\ \ V^c_{\mu_1,\mu_2}(\omega,x) := \bigoplus_{\mu_1<\lambda_i<\mu_2} E_i(\omega,x),\ \ V^u_{\mu_2}(\omega,x) := \bigoplus_{\lambda_i>\mu_2} E_i(\omega,x)\ .$$

Then there exists a set $\Lambda \subseteq \Gamma$ such that $\nu(\Lambda)=1$, a measurable function $\rho:\ \Lambda\times\mathbf{N}\times\mathbf{N}\to(0,\infty)$ and a family $\{W^c_{\mu_1,\mu_2}(\omega,x)|(\omega,x)\in\Lambda\}$ of immersed C^1-submanifolds such that

i) $\varphi(n,\omega,\cdot)(W^c_{\mu_1,\mu_2}(\omega,x)\cap B(x,\rho(\omega,x,N,M)))\subset W^c_{\mu_1,\mu_2}(\Theta_n(\omega,x))$ *for* $n\le N$,
$\varphi(-m,\omega,\cdot)(W^c_{\mu_1,\mu_2}(\omega,x)\cap B(x,\rho(\omega,x,N,M)))\subset W^c_{\mu_1,\mu_2}(\Theta_{-m}(\omega,x))$ *for* $m\le M$,

ii) $x\in W^c_{\mu_1,\mu_2}(\omega,x)$ *and* $T_xW^c_{\mu_1,\mu_2}(\omega,x)=V^c_{\mu_1,\mu_2}(\omega,x)$.

Proof: Once again, the proof can be performed by using global results on invariant manifolds in Banach spaces. Consider the splitting

$$B^s_{\mu_1}:=\bigoplus_{\lambda_i<\mu_1}B_i,\quad B^c_{\mu_1,\mu_2}:=\bigoplus_{\mu_1<\lambda_1<\mu_2}B_i,\quad B^u_{\mu_2}:=\bigoplus_{\mu_2<\lambda_i}B_i.$$

If we set $\lambda_m:=\max_{\lambda_i<\mu_1}\{\lambda_i\}$, $\lambda_n:=\max_{\lambda_i<\mu_2}\{\lambda_i\}$, then we obtain by Eq. (3.7)

$$\begin{aligned}|\mathcal{T}|_{B^s_{\mu_1}}| &\le e^{(\lambda_m+a)}<e^{\mu_1}<e^{(\lambda_{m+1}-a)}\le|\mathcal{T}^{-1}|_{B^c_{\mu_1,\mu_2}}|^{-1}\\ &\le |\mathcal{T}|_{B^c_{\mu_1,\mu_2}}|\le e^{(\lambda_n+a)}<e^{\mu_2}<e^{(\lambda_{n+1}-a)}\le|\mathcal{T}^{-1}|_{B^u_{\mu_2}}|^{-1}\ .\end{aligned}\tag{3.21}$$

We want to use the following theorem on invariant manifolds.

Theorem 3.8 *Let T be an isomorphism of a Banach space E with invariant subspaces E_1,E_2 and E_3 such that $E=E_1\oplus E_2\oplus E_3$, we define $T_i=T|_{E_i}$. Suppose that*

$$\|T_1\|<\|T_2^{-1}\|^{-1}\le\|T_2\|<\|T_3^{-1}\|^{-1}$$

and let κ and $\tilde\kappa$ be such that

$$\|T_3^{-1}\|^{-1}>\kappa>\|T_2\|,\quad \|T_1\|^{-1}>\tilde\kappa>\|T_2^{-1}\|.$$

Let $f:\ E\to E$ be a (global) Lipschitz map such that $f(0)=0$ and

$$\begin{aligned}Lip(f-T)&=k<\min(\|T_3^{-1}\|^{-1}-\kappa,\kappa-\|T_2\|),\\ Lip(f^{-1}-T^{-1})&=\tilde k<\min(\|T_1\|^{-1}-\tilde\kappa,\tilde\kappa-\|T_2^{-1}\|).\end{aligned}$$

Then the set

$$\mathcal{W}^c_{\kappa,\tilde\kappa}:=\{x\in E|\sup_{n\ge0}\|\kappa^{-n}f^n(x)\|<\infty\wedge\sup_{n\ge0}\|\tilde\kappa^{-n}f^{-n}(x)\|<\infty\}$$

is the graph of a Lipschitz map $g:\ E_2\to E_1\oplus E_3$ with $Lip(g)<1$. Furthermore, if f is a C^1-map and $k,\tilde k$ sufficiently small, then g is C^1, and if $Df(0)=T$, then $Dg(0)=0$.

A proof of this theorem can be found e.g. in Dahlke (1989). If we choose the parameter ζ in Lemma 3.2 sufficiently small, then it follows by (3.21) that the conditions of Theorem 3.8 are satisfied which implies by Lemma 3.3 the existence of suitable functions

$$\mathcal{C}(\omega,x):V^c_{\mu_1,\mu_2}(\omega,x)\longrightarrow V^s_{\mu_1}(\omega,x)\oplus V^u_{\mu_2}(\omega,x).$$

Therefore, we may set

$$W^c_{\mu_1,\mu_2}(\omega,x):=\mathrm{Exp}_x\{\xi\in\text{graph }\mathcal{C}(\omega,x)|\ |\xi|_{(\omega,x)}<\min(D(\omega,x,\epsilon),L(\omega,x,\epsilon))\}$$

and proceed as in the proofs of the Theorems 3.1 and 3.6. $\square$

Remark 3.4 In the case of one vanishing Lyapunov exponent we obtain for $\mu_1 = -b,\ \mu_2 = b$ the so-called **center manifolds**. For stochastic flows on $\mathbf{R}^d$, and by using a quite different method, the existence of center manifolds was established by Boxler (1989). Clearly, this non-compact case can also be treated in the way suggested here, see Remark 3.2. On $\mathbf{R}^d$ equipped with the canonical connection we obtain

$$f_{(\omega,x)}(\xi) = \mathrm{Exp}^{-1}_{\varphi(1,\omega,x)} \circ \varphi(1,\omega,\cdot) \circ \mathrm{Exp}_x(\xi) = \varphi(1,\omega,x+\xi) - \varphi(1,\omega,x) := \tilde{\varphi}(1,\omega,x,\xi).$$

Let us furthermore define

$$\phi(1,\omega,\cdot) = \varphi(1,\omega,x+\cdot) - \varphi(1,\omega,x) - T\varphi(1,\omega,x)(\cdot).$$

In Boxler (1989), the center manifolds are obtained under boundedness conditions on the derivatives of ϕ. Each point y in the resulting manifold possesses a dynamical characterization of the form

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\tilde{\varphi}(n,\omega,x,y)\| \le \delta, \quad \liminf_{n\to-\infty} \frac{1}{n} \log \|\tilde{\varphi}(n,\omega,x,y)\| \ge -\delta \tag{3.22}$$

for some sufficiently small δ. This result can be obtained in our setting as follows. Suppose that

$$\|D\phi(1,\omega,\cdot)\| \le \frac{\zeta}{rC(\Theta(\omega,x),\epsilon)}.$$

Then

$$\begin{aligned}|f_{(\omega,x)}(\xi) - T\varphi(1,\omega,x)\xi - (f_{(\omega,x)}(\eta) - T\varphi(1,\omega,x)\eta)|_{\Theta(\omega,x)} &\le C(\Theta(\omega,x),\epsilon)\|\phi(1,\omega,\xi) - \phi(1,\omega,\eta)\| \\ &\le \frac{\zeta}{r}\|\xi-\eta\| \le \zeta|\xi-\eta|_{(\omega,x)}.\end{aligned}$$

Therefore, for ζ sufficiently small, we can use Theorem 3.7 to obtain the center manifolds as graphs of the mappings

$$\mathcal{C}(\omega,x):\ E_0(\omega,x) \longrightarrow V^s_{-b}(\omega,x) \oplus V^u_b(\omega,x).$$

Every point $\xi \in$ graph $\mathcal{C}(\omega,x)$ satisfies

$$F_{\Theta_{n-1}(\omega,x)} \circ \ldots \circ F_{(\omega,x)}(\xi) = f_{\Theta_{n-1}(\omega,x)} \circ \ldots \circ f_{(\omega,x)}(\xi) = \tilde{\varphi}(n,\omega,x,\xi)\ .$$

Hence Lemma 3.1 and Theorem 3.2 yield

$$\begin{aligned}\|\tilde{\varphi}(n,\omega,x,\xi)\| &\le r|\tilde{\varphi}(n,\omega,x,\xi)|_{\Theta_n(\omega,x)} \\ &\le r|F_{\Theta_{n-1}(\omega,x)} \circ \ldots \circ F_{(\omega,x)}(\xi)|_{\Theta_n(\omega,x)} \\ &\le r(\mathrm{Lip}\mathcal{F}|_{\mathcal{W}^s_b})^n|\xi|_{(\omega,x)} \\ &\le r(|\mathcal{T}|_{B^s_b} + \zeta)^n|\xi|_{(\omega,x)} \\ &\le rC(\omega,x,\epsilon)(e^a+\zeta)^n\|\xi\|,\end{aligned}$$

and therefore

$$\frac{1}{n}\log\|\tilde{\varphi}(n,\omega,x,\xi)\| \le \log(e^a+\zeta) + \frac{1}{n}\log(rC(\omega,x,\epsilon)\|\xi\|)\ ,$$

which means that in the forward direction our construction also gives rise to a dynamical characterization of the form (3.22) with $\delta = \log(e^a+\zeta)$. The backward direction can be treated analogously.

3.3 Proof of the Lemmata 3.1 and 3.2

Proof of Lemma 3.1

First of all, it can be checked that for all μ disjoint from the Lyapunov spectrum and $\epsilon > 0$ sufficiently small there exists a function $A(\omega, x, \epsilon, \mu)$ satisfying

$$\begin{aligned}
\|T\varphi(n,\omega,x)\xi\| &\leq A(\omega,x,\epsilon,\mu)\|\xi\|e^{(\mu n)} \text{ for all } \xi \in V^s_\mu(\omega,x),\ n \geq 0, && (3.23)\\
\|T\varphi(-n,\omega,x)\xi\| &\leq A(\omega,x,\epsilon,\mu)\|\xi\|e^{(-\mu n)} \text{ for all } \xi \in V^u_\mu(\omega,x),\ n \geq 0, && (3.24)\\
A(\Theta_n(\omega,x),\epsilon,\mu) &\leq A(\omega,x,\epsilon,\mu)e^{(\epsilon|n|)}\ ,\quad n \in \mathbf{Z}\ . && (3.25)
\end{aligned}$$

For deterministic systems, the existence of such a function was established by Fathi *et al.* (1983). However, their proof immediately carries over to the stochastic situation since it is only based on Theorem 2.1 and Hadamard's inequality. Using (3.23), (3.24) and the definition of the Lyapunov metric in (3.4) we obtain for $\xi_i \in E_i(\omega,x)$

$$|\xi|_{(\omega,x)} \leq \left(\frac{A(\omega,x,\epsilon,\lambda_i+\frac{a}{2}) + A(\omega,x,\epsilon,\lambda_i-\frac{a}{2})}{1-e^{(-\frac{a}{2})}}\right)\|\xi\|$$

which implies for $\xi = \sum_{i=1}^r \xi_i$

$$|\xi|_{(\omega,x)} = \max_i |\xi_i|_{(\omega,x)} \leq \max_i \left(\frac{A(\omega,x,\epsilon,\lambda_i+\frac{a}{2}) + A(\omega,x,\epsilon,\lambda_i-\frac{a}{2})}{1-e^{(-\frac{a}{2})}}\|\xi_i\|\right)\ . \qquad (3.26)$$

The expression on the right-hand side can be estimated further by employing the angle $\psi_\mu(\omega,x)$ between two subspaces $V^s_\mu(\omega,x)$ and $V^u_\mu(\omega,x)$ which is defined by

$$\cos(\psi_\mu(\omega,x)) := \sup\left\{\frac{|\langle\xi,\eta\rangle|}{\|\xi\|\ \|\eta\|} \mid \xi \in V^s_\mu(\omega,x),\ \eta \in V^u_\mu(\omega,x)\right\}\ , \qquad (3.27)$$

where μ is disjoint from the Lyapunov spectrum. The asymptotic behaviour of such an angle is described by the following lemma which is a generalization of the corresponding deterministic result of Pesin (1976), see Dahlke (1989) for details.

Lemma 3.4 *For every $\epsilon > 0$ there exists a measurable function $M(\omega,x,\epsilon)$ on Γ such that*

$$\begin{aligned}
M(\omega,x,\epsilon,\mu) &\leq \sin(\frac{\psi_\mu(\omega,x)}{2}), && (3.28)\\
M(\Theta_n(\omega,x),\epsilon,\mu) &\geq M(\omega,x,\epsilon,\mu)e^{(-\epsilon|n|)} \ \textit{for all } n \in \mathbf{Z}. && (3.29)
\end{aligned}$$

The norm of one component ξ_i can be estimated from above by

$$\|\xi_i\| \leq \prod_{j=1}^{r-1}(1-\cos(\psi_{\lambda_j+a}(\omega,x)))^{-\frac{1}{2}}\|\xi\|, \qquad (3.30)$$

see the proof of Lemma 4.7 for details. Therefore, combining (3.30) and (3.28) we obtain

$$\|\xi_i\| \leq 2^{\frac{1-r}{2}} \prod_{j=1}^{r-1} \sin(\frac{\psi_{\lambda_j+a}(\omega,x)}{2})^{-1}\|\xi\| \leq 2^{\frac{1-r}{2}} \prod_{j=1}^{r-1} M(\omega,x,\epsilon,\lambda_j+a)^{-1}\|\xi\| \ ,$$

so that, by inserting this expression into (3.26), we see that

$$C(\omega,x,\epsilon) := \frac{2^{\frac{1-r}{2}}}{1-e^{-\frac{a}{2}}}\left(\max_i(A(\omega,x,\frac{\epsilon}{r},\lambda_i+\frac{a}{2})+A(\omega,x,\frac{\epsilon}{r},\lambda_i-\frac{a}{2}))\right)\prod_{j=1}^{r-1} M(\omega,x,\frac{\epsilon}{r},\lambda_j+a)^{-1} \tag{3.31}$$

does the job since

$$\begin{aligned}
&C(\Theta_n(\omega,x),\epsilon)\\
&= \frac{2^{\frac{1-r}{2}}}{1-e^{-\frac{a}{2}}}\left(\max_i(A(\Theta_n(\omega,x),\frac{\epsilon}{r},\lambda_i+\frac{a}{2})+A(\Theta_n(\omega,x),\frac{\epsilon}{r},\lambda_i-\frac{a}{2}))\right)\prod_{j=1}^{r-1} M(\Theta_n(\omega,x),\frac{\epsilon}{r},\lambda_j+a)^{-1}\\
&\leq \frac{2^{\frac{1-r}{2}}}{1-e^{-\frac{a}{2}}}\left(\max_i(A(\omega,x,\frac{\epsilon}{r},\lambda_i+\frac{a}{2})+A(\omega,x,\frac{\epsilon}{r},\lambda_i-\frac{a}{2}))\right) e^{(\frac{\epsilon}{r}|n|)}\prod_{j=1}^{r-1} M(\omega,x,\frac{\epsilon}{r},\lambda_j+a)^{-1}e^{(\frac{\epsilon}{r}|n|)}\\
&\leq \quad C(\omega,x,\epsilon)e^{(\epsilon|n|)}.
\end{aligned}$$

□

Proof of Lemma 3.2:

Essentially, this lemma is the generalization to the stochastic situation of a corresponding deterministic result proved by Fathi *et al.* (1983). In Dahlke (1989) a detailed description of this generalization is given, so we will be brief and restrict ourselves to the presentation of the main ideas. For further information, the reader is referred to Fathi *et al.* (1983) and Dahlke (1989).

Let $(U_i,\psi_i)_{i=1,\ldots,m}$ be a fixed finite atlas of M and let $\tilde{r} > 0$ be a sufficiently small number less than the radius of injectivity such that for $x \in M$ the ball $B(x,\tilde{r})$ lies strictly inside a domain $U_{i(x)}$. This is always possible by Lebesque's covering lemma, see e.g. Walters (1982) for details. First step is to show that for all $\tilde{\epsilon}$ there exist functions $E(\omega,\tilde{\epsilon}), I(\omega,\tilde{\epsilon})$ such that for all (ω,x) in a set $\Lambda \subset \Gamma$, $\nu(\Lambda) = 1$ the mapping $f_{(\omega,x)}$ is well-defined for $\xi \in T_xM$ with $\|\xi\| \leq \tilde{r}E(\omega,\tilde{\epsilon})$ and satisfies

$$\|D_\xi f_{(\omega,x)} - D_\eta f_{(\omega,x)}\| \leq I(\omega,\tilde{\epsilon})\|\xi-\eta\| \ .$$

Furthermore, we want to show that E and I can be chosen such that

$$\begin{aligned}
E(\vartheta_n(\omega),\tilde{\epsilon}) &\geq E(\omega,\tilde{\epsilon})e^{(-\tilde{\epsilon}|n|)}\\
I(\vartheta_n(\omega),\tilde{\epsilon}) &\leq I(\omega,\tilde{\epsilon})e^{(\tilde{\epsilon}|n|)}
\end{aligned}$$

for all $n \in \mathbf{Z}$. Moreover, the function $E(\omega,\tilde{\epsilon})$ has to satisfy $E(\omega,\tilde{\epsilon}) \leq 1$. However, the construction presented below provides this property in a natural way.

From the estimate

$$d(\varphi(1,\omega,x),\varphi(1,\omega,y)) \leq \sup_{\tilde{x}\in M} \|T\varphi(1,\omega,\tilde{x})\| d(x,y) \leq G(\omega)d(x,y) \leq \max(G(\omega),1)d(x,y)$$

we see that $f_{(\omega,x)}$ is well-defined for all $\xi \in T_xM$ with $\|\xi\| \leq \tilde{r}(\max(1,G(\omega)))^{-1}$ and that there exist charts $(U_i,\psi_i),(U_j,\psi_j)$ such that $\mathrm{Exp}_x(\xi) \in U_i,\ \varphi(1,\omega,\mathrm{Exp}_x(\xi)) \in U_j$. By employing this fact, a long-winded, but standard computation using local coordinates shows that

$$\|D_\xi f_{(\omega,x)} - D_\eta f_{(\omega,x)}\| \leq c[(\max(1,G(\omega)))^2 + \max(1,G(\omega)) + \max(1,H(\omega))]\|\xi-\eta\| =: J(\omega)\|\xi-\eta\|,$$

where c only depends on the geometry of M. Let us now consider e.g. the term $\tilde{G}(\omega) := c\max(1,G(\omega))$ in more detail. Our integrability assumptions clearly imply that $E\log\tilde{G}(\omega) < \infty$. Therefore, since the shift by ϑ is measure-preserving, we obtain by Birkhoff's ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\tilde{G}(\vartheta_j(\omega)) \quad \text{exists } P-\text{a.e.},$$

and hence

$$\lim_{n\to\pm\infty} \frac{1}{|n|}\log\tilde{G}(\vartheta_n(\omega)) = 0 \quad P-\text{a.e.}.$$

Consequently, for every $\tilde{\epsilon} > 0$ there exists a number $N(\omega)$ such that

$$\tilde{G}(\vartheta_n(\omega)) \leq e^{(\tilde{\epsilon}|n|)} \quad \text{for } |n| \geq N(\omega).$$

The other terms can be estimated analogously, so that we have established the existence of a measurable function $\tilde{J}(\omega)$ satisfying

$$J(\vartheta_n(\omega)) \leq \tilde{J}(\omega)e^{(\tilde{\epsilon}|n|)} \quad \text{for all } n \in \mathbf{Z},\ \omega \in \Omega_1,\ \Omega_1 \subset \Omega,\ P(\Omega_1) = 1.$$

An application of Lemma 1.1.1 in Pesin (1976) yields the existence of a function $I(\omega,\tilde{\epsilon})$ satisfying

$$\begin{aligned} J(\omega) &\leq I(\omega,\tilde{\epsilon}), \\ I(\vartheta_n(\omega),\tilde{\epsilon}) &\leq I(\omega,\tilde{\epsilon})e^{(\tilde{\epsilon}|n|)}, \quad n \in \mathbf{Z},\ \omega \in \Omega_1. \end{aligned} \tag{3.32}$$

The function $E(\omega,\tilde{\epsilon})$ can be constructed analogously with respect to a set Ω_2. Therefore, by setting

$$\Lambda := ((\Omega_1 \cap \Omega_2) \times M) \cap \Gamma$$

we have proved our first claim.

After these preliminaries, we are now ready to prove the estimates stated in Lemma 3.2. We set for $\|\xi\| \leq \tilde{r}E(\omega,\tilde{\epsilon})$

$$h_{(\omega,x)}(\xi) := f_{(\omega,x)}(\xi) - T\varphi(1,\omega,x)\xi.$$

Next we choose a C^∞–function $g(t)$ in such a way that

$$g(t) = \begin{cases} 1 & \text{if } \ t \le \tilde{r}^2/2 \\ 0 & \text{if } \ t \ge \tilde{r}^2 \ . \end{cases}$$

Then, using g and an appropriate function $K(\omega, x, \tilde{\epsilon}) \le 1$ which will be constructed below the mapping

$$H_{(\omega,x)}(\xi) := h_{(\omega,x)}(\xi) g\left(\frac{\|\xi\|^2}{E^2(\omega,\tilde{\epsilon})K^2(\omega,x,\tilde{\epsilon})}\right)$$

is well-defined for all $\xi \in T_xM$. (In the sequel, we will sometimes drop the arguments ω, x and $\tilde{\epsilon}$.) We will proceed as follows. First of all, we show that for $\xi, \eta \in T_xM$

i) $\|H(\xi) - H(\eta)\| \le b_1 JKE^{-1}\|\xi - \eta\|$

ii) $\|D_\xi H - D_\eta H\| \le b_2 JK^{\frac{1}{2}}E^{-\frac{5}{2}}\|\xi - \eta\|^{\frac{1}{2}}$

with some constants b_1, b_2. Next step is to rewrite these inequalities in terms of the Lyapunov metric. From the resulting estimates the function $K(\omega, x, \tilde{\epsilon})$ can be derived. Finally, an appropriate modification of $K(\omega, x, \tilde{\epsilon})$ gives rise to the function $D(\omega, x, \epsilon)$.

We only want to show ii) in detail. Statement i) can be proved analogously. First of all, it is easy to check that for $\|\xi\| \le \tilde{r}E(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})$

α) $\|h_{(\omega,x)}(\xi)\| \le \tilde{r}^2 J(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})^2$,

β) $\|D_\xi h_{(\omega,x)}\| \le \tilde{r}J(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})$,

γ) Lip $h_{(\omega,x)} \le \tilde{r}J(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})$,

δ) $\mathrm{Lip}^{\frac{1}{2}} Dh_{(\omega,x)} \le (2\tilde{r})^{\frac{1}{2}} J(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})^{\frac{1}{2}}$.

We have to study

$$D_\xi H_{(\omega,x)}(\zeta) = D_\xi\left(h_{(\omega,x)}(\cdot)g(\frac{\|\cdot\|^2}{E^2K^2})\right)(\zeta) = \frac{2}{E^2K^2}g'(\frac{\|\xi\|^2}{E^2K^2})h(\xi)\langle\xi,\zeta\rangle + g(\frac{\|\xi\|^2}{E^2K^2})D_\xi h(\zeta)\ .$$

We want to estimate these two terms separately. To do that, we have to distinguish between the following cases.

- $\xi, \eta \in B(0, \tilde{r}EK)$,
- $\xi \in B(0, \tilde{r}EK),\ \eta \notin B(0, \tilde{r}EK)$,
- $\xi, \eta \notin B(0, \tilde{r}EK)$.

We will only study the first case in detail. Using β) and δ), the second term can be estimated as follows.

$$\begin{aligned}
\|g(\frac{\|\xi\|^2}{E^2K^2})D_\xi h - g(\frac{\|\eta\|^2}{E^2K^2})D_\eta h\| &\leq \|D_\xi h\||g(\frac{\|\xi\|^2}{E^2K^2}) - g(\frac{\|\eta\|^2}{E^2K^2})| + \|D_\xi h - D_\eta h\| \\
&\leq \tilde{r}JK \ \mathrm{Lip}^{\frac{1}{2}}g \ |\frac{\|\xi\|^2}{E^2K^2} - \frac{\|\eta\|^2}{E^2K^2}|^{\frac{1}{2}} + (2\tilde{r})^{\frac{1}{2}}JK^{\frac{1}{2}}\|\xi-\eta\|^{\frac{1}{2}} \\
&\leq \tilde{r}JK \ \mathrm{Lip}^{\frac{1}{2}}g \ |\frac{(\|\xi\|+\|\eta\|)(\|\xi\|-\|\eta\|)}{EK \quad EK}|^{\frac{1}{2}} + (2\tilde{r})^{\frac{1}{2}}JK^{\frac{1}{2}}\|\xi-\eta\|^{\frac{1}{2}} \\
&\leq \tilde{r}JK \ \mathrm{Lip}^{\frac{1}{2}}g \ (2\tilde{r})^{\frac{1}{2}}\frac{\|\xi-\eta\|^{\frac{1}{2}}}{(EK)^{\frac{1}{2}}} + (2\tilde{r})^{\frac{1}{2}}JK^{\frac{1}{2}}\|\xi-\eta\|^{\frac{1}{2}} \\
&\leq \frac{JK^{\frac{1}{2}}}{E^{\frac{1}{2}}}[\tilde{r}\mathrm{Lip}^{\frac{1}{2}}g \ (2\tilde{r})^{\frac{1}{2}} + (2\tilde{r})^{\frac{1}{2}}]\|\xi-\eta\|^{\frac{1}{2}}.
\end{aligned}$$

The treatment of the first term is a little bit more involved. We get

$$\begin{aligned}
&\|g'(\frac{\|\xi\|^2}{E^2K^2})\frac{2}{E^2K^2}h(\xi)\langle\xi,\cdot\rangle - g'(\frac{\|\eta\|^2}{E^2K^2})\frac{2}{E^2K^2}h(\eta)\langle\eta,\cdot\rangle\| \\
&\quad \leq \max|g'|\|\frac{2}{E^2K^2}h(\xi)\langle\xi,\cdot\rangle - \frac{2}{E^2K^2}h(\eta)\langle\eta,\cdot\rangle\| + \|\frac{2}{E^2K^2}h(\eta)\langle\eta,\cdot\rangle\||g'(\frac{\|\xi\|^2}{E^2K^2}) - g'(\frac{\|\eta\|^2}{E^2K^2})| \ .
\end{aligned}$$

The right-hand side can be estimated further by using the following facts which will be proved below.

a) $\|2E^{-2}K^{-2}h(\xi)\langle\xi,\cdot\rangle - 2E^{-2}K^{-2}h(\eta)\langle\eta,\cdot\rangle\| \leq (2\tilde{r})^{\frac{5}{2}}JK^{\frac{1}{2}}E^{-2}\|\xi-\eta\|^{\frac{1}{2}}$,

b) $\|2E^{-2}K^{-2}h(\xi)\langle\xi,\cdot\rangle\| \leq 4\tilde{r}^3JKE^{-2}$,

for $\|\xi\|, \|\eta\| \leq \tilde{r}EK$. Using a) and b), we obtain

$$\begin{aligned}
&\|g'(\frac{\|\xi\|^2}{E^2K^2})\frac{2}{E^2K^2}h(\xi)\langle\xi,\cdot\rangle - g'(\frac{\|\eta\|^2}{E^2K^2})\frac{2}{E^2K^2}h(\eta)\langle\eta,\cdot\rangle\| \\
&\quad \leq \max|g'| \ \frac{(2\tilde{r})^{\frac{5}{2}}}{E^2}JK^{\frac{1}{2}}\|\xi-\eta\|^{\frac{1}{2}} + \frac{4\tilde{r}^3JK}{E^2}\mathrm{Lip}^{\frac{1}{2}}g' \ |\frac{\|\xi\|^2}{E^2K^2} - \frac{\|\eta\|^2}{E^2K^2}|^{\frac{1}{2}} \\
&\quad \leq \frac{JK^{\frac{1}{2}}}{E^{\frac{5}{2}}}[\max|g'|(2\tilde{r})^{\frac{5}{2}} + 4\tilde{r}^3(2\tilde{r})^{\frac{1}{2}}\mathrm{Lip}^{\frac{1}{2}}g']\|\xi-\eta\|^{\frac{1}{2}},
\end{aligned}$$

so that, by combining the estimates of both terms, we finally get

$$\|D_\xi H_{(\omega,x)} - D_\eta H_{(\omega,x)}\| \leq b_2\frac{JK^{\frac{1}{2}}}{E^{\frac{5}{2}}}\|\xi-\eta\|^{\frac{1}{2}}$$

as stated in ii).

Let us carry on by proving a) and b). Combining α), γ) and the fact that $\|\xi\| \leq \tilde{r}EK$ we obtain for an arbitrary $\zeta \in T_xM$

$$\begin{aligned}
\|h(\xi)\langle\xi,\zeta\rangle - h(\eta)\langle\eta,\zeta\rangle\| &\leq \|h(\xi)-h(\eta)\|\|\xi\|\|\zeta\| + \|h(\eta)\||\langle\xi-\eta,\zeta\rangle| \\
&\leq \tilde{r}JK\|\xi-\eta\|\|\xi\|\|\zeta\| + \tilde{r}^2JK^2\|\xi-\eta\|\|\zeta\| \\
&\leq 2\tilde{r}^2JK^2\|\xi-\eta\|\|\zeta\|,
\end{aligned}$$

and therefore

$$\|\frac{2}{E^2K^2}h(\xi)\langle\xi,\cdot\rangle - \frac{2}{E^2K^2}h(\eta)\langle\eta,\cdot\rangle\| \;\leq\; \frac{4\tilde{r}^2J}{E^2}(2\tilde{r}K)^{\frac{1}{2}}\|\xi-\eta\|^{\frac{1}{2}} = \frac{(2\tilde{r})^{\frac{5}{2}}JK^{\frac{1}{2}}}{E^2}\|\xi-\eta\|^{\frac{1}{2}}$$

proving a). Statement b) follows immediately from the estimates

$$\|\frac{2}{E^2K^2}h(\xi)\langle\xi,\cdot\rangle\| \leq \frac{4\tilde{r}^2J}{E^2}\|\xi\| \leq \frac{4\tilde{r}^2J}{E^2}(\tilde{r}K) \leq \frac{4\tilde{r}^3JK}{E^2} \ .$$

Now let us return to our original goal. Using Lemma 3.1, the statements i) and ii) can be transformed to the Lyapunov metric as follows

iii) $|H_{(\omega,x)}(\xi) - H_{(\omega,x)}(\eta)|_{\Theta(\omega,x)} \leq rb_1e^{\tilde{\epsilon}}C(\omega,x,\tilde{\epsilon})JKE^{-1}|\xi-\eta|_{(\omega,x)}$,

iv) $|D_\xi H_{(\omega,x)} - D_\eta H_{(\omega,x)}|_{\Theta(\omega,x)} \leq r^{\frac{3}{2}}b_2e^{\tilde{\epsilon}}C(\omega,x,\tilde{\epsilon})JK^{\frac{1}{2}}E^{-\frac{5}{2}}|\xi-\eta|^{\frac{1}{2}}_{(\omega,x)}$.

From iii) and iv) we can guess how to choose the functions $K(\omega,x,\tilde{\epsilon})$ and $D(\omega,x,\epsilon)$. We set

$$K(\omega,x,\tilde{\epsilon}) := \min\left(\frac{E^5}{r^3b_2e^{2\tilde{\epsilon}}C(\omega,x,\tilde{\epsilon})^2J^2}, \frac{\zeta E}{rb_1e^{\tilde{\epsilon}}C(\omega,x,\tilde{\epsilon})J}, 1\right) \ .$$

Then (3.32) and Lemma 3.1 imply that

$$K(\Theta_n(\omega,x),\tilde{\epsilon}) \geq K(\omega,x,\tilde{\epsilon})e^{-9\tilde{\epsilon}|n|} \ .$$

$K(\omega,x,\tilde{\epsilon})$ can now be used to construct the function $D(\omega,x,\epsilon)$. We define for $\tilde{\epsilon} = \frac{\epsilon}{10}$

$$D(\omega,x,\epsilon) := \frac{\tilde{r}E(\omega,\tilde{\epsilon})K(\omega,x,\tilde{\epsilon})}{2^{\frac{1}{2}}r} \ .$$

Then $D(\omega,x,\epsilon)$ satisfies

$$D(\Theta_n(\omega,x),\epsilon) \geq D(\omega,x,\epsilon)e^{(-\epsilon|n|)},$$

i.e., statement c) in Lemma 3.2 is proved. It is easy to see that for this choice of $D(\omega,x,\epsilon)$ the other statements of Lemma 3.2 also hold. Property a) is a consequence of iii) since

$$\begin{aligned}
\mathrm{Lip}_{|\,|}(F_{(\omega,x)} - T\varphi(1,\omega,x)) &= \mathrm{Lip}_{|\,|}\left((f_{(\omega,x)} - T\varphi(1,\omega,x))g(\frac{\tilde{r}^2\|\cdot\|^2}{2r^2D(\omega,x,\epsilon)^2})\right) \\
&= \mathrm{Lip}_{|\,|}\left((f_{(\omega,x)} - T\varphi(1,\omega,x))g(\frac{\|\cdot\|^2}{E^2K^2})\right) \\
&= \mathrm{Lip}_{|\,|}H_{(\omega,x)} \leq \zeta \ .
\end{aligned}$$

d) follows from iv) in a similar way. It remains to check b). However, using (3.6) we observe that $|\xi|_{(\omega,x)} \leq D(\omega,x,\epsilon)$ implies

$$\|\xi\|^2E^{-2}K^{-2} \leq \tilde{r}^22^{-1}$$

and therefore

$$g(\frac{\|\xi\|^2}{E^2K^2}) = 1 \ .$$

□

4 The Globalization Problem

In Section 3 we have constructed local invariant manifolds with respect to almost arbitrary parameters disjoint from the Lyapunov spectrum. It seems natural to ask if these local objects can be "glued together" to well-defined global foliations. It was shown by Pesin (1977a) and Carverhill (1985) that this globalization procedure can be carried out for strongly stable manifolds, i.e., for $\mu < 0$ the local stable manifolds give rise to integral manifolds of the distribution $V_\mu^s(\omega, x)$, see also Ruelle (1979). However, the proofs are always based on the dynamical characterization of the local manifolds as described in Corollary 3.1. For $\mu > 0$, such a dynamical characterization does no longer hold, so that this case is much more complicated. Without a dynamical characterization, the structure of the local stable manifolds can depend e.g. on the chosen parameters and on the C^∞−function. Similar problems occur already for the construction of deterministic center manifolds, see e.g. Carr (1981) for details.

Nevertheless, even for $\mu > 0$, the construction is by no means arbitrary since the local manifolds are derived from a global problem with a unique solution. Therefore, all the properties of the family $\{W_\mu^s(\omega, x) | \, (\omega, x) \in \Lambda\}$ are hidden in the manifold $\mathcal{W}_\mu^s$, and the globalization problem corresponds to the study of this huge object. This study is performed in two directions. First of all, we try to describe in more detail the general properties of invariant manifolds for hyperbolic fixed points in Banach spaces. Secondly, we try to extract more information from the special structure of the Banach space we are working with. A combination of both directions yields the main result of this section which says that there is no arbitrariness along the strongly stable manifolds, even not for the generalized stable manifolds, i.e., for $\mu_1 < 0, \ \mu_2 > 0$ and $y \in W_{\mu_1}^s(\omega, x)$ we have $T_y W_{\mu_2}^s(\omega, x) = V_{\mu_2}^s(\omega, y)$.

To derive the properties of invariant manifolds, one has to study the proofs of the corresponding existence theorems. There are more or less two basic approaches. The first one, derived by Irwin (1972), is not suitable for our purpose. The second one is the more geometric approach developed by Hirsch and Pugh (1970) called the **graph transform method**. As we will see, this geometric method can be used to prove our result. In Section 4.1, we will state its main properties.

4.1 The Graph Transform Method

Let E be a Banach space, T an isomorphism with invariant subspaces E_1, E_2. Furthermore, let $f : E \to E$ be a global Lipschitz map with $f(0) = 0$. We want to find invariant manifolds for f tangent to E_1 that can be represented as the graphs of functions $h : \ E_1 \to E_2$. To this end, let $g : \ E_1 \to E_2, \ g(0) = 0$ be a Lipschitz map. Under quite natural assumptions, $f(\text{graph } g)$ is again the graph of a Lipschitz map $\Gamma_f g$. It can be checked that

$$\Gamma_f g(x) = (\pi_2 \circ f \circ (id, g)) \circ (\pi_1 \circ f \circ (id, g))^{-1}(x) \ . \tag{4.33}$$

Obviously, graph g represents the desired invariant manifold if

$$\Gamma_f g(x) = g(x).$$

Under certain conditions, the map $g \mapsto \Gamma_f g$ will be a well-defined contraction in a specific function space which implies the existence and uniqueness of the invariant manifold.

By this method it is only possible to find unstable manifolds since they behave in general as attractors. Therefore, a proof of the central Theorem 3.2 has to be performed by applying the graph transform method to f^{-1}. In our case, the setting is always chosen in such a way that this process converges. A direct application of the graph transform $\Gamma_{\mathcal{F}}$ would yield a different proof of Theorem 3.6 without the inversion process described in Section 3.2. Both approaches produce the same objects. Because of these observations, it would have been also possible to set up the investigations systematically on the unstable manifolds. However, this would yield some unexpected difficulties. For instance, it would be harder to describe the relations to Carverhill's results since by his approach it is only possible to construct stable manifolds.

As stated above, the graph transform can be interpreted as a geometric approach. One might think that a method that takes more advantage of the dynamical properties of the points in the invariant manifolds could be more powerful since the dynamic of an attractor might be more important than its geometric structure. However, it seems that this is not the case. For instance, the deep results of Pugh and Shub (1989) are only available by using the graph transform method.

We want to use the graph transform method to treat functions in Banach spaces of sections. Especially, we are interested in the class of functions $\mathcal{G}: \ B^u_\mu \longrightarrow B^s_\mu$ which induce mappings

$$\mathcal{G}(\omega, x): \ V^u_\mu(\omega, x) \longrightarrow V^s_\mu(\omega, x)$$

such that

$$\mathcal{G}(S)(\omega, x) = (\omega, \mathcal{G}(\omega, x)\sigma(\omega, x)) \ .$$

Such a function will be called **ponctual** in the sequel. The remainder of this section is devoted to the properties of the graph transform method in the space of ponctual functions. We will only state the basic results needed in the following section, for the proofs and more detailed informations the reader is referred to Dahlke (1989).

Lemma 4.1 *i) With respect to a suitable metric, the set* $P := \{\mathcal{G} \in C^0(B^u_\mu, B^s_\mu) | \ \mathcal{G} \text{ is ponctual}, \ \mathcal{G}(0) = 0, Lip\ \mathcal{G} \leq 1\}$ *is a closed subset of* $\{\mathcal{G} \in C^0(B^u_\mu, B^s_\mu) | \ \mathcal{G}(0) = 0, Lip\ \mathcal{G} \leq 1\}$.

ii) The graph transform $\Gamma_{\mathcal{F}}$ with respect to $\mathcal{F}$ maps P into itself.

Lemma 4.2 *Let $\mathcal{G} \in P$. Then*

$$graph(\Gamma_{\mathcal{F}}\mathcal{G}(\Theta(\omega, x))) = F_{(\omega,x)}(graph\ \mathcal{G}(\omega, x)) \ .$$

Furthermore, one has the following lemma which describes the behaviour of the derivatives of ponctual maps.

Lemma 4.3 *Let $\mathcal{G} \in P$ and suppose that $\mathcal{G}$ is differentiable at 0.*

i) If $D_0\mathcal{G}(\omega, x) = 0$, then $D_0\Gamma_{\mathcal{F}}\mathcal{G}(\Theta(\omega, x)) = 0$.

ii) If $|D_0\mathcal{G}| > 0$, then $|D_0\Gamma_{\mathcal{F}}\mathcal{G}| < |D_0\mathcal{G}|$.

We will finish this section with a short

Proof of Lemma 3.3:

As indicated above, the invariant manifold $\mathcal{W}^s_\mu$ can be obtained by applying the graph transform method to $\mathcal{F}^{-1}$ in the space $\{\mathcal{G} \in C^0(B^s_\mu, B^u_\mu) | \ \mathcal{G}(0) = 0, Lip\ \mathcal{G} \leq 1\}$. The graph transform is a contractive mapping in this space, see e.g. Dahlke (1988) for details. However, since by Lemma 4.1 the space P is a closed and invariant subset, the fixed point necessarily lies in this set. $\square$

4.2 A Globalization Theorem

So far, the whole construction was based on a given cocycle $\varphi(n, \omega, x)$. This cocycle was constructed by means of a measurable mapping

$$\Upsilon : \ Y \longrightarrow \mathrm{Diff}^2(M),$$

compare with Section 2. To show the main result of this section, we need a condition on the distribution Q of Υ. Since the proof is based on the graph transform method, we have formulated this result for the families of unstable manifolds.

Theorem 4.1 *Suppose that the conditions of Theorem 3.6 hold. Let $\mu_1, \mu_2 \in \mathbf{R}$ be disjoint from all intervals $[\lambda_i - a, \lambda_i + a]$ and suppose that $\mu_1 < 0 < \mu_2$. If supp $Q \subset Diff^2(M)$ is compact, then there exists a set $\tilde{\Lambda} \subseteq \Lambda \subseteq \Gamma$, $\nu(\tilde{\Lambda}) = 1$ such that for $(\omega, x), (\omega, y) \in \tilde{\Lambda}$*

$$y \in W^u_{\mu_2}(\omega, x) \cap B(x, \beta(\omega, x)) \ \ \textit{implies}\ T_y W^u_{\mu_1}(\omega, x) = V^u_{\mu_1}(\omega, y)\ ,$$

where $\beta(\omega, x)$ denotes a function constructed by means of Corollary 3.1 for the family of unstable manifolds.

Proof: We have to show that

$Exp_y^{-1}(W^u_{\mu_1}(\omega, x))$ is locally the graph of a C^1-map $\mathcal{L}(\omega, y) : \ V^u_{\mu_1}(\omega, y) \rightarrow V^s_{\mu_1}(\omega, y)$ satisfying $D_0\mathcal{L}(\omega, y) = 0$.

We want to prove this fact by using the properties of the graph transform method described in Section 4.1. The proof consists of the following three steps.

Claim 1: For n be sufficiently large we consider in $T_{\varphi(-n,\omega,y)}M$ the coordinate system that is obtained by a parallel translation of the spaces $V^u_{\mu_1}(\Theta_{-n}(\omega, x))$ and $V^s_{\mu_1}(\Theta_{-n}(\omega, x))$ from $\varphi(-n, \omega, x)$ to $\varphi(-n, \omega, y)$. We show that $\mathrm{Exp}^{-1}_{\varphi(-n,\omega,y)}(W^u_{\mu_1}(\Theta_{-n}(\omega, x)))$ is the graph of a mapping $\mathcal{I}(\Theta_{-n}(\omega, y))$ with respect to this coordinate system and we estimate its

Lipschitz constant.

Claim 2: We show that the facts proved in claim 1 also imply that $\mathrm{Exp}^{-1}_{\varphi(-n,\omega,y)}(W^u_{\mu_1}(\Theta_{-n}(\omega,x)))$ is the graph of a Lipschitz map $\mathcal{K}(\Theta_{-n}(\omega,y)):\ V^u_{\mu_1}(\Theta_{-n}(\omega,y)) \to V^s_{\mu_1}(\Theta_{-n}(\omega,y))$ and we estimate once again its Lipschitz constant. To show this part of the proof, we need the compactness of supp Q which yields a certain continuity of the Oseledec spaces.

Claim 3: The mappings $\mathcal{K}(\Theta_{-n}(\omega,y))$ are composed to a global function $\mathcal{L}$ in the space P. We show that the invariance of the family $\{W^u_{\mu_1}(\omega,x) \mid (\omega,x)\in\Lambda\}$ implies that the derivative of $\mathcal{L}$ at the zero section increases. Therefore, by Lemma 4.3, it has to be zero which yields the desired tangentiality.

Once again, the proof is based on several very technical lemmata. For the proof of these lemmata, the reader is referred to Dahlke (1989) and to Section 4.3, respectively.

Proof of Claim 1: First of all, we have to construct appropriate neighbourhoods $U(\Theta_{-n}(\omega,x))$ in $V^u_{\mu_1}(\Theta_{-n}(\omega,x))$ having the property that Lip $\mathcal{H}(\Theta_{-n}(\omega,x))|_{U(\Theta_{-n}(\omega,x))}$ is sufficiently small and that $\mathrm{Exp}^{-1}_{\varphi(-n,\omega,x)}\varphi(-n,\omega,y) = (\eta, \mathcal{H}(\Theta_{-n}(\omega,x))(\eta))$ for some $\eta\in U(\Theta_{-n}(\omega,x))$. Clearly, $\mathcal{H}(\omega,x)$ denotes the function $V^u_{\mu_1}(\omega,x) \longrightarrow V^s_{\mu_1}(\omega,x)$ which determines the unstable manifolds, see Section 3.

It can be checked that the functions involved in the construction of the invariant manifolds can be modified in such a way that

$$|D\mathcal{H}(S) - D\mathcal{H}(\tilde{S})| \le |S-\tilde{S}|^\alpha\ . \tag{4.34}$$

This is because the proof of Lemma 3.2 shows that the function $D(\omega,x,\epsilon)$ can be chosen such that

$$\mathrm{Lip}^\alpha_{|\ |} DF_{(\omega,x)} < \rho \quad \text{for } \rho>0,\ 0<\alpha<1. \tag{4.35}$$

(We have only restricted ourselves to the case $\alpha=\frac{1}{2}, \rho=1$ to avoid unnecessary technical difficulties). Eq. (4.35) implies that $D\mathcal{F}$ is also Lipschitz continuous with the same constants α,ρ, and it can be checked that this Lipschitz continuity carries over to the resulting invariant manifold, see Dahlke (1989) for details.

Eq. (4.34) implies

$$|D_\xi \mathcal{H}(\omega,x)|_{(\omega,x)} \le |\xi|^\alpha_{(\omega,x)}$$

which means that

$$\mathrm{Lip}_{|\ |_{(\omega,x)}} \mathcal{H}(\omega,x)|_{B(0,R(\omega,x))} \le R(\omega,x)^\alpha$$

for some suitable function R. According to Lemma 3.1, we therefore obtain for the original Riemannian metric

$$\mathrm{Lip}_{\|\ \|}\mathcal{H}(\omega,x)|_{B(0,R(\omega,x)C(\omega,x,\epsilon)^{-1})} \le rC(\omega,x,\epsilon)R(\omega,x)^\alpha\ . \tag{4.36}$$

We want to define $R(\omega,x)$ in such a way that it tends to zero along the backward orbit $\varphi(-n,\omega,x)$, but this convergence has to be slowly enough to make sure that

$\mathrm{Exp}^{-1}_{\varphi(-n,\omega,x)}\varphi(-n,\omega,y) = (\eta,\mathcal{H}(\Theta_{-n}(\omega,x))(\eta))$ for some $\eta\in B(0,R(\Theta_{-n}(\omega,x))C(\Theta_{-n}(\omega,x),\epsilon)^{-1})$, for then the desired neighbourhoods are given by

$$U(\Theta_{-n}(\omega,x)) := B(0, R(\Theta_{-n}(\omega,x))C(\Theta_{-n}(\omega,x),\epsilon)^{-1}) \ .$$

To find the function R, observe first that for some $\xi = (\eta, \mathcal{H}(\omega,x)(\eta))$

$$\|\xi\| \le R(\omega,x)C(\omega,x,\epsilon)^{-1}(1-\cos(\psi(\omega,x)))^{\frac{1}{2}} \ \text{implies} \ \|\eta\| \le R(\omega,x)C(\omega,x,\epsilon)^{-1}, \tag{4.37}$$

i.e., $\xi \in$ graph $\mathcal{H}(\omega,x)|_{B(0,R(\omega,x)C(\omega,x,\epsilon)^{-1})}$, see the proof of Lemma 4.7 for details. Once again, $\psi(\omega,x)$ denotes the angle between $V^u_{\mu_1}(\omega,x)$ and $V^s_{\mu_1}(\omega,x)$. We set

$$\begin{aligned} R(\Theta_{-n}(\omega,x)) &:= 2\gamma(\omega,x)C(\Theta_{-n}(\omega,x),\epsilon)(1-\cos(\psi(\Theta_{-n}(\omega,x)))^{-\frac{1}{2}}e^{(-(\mu_2+\epsilon)n)} \quad (4.38)\\ &= 2\gamma(\omega,x)C(\Theta_{-n}(\omega,x),\epsilon)2^{-\frac{1}{2}}\sin(\frac{\psi(\Theta_{-n}(\omega,x))}{2})^{-1}e^{(-(\mu_2+\epsilon)n)} \end{aligned}$$

where γ is a function defined according to Corollary 3.1 for the unstable manifolds. Then we obtain (for simplicity, we will sometimes use the abbreviations $x_n := \varphi(-n,\omega,x)$, $y_n := \varphi(-n,\omega,y)$)

$$\begin{aligned} \|\mathrm{Exp}^{-1}_{x_n}(y_n)\| &= d(x_n,y_n) \\ &\le \gamma(\omega,x)e^{(-(\mu_2+\epsilon)n)} \\ &= 2^{-\frac{1}{2}}[R(\Theta_{-n}(\omega,x))C(\Theta_{-n}(\omega,x),\epsilon)^{-1}\sin(\frac{\psi(\Theta_{-n}(\omega,x))}{2})] \ , \end{aligned}$$

so that (4.37) implies

$$\mathrm{Exp}^{-1}_{x_n}(y_n) = (\eta, \mathcal{H}(\Theta_{-n}(\omega,x))(\eta)) \ \text{for some} \ \eta \in B(0, R(\Theta_{-n}(\omega,x))C(\Theta_{-n}(\omega,x),\epsilon)^{-1}).$$

By using Lemma 3.4, the special form of the function $R(\omega,x)$ enables us to estimate the Lipschitz constant of $\mathcal{H}$ uniformly for all points in the backward orbit. We obtain

$$\begin{aligned} \mathrm{Lip}_{\|\ \|}\mathcal{H}(\Theta_{-n}(\omega,x))|_{B(0,RC^{-1})} &\le rC(\Theta_{-n}(\omega,x),\epsilon)R(\Theta_{-n}(\omega,x))^{\alpha} \\ &\le 2^{\frac{\alpha}{2}}r\gamma(\omega,x)^{\alpha}C(\Theta_{-n}(\omega,x),\epsilon))^{1+\alpha}\sin(\frac{\psi(\Theta_{-n}(\omega,x))}{2})^{-\alpha}e^{(-(\mu_2+\epsilon)\alpha n)} \\ &\le 2^{\frac{\alpha}{2}}r\gamma(\omega,x)^{\alpha}C(\omega,x,\epsilon)^{1+\alpha}e^{((1+\alpha)\epsilon n)}M(\Theta_{-n}(\omega,x),\epsilon)^{-\alpha}e^{(-(\mu_2+\epsilon)\alpha n)} \\ &\le 2^{\frac{\alpha}{2}}r\gamma(\omega,x)^{\alpha}C(\omega,x,\epsilon)^{1+\alpha}e^{((1+\alpha)\epsilon n)}M(\omega,x,\epsilon)^{-\alpha}e^{(\epsilon\alpha n)}e^{(-(\mu_2+\epsilon)\alpha n)} \\ &\le c_1e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)} \ , \end{aligned}$$

where c_1 depends on (ω,x), but not on the other points in the backward orbit.

After these preliminaries, we are now ready to prove the claim. Let $P(x_n,y_n)$ denote the parallel translation from x_n to y_n along the unique shortest geodesic. We want to show: For $n \ge N(\omega,x,y)$ sufficiently large there exist neighbourhoods U_{x_n} of $\mathrm{Exp}^{-1}_{x_n}(y_n)$ in $T_{x_n}M$ and V_{y_n} of 0 in $P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x))$ such that

$$\mathrm{Exp}^{-1}_{y_n}\circ\mathrm{Exp}_{x_n}(\text{graph } \mathcal{H}(\Theta_{-n}(\omega,x))\cap U_{x_n}) \tag{4.39}$$

is the graph of a function

$$\mathcal{I}(\Theta_{-n}(\omega,y)):\ V_{y_n}\longrightarrow P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x))$$

satisfying

$$\text{Lip }\mathcal{I}(\Theta_{-n}(\omega,y))\le c_2e^{(-(\mu_2\alpha-(1+a)\epsilon)n)}\ .$$

The modification of $\mathcal{H}(\Theta_{-n}(\omega,x))$ described in (4.39) can be interpreted as a generalized graph transform. The properties of such a transform needed for our purpose are summarized in the following lemma. The proof is more or less straightforward and can be found e.g. in Dahlke (1989).

Lemma 4.4 *Let E and F be Banach spaces with decompositions $E=E_1\oplus E_2,\ F=F_1\oplus F_2$ and equipped with the corresponding max-norms. Furthermore, let $T:\ E\longrightarrow F$ be an isomorphism satisfying $T(E_1)=F_1,\ T(E_2)=F_2$ and let $f:\ E\supset U\to F$ be a Lipschitz map. Suppose that $Lip(T-f)\le\ell$ is sufficiently small. Then, for a Lipschitz map $g:\ E_1\supset V\longrightarrow E_2$ with $Lip\ g\le k<1$, $f(graph g)$ is also the graph of a Lipschitz map $\Gamma_f g:\ F_1\supset W\longrightarrow F_2$ with $Lip\ \Gamma_f g\le(\|T_2\|k+\ell)(\|T_1^{-1}\|^{-1}-\ell)^{-1}$.*

We want to use this Lemma 4.4 for the special case

$$\begin{array}{lll} E=T_{x_n}M, & E_1=V^u_{\mu_1}(\Theta_{-n}(\omega,x)), & E_2=V^s_{\mu_1}(\Theta_{-n}(\omega,x)),\\ F=T_{y_n}M, & F_1=P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x)), & F_2=P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x)),\\ f=\text{Exp}^{-1}_{y_n}\circ\text{Exp}_{x_n}, & T=P(x_n,y_n). & \end{array}$$

The following lemma ensures that it is indeed possible to satisfy the conditions of Lemma 4.4 with these objects. Its proof is based on local coordinates and can be found in Dahlke (1989).

Lemma 4.5 *Let M be a compact Riemannian manifold with Levi-Civita connection. Then there exists a constant c such that for all $x,y\in M$ with $d(x,y)$ sufficiently small*

$$Lip\{(Exp^{-1}_y\circ Exp_x-P(x,y))|_{B(0,2d(x,y))}\}\le cd(x,y)\ .$$

However, we have to take into account the fact that Lemma 4.5 is stated in terms of the Riemannian metric whereas Lemma 4.4 is based on the max-norms induced by the decompositions. Let $|||\cdot|||_{x_n}$ denote the max-norm on $T_{x_n}M$, then

$$\frac{\|\xi\|}{2}\le|||\xi|||_{x_n}\le\frac{\|\xi\|}{(1-\cos(\psi(\Theta_{-n}(\omega,x))))^{\frac12}}\ , \tag{4.40}$$

compare with the proof of Lemma 4.7. Therefore, by using the fact that the parallel translation preserves the angles between subspaces, we get

$$\begin{aligned}\text{Lip}_{|||\ |||}\{\text{Exp}^{-1}_{y_n}\circ\text{Exp}_{x_n}-P(x_n,y_n))|_{B(0,2d(x_n,y_n))}\}\ &\le\ \frac{2cd(x_n,y_n)}{(1-\cos(\psi(\Theta_{-n}(\omega,x))))^{\frac12}}\\ &\le\ 2c\gamma(\omega,x)e^{(-(\mu_2+\epsilon)n)}2^{-\frac12}\sin(\frac{\psi(\Theta_{-n}(\omega,x))}{2})^{-1}\\ &\le\ 2^{\frac12}c\gamma(\omega,x)M(\omega,x,\epsilon)^{-1}e^{(-\mu_2 n)}\\ &\le\ c_3e^{(-\mu_2 n)}\ .\end{aligned}$$

Consequently, since $\|T_1\| = \|T_2\| = 1$, we obtain by Lemma 4.4 that for n sufficiently large the transformation described in (4.39) gives rise to a function

$$\mathcal{I}(\Theta_{-n}(\omega,y)):\ V_{y_n} \longrightarrow P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x))$$

satisfying

$$\text{Lip}\ \mathcal{I}(\Theta_{-n}(\omega,y)) \le \left(\frac{c_1 e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)} + c_3 e^{(-\mu_2 n)}}{1-c_3 e^{(-\mu_2 n)}}\right) .$$

This finishes the proof of claim 1.

Proof of Claim 2: Later on, we want to use the properties of the graph transform method described in Section 4.1. To do that, the local mappings $\mathcal{I}$ have to be extended to the whole spaces $P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x))$. However, by using e.g. suitable $C^\infty-$functions it is easy to see that there exist mappings

$$\mathcal{J}(\Theta_{-n}(\omega,y)):\ P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x)) \longrightarrow P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x))$$

and appropriate neighbourhoods $W_{y_n} \subset V_{y_n}$ such that

$$\mathcal{I}(\Theta_{-n}(\omega,y))|_{W_{y_n}} = \mathcal{J}(\Theta_{-n}(\omega,y))|_{W_{y_n}}$$

and

$$\text{Lip}\ \mathcal{J}(\Theta_{-n}(\omega,y)) \le c_2 e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)} .$$

We want to show that graph$\mathcal{J}(\Theta_{-n}(\omega,y))$ can be interpreted as the graph of a function $\mathcal{K}(\Theta_{-n}(\omega,y)):\ V^u_{\mu_1}(\Theta_{-n}(\omega,y)) \longrightarrow V^s_{\mu_1}(\Theta_{-n}(\omega,y))$. To this end, we will use the following lemma which will be proved in Section 4.3.

Lemma 4.6 *Let E be a finite-dimensional Banach space equipped with a scalar product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Suppose that E possesses decompositions $E = E_1 \oplus E_2$, $E = F_1 \oplus F_2$ with norms $\|\cdot\|^i_E, \|\cdot\|^i_F$ on $E_i, F_i, i=1,2$, and let $\|\cdot\|_E, \|\cdot\|_F$ denote the associated max-norms. Let $g_E:\ E_1 \longrightarrow E_2$ be a Lipschitz map with Lip $g_E \le \delta \le 1$. Suppose that for ϱ sufficiently small*

$$\max(\|P_{E_1}-P_{F_1}\|, \|P_{E_2}-P_{F_2}\|) < \varrho ,$$

where $P_{E_i}, P_{F_i},\ i=1,2$, denote the projections onto the corresponding subspaces. Then there exists a function

$$g_F:\ F_1 \longrightarrow F_2$$

such that

i) graph $g_F =$ graph g_E,

ii) $Lip_{\|\ \|_F} g_F \le (c_1(E)c_2(E)\varrho+\delta)(1-c_1(E)c_2(E)\varrho)^{-1}c_1(E)c_1(F)c_2(E)c_2(F)$, *where the constants* $c_i(E), c_i(F), i = 1, 2,$ *are defined by*

$$\begin{aligned}\frac{1}{c_1(E)}\|\xi\|_E &\leq \|\xi\| \leq c_2(E)\|\xi\|_E \ ,\\ \frac{1}{c_1(F)}\|\xi\|_F &\leq \|\xi\| \leq c_2(F)\|\xi\|_F \ .\end{aligned}$$

We want to apply this lemma to the case

$$\begin{array}{lll} E = T_{y_n}M, \quad E_1 = P(x_n, y_n)V^u_{\mu_1}(\Theta_{-n}(\omega, x)), & E_2 = P(x_n, y_n)V^s_{\mu_1}(\Theta_{-n}(\omega, x)), & \|\cdot\|_E = |||\cdot|||_{y_n}, \\ \qquad\qquad F_1 = V^u_{\mu_1}(\Theta_{-n}(\omega, y)), & F_2 = V^s_{\mu_1}(\Theta_{-n}(\omega, y)), & \|\cdot\|_F = |\cdot|_{\Theta_{-n}(\omega,y)}. \end{array}$$

The constants describing the relations of the different norms are given by (3.6) and (4.40). Therefore, using Lemma 4.6, Lemma 3.1 and Lemma 3.4, we deduce that there exist functions

$$\mathcal{K}(\Theta_{-n}(\omega, y)) : \ V^u_{\mu_1}(\Theta_{-n}(\omega, y)) \longrightarrow V^s_{\mu_1}(\Theta_{-n}(\omega, y))$$

such that

$$\text{graph}\mathcal{K}(\Theta_{-n}(\omega, y)) = \text{graph}\mathcal{J}(\Theta_{-n}(\omega, y))$$

and

$$\begin{aligned}\text{Lip}_{|\ |_{\Theta_{-n}(\omega,y)}}\mathcal{K}(\Theta_{-n}(\omega, y)) &\leq [2(1-\cos(\psi(\Theta_{-n}(\omega, x))))^{-\frac{1}{2}}\varrho(\Theta_{-n}(\omega, y)) + c_2 e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)}]\\ &\quad \cdot[1 - 2(1-\cos(\psi(\Theta_{-n}(\omega, x))))^{-\frac{1}{2}}\varrho(\Theta_{-n}(\omega, y))]^{-1} \\ &\quad \cdot[2r(1-\cos(\psi(\Theta_{-n}(\omega, x))))^{-\frac{1}{2}}C(\Theta_{-n}(\omega, y), \epsilon)]\\ &\leq [2^{\frac{1}{2}}M(\omega, x, \epsilon)^{-1}e^{(\epsilon n)}\varrho(\Theta_{-n}(\omega, y)) + c_2 e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)}]\\ &\quad \cdot[1 - 2^{\frac{1}{2}}M(\omega, x, \epsilon)^{-1}e^{(\epsilon n)}\varrho(\Theta_{-n}(\omega, y))]^{-1}2^{\frac{1}{2}}rM(\omega, x, \epsilon)^{-1}e^{(\epsilon n)}C(\omega, y, \epsilon)e^{(\epsilon n)}\\ &\leq [c_4\varrho(\Theta_{-n}(\omega, y))e^{(\epsilon n)} + c_2 e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)}]\\ &\quad \cdot[1 - c_4\varrho(\Theta_{-n}(\omega, y))e^{(\epsilon n)}]^{-1}c_5 e^{(2\epsilon n)} \ .\end{aligned} \tag{4.41}$$

From (4.41) we observe that we have reduced our problem to the study of the function ϱ. This is performed by estimating the distance between the corresponding subspaces. In general, the distance between two subspaces is defined by

$$\text{dist}(E_1, E_2) := \max(\sup_{\xi\in E_1, \|\xi\|=1}\inf_{\eta\in E_2}\|\xi - \eta\|, \sup_{\eta\in E_2, \|\eta\|=1}\inf_{\xi\in E_1}\|\xi - \eta\|) \ . \tag{4.42}$$

The relations between distances and norms of projections are clarified by the following lemma which will also be proved in Section 4.3.

Lemma 4.7 *Let E be a finite dimensional vector space with scalar product that possesses the decompositions $E = E_1 \oplus E_2, \ E = F_1 \oplus F_2, \ \dim E_1 = \dim F_1 = d$. Let α denote the angle between E_1 and E_2 and let β denote the angle between F_1 and F_2. Then*

$$\|P_{E_1} - P_{F_1}\| \leq \frac{2d^{\frac{1}{2}}(dist(E_1, F_1) + dist(E_2, F_2))}{(1 - cos\alpha)^{\frac{1}{2}}(1 - cos\beta)^{\frac{1}{2}}} \ .$$

In our case we therefore obtain

$$\begin{aligned}\varrho(\Theta_{-n}(\omega,y)) &\leq c_6[\operatorname{dist}(V^u_{\mu_1}(\Theta_{-n}(\omega,y)),P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x)))\\ &\quad +\operatorname{dist}(V^s_{\mu_1}(\Theta_{-n}(\omega,y)),P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x)))]\\ &\quad M(\omega,x,\epsilon)^{-1}M(\omega,y,\epsilon)^{-1}e^{(2\epsilon n)}\ .\end{aligned}$$

For a further estimation of the distances, we have to invoke the compactness condition on supp Q, for then one has the following lemma which will once again be proved later.

Lemma 4.8 *Suppose that the conditions of Theorem 4.1 are satisfied . Then there exists a set $\tilde\Lambda\subseteq\Lambda,\nu(\tilde\Lambda)=1$, such that for every $(\omega,x)\in\tilde\Lambda$ the following holds: For every $y\in W^u_{\mu_2}(\omega,x)\cap B(x,\beta(\omega,x))$ there exists a function $N(x_n,y_n)$ and a number $b>0$ such that*

$$\begin{aligned}\mathit{dist}(V^u_{\mu_1}(\Theta_{-n}(\omega,y)),P(x_n,y_n)V^u_{\mu_1}(\Theta_{-n}(\omega,x))) &\leq N(x_n,y_n)d(x_n,y_n)^b,\\ \mathit{dist}(V^s_{\mu_1}(\Theta_{-n}(\omega,y)),P(x_n,y_n)V^s_{\mu_1}(\Theta_{-n}(\omega,x))) &\leq N(x_n,y_n)d(x_n,y_n)^b,\end{aligned}$$

and

$$N(x_{n+m},y_{n+m})\leq N(x_n,y_n)e^{(2\epsilon m)}\ .$$

Lemma 4.8 yields

$$\begin{aligned}\varrho(\Theta_{-n}(\omega,y)) &\leq c_7[2N(x_n,y_n)d(x_n,y_n)^b]e^{(2\epsilon n)}\\ &\leq c_7[2N(x,y)e^{(2\epsilon n)}\gamma(\omega,x)^b e^{(-(\mu_2+\epsilon))bn)}]e^{(2\epsilon n)}\\ &\leq c_8 e^{(-(\mu_2 b-(4-b)\epsilon)n)}\end{aligned}$$

so that we finally obtain

$$\operatorname{Lip}_{|\ |_{\Theta_{-n}(\omega,y)}}\mathcal{K}(\Theta_{-n}(\omega,y))\leq[c_9e^{(-(\mu_2b-(5-b)\epsilon)n)}+c_2e^{(-(\mu_2\alpha-(1+\alpha)\epsilon)n)}][1-c_9e^{(-(\mu_2b-(5-b)\epsilon)n)}]^{-1}c_5e^{(2\epsilon n)}, \tag{4.43}$$

and the proof of claim 2 is finished.

Proof of Claim 3: For $\epsilon>0$ sufficiently small, formula (4.43) implies that

$$\lim_{n\to\infty}\operatorname{Lip}_{|\ |_{\Theta_{-n}(\omega,y)}}\mathcal{K}(\Theta_{-n}(\omega,y))=0\ . \tag{4.44}$$

Therefore, we can find a number $M(\omega,x,y)\in\mathbf{N}$ such that

$$\operatorname{Lip}_{|\ |_{\Theta_{-n}(\omega,y)}}\mathcal{K}(\Theta_{-n}(\omega,y))<1\quad\text{for } n\geq M(\omega,x,y)\ .$$

Consequently, if we define the mapping $\mathcal{L}:\ B^u_{\mu_1}\longrightarrow B^s_{\mu_1}$ by

$$(\mathcal{L}S)(\tilde\omega,\tilde x):=\begin{cases}(\tilde\omega,\mathcal{K}(\Theta_{-n}(\omega,y))\sigma(\tilde\omega,\tilde x)) & \text{if } (\tilde\omega,\tilde x)=\Theta_{-n}(\omega,y) \text{ and } n\geq M(\omega,x,y)\\ (\tilde\omega,0) & \text{otherwise ,}\end{cases} \tag{4.45}$$

then $\mathcal{L} \in P$, compare with Section 4.1, and $\Gamma_{\mathcal{F}}\mathcal{L}$ is well-defined. Let us study the graph transform of $\mathcal{L}$ in more detail. First of all, Lemma 4.2 implies that

$$\text{graph}(\Gamma_{\mathcal{F}}\mathcal{L}(\Theta(\tilde{\omega},\tilde{x}))) = F_{(\tilde{\omega},\tilde{x})}(\text{graph } \mathcal{L}(\tilde{\omega},\tilde{x})) \ .$$

Let $\xi \in \text{graph } \mathcal{K}(\Theta_{-n}(\omega,y))$, $|\xi|_{\Theta_{-n}(\omega,y)}$ sufficiently small and $n > M(\omega,x,y)$. Then the definition of $\mathcal{K}$ implies that

$$F_{\Theta_{-n}(\omega,y)}(\xi) = f_{\Theta_{-n}(\omega,y)}(\xi) = f_{\Theta_{-n}(\omega,y)} \circ \text{Exp}_{y_n}^{-1} \circ \text{Exp}_{x_n}(\eta)$$

for some suitable $\eta \in \text{graph } \mathcal{H}(\Theta_{-n}(\omega,x))$. Employing (3.8) yields

$$\begin{aligned} F_{\Theta_{-n}(\omega,y)}(\xi) &= \text{Exp}_{y_{n-1}}^{-1} \circ \varphi(1,\vartheta_{-n}(\omega),\cdot) \circ \text{Exp}_{y_n} \circ \text{Exp}_{y_n}^{-1} \circ \text{Exp}_{x_n}(\eta) \\ &= \text{Exp}_{y_{n-1}}^{-1} \circ \text{Exp}_{x_{n-1}} \circ \text{Exp}_{x_{n-1}}^{-1} \circ \varphi(1,\vartheta_{-n}(\omega),\cdot) \circ \text{Exp}_{x_n}(\eta) \ . \end{aligned}$$

Consequently, since $\text{Exp}_x(\text{graph } \mathcal{H}(\omega,x)) = W^u_{\mu_1}(\omega,x)$ and the family $\{W^u_{\mu_1}(\omega,x) | (\omega,x) \in \Lambda\}$ is invariant with respect to φ, we obtain

$$F_{\Theta_{-n}(\omega,y)}(\xi) = \text{Exp}_{y_{n-1}}^{-1} \circ \text{Exp}_{x_{n-1}}(\zeta)$$

for some $\zeta \in \text{graph } \mathcal{H}(\Theta_{-(n-1)}(\omega,x))$ and therefore

$$F_{\Theta_{-n}(\omega,y)}(\xi) \in \text{graph } \mathcal{K}(\Theta_{-(n-1)}(\omega,y)) \ .$$

Consequently, for $n \geq M(\omega,x,y)$ and sufficiently small neighbourhoods $U_{\Theta_{-n}(\omega,y)} \subset V^u_{\mu_1}(\Theta_{-n}(\omega,y))$ we can deduce that

$$\Gamma_{\mathcal{F}}\mathcal{L}(\Theta_{-n}(\omega,y))|_{U_{\Theta_{-n}(\omega,y)}} = \mathcal{L}(\Theta_{-n}(\omega,y))|_{U_{\Theta_{-n}(\omega,y)}} ,$$

and hence

$$D_0\Gamma_{\mathcal{F}}\mathcal{L}(\Theta_{-n}(\omega,y)) = D_0\mathcal{L}(\Theta_{-n}(\omega,y)) \ .$$

Therefore, employing the definition (4.45) of $\mathcal{L}$, we obtain that the application of the graph transform to $\mathcal{L}(\tilde{\omega},\tilde{x})$ increases its derivative at 0, i.e.,

$$|D_0\Gamma_{\mathcal{F}}\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \geq |D_0\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \ . \tag{4.46}$$

We want to prove that this fact carries over to the global mapping $\mathcal{L}$. Eq. (4.44) implies that for all $\tilde{\epsilon} > 0$ there exists $\tilde{\delta} > 0$ such that

$$|\mathcal{L}(\tilde{\omega},\tilde{x})(\xi) - D_0\mathcal{L}(\tilde{\omega},\tilde{x})\xi|_{(\tilde{\omega},\tilde{x})} \leq \tilde{\epsilon}|\xi|_{(\tilde{\omega},\tilde{x})} \ \text{ for } |\xi|_{(\tilde{\omega},\tilde{x})} < \tilde{\delta}$$

which shows that $\mathcal{L}$ is differentiable at 0 and

$$D_0\mathcal{L}(S)(\tilde{\omega},\tilde{x}) = (\tilde{\omega}, D_0\mathcal{L}(\tilde{\omega},\tilde{x})\sigma(\tilde{\omega},\tilde{x})) \ .$$

The derivative can be estimated by

$$|D_0\mathcal{L}| = \sup_{|S|\leq 1} |D_0\mathcal{L}(S)| \leq \sup_{|S|\leq 1} \sup_{(\tilde{\omega},\tilde{x})} |D_0\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} |\sigma(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \leq \sup_{(\tilde{\omega},\tilde{x})} |D_0\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \ . \tag{4.47}$$

Furthermore, one has

$$|D_0\Gamma_{\mathcal{F}}\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} = |D_0(\pi_{(\tilde{\omega},\tilde{x})}\circ\Gamma_{\mathcal{F}}\mathcal{L}\circ i_{(\tilde{\omega},\tilde{x})})|_{(\tilde{\omega},\tilde{x})} \le |D_0(\Gamma_{\mathcal{F}}\mathcal{L})|, \tag{4.48}$$

where $\pi_{(\tilde{\omega},\tilde{x})}, i_{(\tilde{\omega},\tilde{x})}$ denote the linear operators introduced at the end of the proof of Theorem 3.1. Combining (4.48), (4.46) and (4.47) we obtain

$$|D_0\Gamma_{\mathcal{F}}\mathcal{L}| \ge \sup_{(\tilde{\omega},\tilde{x})} |D_0\Gamma_{\mathcal{F}}\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \ge \sup_{(\tilde{\omega},\tilde{x})} |D_0\mathcal{L}(\tilde{\omega},\tilde{x})|_{(\tilde{\omega},\tilde{x})} \ge |D_0\mathcal{L}|,$$

i.e., the derivative is indeed increasing, which implies by Lemma 4.3 that

$$D_0\mathcal{L} = 0$$

and therefore

$$D_0\mathcal{K}(\Theta_{-n}(\omega,y)) = 0 \quad \text{for } n \ge M(\omega,x,y)\ .$$

We have shown that for n sufficiently large the function $\mathcal{K}(\Theta_{-n}(\omega,y))$ is tangent to $V^u_{\mu_1}(\Theta_{-n}(\omega,y))$. However, an application of Lemma 4.3 yields

$$D_0\Gamma^m_{\mathcal{F}}\mathcal{L}(\Theta_{m-n}(\omega,y)) = 0$$

so that

$$D_0\Gamma^n_{\mathcal{F}}\mathcal{L}(\omega,y) = 0.$$

If we now use Lemma 4.2 and take into account the definition of the functions $\mathcal{K}(\Theta_{-n}(\omega,y))$ (they are nothing else but the original local manifolds considered in another coordinate system), then the invariance of the family $\{W^u_{\mu_1}(\omega,x)|(\omega,x)\in\Lambda\}$ with respect to φ implies that

$$V^u_{\mu_1}(\omega,y) = T_y(\mathrm{Exp}_y\{\text{graph } \Gamma^n_{\mathcal{F}}\mathcal{L}(\omega,y)\cap U_{(\omega,y)}\}) = T_yW^u_{\mu_1}(\omega,x),$$

where $U_{(\omega,y)}$ is a suitable neighbourhood in T_yM. The theorem is proved. □

Remark 4.1

i) A similar result can be shown for the stable manifolds by mimicking the proof of Theorem 4.1 for $\mathcal{F}^{-1}$ instead of $\mathcal{F}$.

ii) As mentioned above, it was shown by Carverhill (1985) that for $\mu < 0$ the stochastic stable manifolds give rise to a foliation. In a weaker sense, this result can be derived by applying a version of Theorem 4.1 for $\mathcal{F}^{-1}$ to the case $\mu_1 = \mu_2 < 0$. Then we obtain that the strongly stable manifolds can be glued together for all points $(\omega,x)\in\tilde{\Lambda}$. However, Carverhill's results can be shown easier by using the dynamical characterization according to Corollary 3.1.

4.3 Proof of the Lemmata 4.6–4.8

Proof of Lemma 4.6

Let us start by showing i). We have to prove that $P_{F_1}|_{\text{graph } g_E}$ is a bijection, for then $g_F := P_{F_2} \circ (P_{F_1}|_{\text{graph } g_E})^{-1}$ does the job. First we show that $P_{F_1}|_{\text{graph } g_E}$ is injective. For two points $\xi, \eta \in E_1$ we obtain

$$\begin{aligned}\|P_{F_1}(\xi, g_E(\xi)) - P_{F_1}(\eta, g_E(\eta))\|_E &\geq \|P_{E_1}((\xi, g_E(\xi)) - (\eta, g_E(\eta)))\|_E \\ &\quad -\|P_{F_1} - P_{E_1}\|_E \|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E \\ &\geq \|\xi - \eta\|_E - c_1(E)c_2(E)\varrho\|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E.\end{aligned}$$

Since

$$\|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E = \max(\|\xi - \eta\|_E, \|g_E(\xi) - g_E(\eta)\|_E) = \|\xi - \eta\|_E$$

it follows that

$$\|P_{F_1}(\xi, g_E(\xi)) - P_{F_1}(\eta, g_E(\eta))\|_E \geq (1 - c_1(E)c_2(E)\varrho)\|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E$$

which shows that $P_{F_1}|_{\text{graph } g_E}$ is indeed injective and

$$\mathrm{Lip}_{\| \|_E}(P_{F_1}|_{\text{graph } g_E})^{-1} \leq (1 - c_1(E)c_2(E)\varrho)^{-1} .$$

It remains to show that $P_{F_1}|_{\text{graph } g_E}$ is onto. We will use the following fact which can be proved easily by using standard arguments.

Suppose that E is a complete metric space, B a Banach space, and $f: E \longrightarrow B$ a bijection whose inverse is Lipschitz. If a mapping $g: E \longrightarrow B$ is Lipschitz-close to f, i.e., $\mathrm{Lip}(f - g) \leq \ell$ sufficiently small, then g is onto.

Obviously, $P_{F_1} \circ P_{E_1}|_{\text{graph}_{g_E}}$ is a bijection whose inverse is Lipschitz, and we have

$$\begin{aligned}&\|P_{F_1} \circ P_{E_1}(\xi, g_E(\xi)) - P_{F_1}(\xi, g_E(\xi)) - (P_{F_1} \circ P_{E_1}(\eta, g_E(\eta)) - P_{F_1}(\eta, g_E(\eta)))\|_E \\ &\quad \leq \|P_{F_1}\|_E \|P_{E_1} - P_{F_1}\|_E \|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E \\ &\quad \leq (1 + \|P_{E_1} - P_{F_1}\|_E)\|P_{E_1} - P_{F_1}\|_E \|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E .\end{aligned}$$

Therefore

$$\mathrm{Lip}_{\| \|_E}(P_{F_1} - P_{F_1} \circ P_{E_1})|_{\text{graph } g_E} \leq (1 + c_1(E)c_2(E)\varrho)c_1(E)c_2(E)\varrho ,$$

and the result follows by the fact stated above.

Now let us attack ii). So far, we have shown that $g_F = P_{F_2} \circ (P_{F_1}|_{\text{graph } g_E})^{-1}$ is a well-defined map. To estimate its Lipschitz constant, we first observe that

$$\begin{aligned}
\mathrm{Lip}_{\|\ \|_E} g_F &\leq \mathrm{Lip}_{\|\ \|_E}((P_{F_2} - P_{E_2}) \circ (P_{F_1}|_{\text{graph } g_E})^{-1}) + \mathrm{Lip}_{\|\ \|_E}(P_{E_2} \circ (P_{F_1}|_{\text{graph } g_E})^{-1}) \\
&\leq c_1(E)c_2(E)\varrho(1 - c_1(E)c_2(E)\varrho)^{-1} + \mathrm{Lip}_{\|\ \|_E}(P_{E_2}|_{\text{graph } g_E}) \cdot (1 - c_1(E)c_2(E)\varrho)^{-1} \ .
\end{aligned}$$

Therefore, since

$$\|P_{E_2}(\xi, g_E(\xi)) - P_{E_2}(\eta, g_E(\eta))\|_E = \|g_E(\xi) - g_E(\eta)\|_E \leq \delta\|\xi - \eta\|_E \leq \delta\|(\xi, g_E(\xi)) - (\eta, g_E(\eta))\|_E$$

we obtain

$$\mathrm{Lip}_{\|\ \|_E} g_F \leq (c_1(E)c_2(E)\varrho + \delta)(1 - c_1(E)c_2(E)\varrho)^{-1} \ .$$

This yields

$$\mathrm{Lip}_{\|\ \|_F} g_F \leq (c_1(E)c_2(E)\varrho + \delta)(1 - c_1(E)c_2(E)\varrho)^{-1} c_1(E)c_2(E)c_1(F)c_2(F) \ ,$$

and the lemma is proved. $\Box$

Proof of Lemma 4.7:

It is sufficient to show

i) $\|(P_{E_1} - P_{F_1})\xi\| \leq \dfrac{\|\xi\|\mathrm{dist}(E_1, F_1)}{(1 - \cos\alpha)^{\frac{1}{2}}}$ for $\xi \in F_1$,

ii) $\|(P_{E_1} - P_{F_1})\xi\| \leq \dfrac{\|\xi\| d^{\frac{1}{2}}\mathrm{dist}(E_2, F_2)}{(1 - \cos\alpha)^{\frac{1}{2}}}$ for $\xi \in F_2$.

This can be seen as follows. Define a new scalar product on E by

$$< \xi, \eta >_F := \langle P_{F_1}\xi, P_{F_1}\eta \rangle + \langle P_{F_2}\xi, P_{F_2}\eta \rangle \tag{4.49}$$

and let $|\cdot|_F$ denote the norm associated with $< \cdot, \cdot >_F$. We will prove below that

$$|\xi|_F \leq \frac{\|\xi\|}{(1 - \cos\beta)^{\frac{1}{2}}} \ . \tag{4.50}$$

Definition (4.49) and Eq. (4.50) imply that

$$\|P_{F_1}\xi\| \leq (1 - \cos\beta)^{-\frac{1}{2}}\|\xi\|, \qquad \|P_{F_2}\xi\| \leq (1 - \cos\beta)^{-\frac{1}{2}}\|\xi\| \ . \tag{4.51}$$

Therefore, by combining i), ii) and (4.51) we obtain

$$\begin{aligned}
\|(P_{E_1} - P_{F_1})\xi\| &= \|(P_{E_1} - P_{F_1})(P_{F_1}(\xi) + P_{F_2}(\xi))\| \\
&\leq \|(P_{E_1} - P_{F_1})(P_{F_1}(\xi))\| + \|(P_{E_1} - P_{F_1})(P_{F_2}(\xi))\| \\
&\leq \frac{\|P_{F_1}\xi\|\mathrm{dist}(E_1, F_1)}{(1 - \cos\alpha)^{\frac{1}{2}}} + \frac{\|P_{F_2}\xi\| d^{\frac{1}{2}}\mathrm{dist}(E_2, F_2)}{(1 - \cos\alpha)^{\frac{1}{2}}} \\
&\leq \frac{\|\xi\|\mathrm{dist}(E_1, F_1)}{(1 - \cos\alpha)^{\frac{1}{2}}(1 - \cos\beta)^{\frac{1}{2}}} + \frac{\|\xi\| d^{\frac{1}{2}}\mathrm{dist}(E_2, F_2)}{(1 - \cos\alpha)^{\frac{1}{2}}(1 - \cos\beta)^{\frac{1}{2}}} \\
&\leq \frac{\|\xi\| 2d^{\frac{1}{2}}(\mathrm{dist}(E_1, F_1) + \mathrm{dist}(E_2, F_2))}{(1 - \cos\beta)^{\frac{1}{2}}(1 - \cos\alpha)^{\frac{1}{2}}} \ .
\end{aligned}$$

Let us carry on by proving i). Let $P_{E_1}^{\perp}$ denote the orthogonal projection onto E_1. Since for $\eta \in E_2$

$$\cos\alpha \geq \frac{|\langle P_{E_1}^{\perp}\eta, \eta\rangle|}{\|P_{E_1}^{\perp}\eta\|\|\eta\|} \geq \frac{\langle P_{E_1}^{\perp}\eta, P_{E_1}^{\perp}\eta\rangle}{\|P_{E_1}^{\perp}\eta\|\|\eta\|} = \frac{\|P_{E_1}^{\perp}\eta\|^2}{\|P_{E_1}^{\perp}\eta\|\|\eta\|}$$

we obtain

$$\|P_{E_1}^{\perp}\eta\| \leq \cos\alpha\|\eta\| \ .$$

Therefore

$$\|\eta\|^2 = \|\eta - P_{E_1}^{\perp}\eta\|^2 + \|P_{E_1}^{\perp}\eta\|^2 \leq \|\eta - P_{E_1}^{\perp}\eta\|^2 + \cos\alpha\|\eta\|^2,$$

so that

$$\|\eta\| \leq \frac{\|\eta - P_{E_1}^{\perp}\eta\|}{(1-\cos\alpha)^{\frac{1}{2}}} \ .$$

Using this expression we obtain for every $\xi \in F_1, \|\xi\| = 1$

$$\begin{aligned}
\|(P_{E_1} - P_{F_1})\xi\| &= \|P_{E_1}\xi - \xi\| \\
&\leq \frac{\|(P_{E_1}\xi - \xi) - P_{E_1}^{\perp}(P_{E_1}\xi - \xi)\|}{(1-\cos\alpha)^{\frac{1}{2}}} \\
&\leq \frac{\|P_{E_1}\xi - \xi - P_{E_1}^{\perp}P_{E_1}\xi + P_{E_1}^{\perp}\xi\|}{(1-\cos\alpha)^{\frac{1}{2}}} \\
&\leq \frac{\|P_{E_1}^{\perp}\xi - \xi\|}{(1-\cos\alpha)^{\frac{1}{2}}} \\
&= (\inf_{\eta\in E_1}\|\xi - \eta\|)(1-\cos\alpha)^{-\frac{1}{2}} \\
&\leq \frac{\mathrm{dist}(E_1, F_1)}{(1-\cos\alpha)^{\frac{1}{2}}} \ ,
\end{aligned}$$

and i) is shown.

Next we attack ii). We may define a scalar product $< \cdot, \cdot >_E$ similar to (4.49) but with E_1, E_2 instead of F_1, F_2. Then the associated norm $|\cdot|_E$ satisfies

$$|\xi|_E \leq \frac{\|\xi\|}{(1-\cos\alpha)^{\frac{1}{2}}} \ . \tag{4.52}$$

Now, let $\{\eta_i\}_{i=1,\ldots d}$ denote an orthonormal basis in E_1 and let $\xi \in F_2, \|\xi\| = 1$. Then we obtain for any arbitrary element $\zeta \in E_2$

$$(P_{E_1} - P_{F_1})(\xi) = P_{E_1}(\xi) = \sum_{i=1}^{d} < \eta_i, \xi >_E \eta_i = \sum_{i=1}^{d} < \eta_i, \zeta - \xi >_E \eta_i \ ,$$

and hence

$$\|(P_{E_1} - P_{F_1})(\xi)\|^2 \leq \sum_{i=1}^{d} | < \eta_i, \zeta - \xi >_E |^2 \leq \sum_{i=1}^{d} |\eta_i|_E^2 |\zeta - \xi|_E^2 \leq d|\zeta - \xi|_E^2 \ .$$

Therefore, by invoking (4.52) we can conclude that

$$\|(P_{E_1}-P_{F_1})(\xi)\| \le d^{\frac{1}{2}}(\inf_{\zeta\in E_2}\|\zeta-\xi\|)(1-\cos\alpha)^{-\frac{1}{2}} \le d^{\frac{1}{2}}(\sup_{\xi\in F_2,\|\xi\|=1}\inf_{\zeta\in E_2}\|\zeta-\xi\|)(1-\cos\alpha)^{-\frac{1}{2}}$$

proving ii).

It remains to show (4.50). To this end, let $\{\xi_i\}_{i=1,\ldots,d}$ be an orthonormal basis of F_1 and $\{\eta_j\}_{j=1,\ldots,m}$ be an orthonormal basis of F_2. Then one has for $\xi = \sum_{i=1}^d a_i\xi_i + \sum_{j=1}^m b_j\eta_j$

$$|\xi|_F^2 =< \sum_{i=1}^d a_i\xi_i + \sum_{j=1}^m b_j\eta_j, \sum_{i=1}^d a_i\xi_i + \sum_{j=1}^m b_j\eta_j >_F = \sum_{i=1}^d a_i^2 + \sum_{j=1}^m b_j^2$$

and therefore

$$\|\xi\|^2 = \sum_{i=1}^d a_i^2 + \sum_{j=1}^m b_j^2 + 2\langle\sum_{i=1}^d a_i\xi_i, \sum_{j=1}^m b_j\eta_j\rangle = |\xi|_F^2 + 2\langle\sum_{i=1}^d a_i\xi_i, \sum_{j=1}^m b_j\eta_j\rangle \ .$$

Since

$$-\langle\sum_{i=1}^d a_i\xi_i, \sum_{j=1}^m b_j\eta_j\rangle \le \cos\beta\|\sum_{i=1}^d a_i\xi_i\|\|\sum_{j=1}^m b_j\eta_j\|$$

this yields

$$\begin{aligned} |\xi|_F^2 &\le \|\xi\|^2 + 2\cos\beta(\sum_{i=1}^d a_i^2)^{\frac{1}{2}}(\sum_{j=1}^m b_j^2)^{\frac{1}{2}} \\ &\le \|\xi\|^2 + \cos\beta(\sum_{i=1}^d a_i^2 + \sum_{j=1}^m b_j^2) \\ &\le \|\xi\|^2 + \cos\beta|\xi|_F^2 \ , \end{aligned}$$

and the lemma is proved. □

Proof of Lemma 4.8:

The proof is based on the following general theorem.

Theorem 4.2 *Let X be a metric space with diam $X \le 1$ and let H be a Hilbert space. Furthermore, let $\{T_i(x)\}, i = 0,1,2,\ldots,\ x\in X$ be sequences of bounded linear operators $T_i(x):\ H \longrightarrow H$. We set*

$$T^m(x) := T_m(x)\circ\ldots\circ T_0(x) \ . \tag{4.53}$$

For some $\varsigma > 1$ and $\tau_1 < \tau_2$ let $\Lambda_{\varsigma,\tau_1,\tau_2} \subset X$ denote the set of points for which there exists a decomposition

$$H = E_1(x)\oplus E_2(x)$$

such that

$$\begin{aligned} \|T^m(x)\xi\| &\le \varsigma e^{\tau_1(m+1)}\|\xi\| \qquad \textit{for } \xi \in E_1(x), & (4.54)\\ \|T^m(x)\xi\| &\ge \varsigma^{-1} e^{\tau_2(m+1)}\|\xi\| \quad \textit{for } \xi \in E_2(x). & (4.55)\end{aligned}$$

If there exists a number $a > 0$ such that

$$\|T^m(x) - T^m(y)\| \le e^{a(m+1)} d(x,y) \tag{4.56}$$

then

$$dist(E_1(x), E_1(y)) \le 3\varsigma^2 e^{(\tau_2-\tau_1)} d(x,y)^{\frac{\tau_1-\tau_2}{\tau_1-a_1}}$$

with $a_1 > \max(\tau_1, a)$.

Under additional orthogonality assumptions, this theorem was proved by Brin and Kifer (1987). However, it can be checked that these additional assumptions are in fact not necessary.

We want to apply Theorem 4.2 to the case

$$\begin{aligned} X &= (x_n, y_n), \qquad\qquad\qquad\qquad H = T_{y_n}M,\\ T_i(y_n) &= I_{i+1}^{-1} \circ T\varphi(-1, \Theta_{-(i+n)}(\omega, y)) \circ I_i,\\ T_i(x_n) &= I_{i+1}^{-1} \circ P(x_{i+1+n}, y_{i+1+n}) \circ T\varphi(-1, \Theta_{-(i+n)}(\omega, x)) \circ P(y_{i+n}, x_{i+n}) \circ I_i,\\ E_1(y_n) &= V^u_{\mu_1}(\Theta_{-n}(\omega, y)), \qquad E_2(y_n) = V^s_{\mu_1}(\Theta_{-n}(\omega, y)),\\ E_1(x_n) &= P(x_n, y_n) V^u_{\mu_1}(\Theta_{-n}(\omega, x)), \qquad E_2(x_n) = P(x_n, y_n) V^s_{\mu_1}(\Theta_{-n}(\omega, x)),\end{aligned}$$

where I_i denotes a linear isometry $T_{y_n}M \to T_{y_{n+i}}M$. First of all, we have to establish (4.54) and (4.55). We will only prove (4.55) for the point y_n in detail, the remaining cases can be treated analogously. Formula (3.23) implies for $\eta \in V^s_{\mu_1}(\Theta_n(\omega, x))$

$$\|\eta\| \le A(\omega, x, \epsilon, \mu_1)\|T_{\varphi(n,\omega,x)}\varphi(n, \omega, \cdot)^{-1}\eta\| e^{\mu_1 n} = A(\omega, x, \epsilon, \mu_1)\|T_{\varphi(n,\omega,x)}\varphi(-n, \vartheta_n(\omega), \cdot)\eta\| e^{\mu_1 n}\ .$$

This yields for $\xi \in V^s_{\mu_1}(\omega, x)$

$$\|\xi\| \le A(\Theta_{-n}(\omega, x), \epsilon, \mu_1)\|T_x\varphi(-n, \omega, \cdot)\xi\| e^{\mu_1 n}$$

and therefore

$$\|T\varphi(-n, \omega, x)\xi\| \ge A(\Theta_{-n}(\omega, x), \epsilon, \mu_1)^{-1} e^{(-\mu_1 n)}\|\xi\| \ge A(\omega, x, \epsilon, \mu_1)^{-1} e^{(-(\mu_1+\epsilon)n)}\|\xi\|\ .$$

However, for ϵ sufficiently small, a similar estimate is valid for $\tilde{\mu}_1 \le \mu_1 - 2\rho,\ \rho \ge \epsilon$, i.e.,

$$\|T\varphi(-n, \omega, x)\xi\| \ge A(\omega, x, \epsilon, \tilde{\mu}_1)^{-1} e^{(-(\tilde{\mu}_1+\epsilon)n)}\|\xi\| \ge A(\omega, x, \epsilon, \tilde{\mu}_1)^{-1} e^{(-(\mu_1-\rho)n)}\|\xi\|.$$

Using this expression and setting

$$\begin{aligned} \varsigma &:= \max(A(\Theta_{-n}(\omega, x), \epsilon, \mu_1), A(\Theta_{-n}(\omega, x), \epsilon, \tilde{\mu}_1), A(\Theta_{-n}(\omega, y), \epsilon, \mu_1), A(\Theta_{-n}(\omega, y), \epsilon, \tilde{\mu}_1)),\\ \tau_1 &:= -\mu_1, \qquad \tau_2 := -(\mu_1 - \rho),\end{aligned}$$

we obtain for $\xi \in V^s_{\mu_1}(\Theta_{-n}(\omega, y))$

$$\begin{aligned}
\|T^m(y_n)\xi\| &= \|I^{-1}_{m+1} \circ T\varphi(-1, \Theta_{-(m+n)}(\omega, y)) \circ I_m \circ I^{-1}_m \circ T\varphi(-1, \Theta_{-(m-1+n)}(\omega, y)) \circ \\
&\quad I_{m-1} \circ \ldots \circ T\varphi(-1, \Theta_{-n}(\omega, y))\xi\| \\
&= \|T\varphi(-1, \Theta_{-(m+n)}(\omega, y)) \circ \ldots \circ T\varphi(-1, \Theta_{-n}(\omega, y))\xi\| \\
&= \|T\varphi(-(m+1), \Theta_{-n}(\omega, y))\xi\| \\
&\geq \varsigma^{-1} e^{\tau_2(m+1)}\|\xi\|.
\end{aligned}$$

It remains to prove (4.56). To this end, we want to use the following estimate which was in a similar form proved by Brin and Kifer (1987).

$$\operatorname{dist}(T\varphi(-m, \Theta_{-n}(\omega, x)), T\varphi(-m, \Theta_{-n}(\omega, y))) \leq e^{am} d(x_n, y_n) \tag{4.57}$$

for some appropriate $a > 0$, where for some diffeomorphism $f \in \operatorname{Diff}^2(M)$

$$\operatorname{dist}(T_x f, T_y f) := \begin{cases} \|T_x f\| + \|T_y f\| & \text{if } \max(d(x,y), d(f(x), f(y))) > R, \\ \|T_x f - P(f(y), f(x)) \circ T_y f \circ P(x,y)\| & \text{else.} \end{cases}$$

Clearly, R denotes the radius of injectivity of M. Observe that this result of Brin and Kifer only holds if supp Q is compact, so that eq. (4.57) is exactly the part of the proof where our compactness assumption is needed. Eq. (4.57) implies

$$\begin{aligned}
\|T^m(y_n) - T^m(x_n)\| &= \|I^{-1}_{m+1} \circ T\varphi(-(m+1), \Theta_{-n}(\omega, y)) - I^{-1}_{m+1} \circ P(x_{m+n+1}, y_{m+n+1}) \\
&\quad \circ T\varphi(-(m+1), \Theta_{-n}(\omega, x)) \circ P(y_n, x_n)\| \\
&\leq \|I^{-1}_{m+1}\| \operatorname{dist}(T\varphi(-(m+1), \Theta_{-n}(\omega, y)), T\varphi(-(m+1), \Theta_{-n}(\omega, x))) \\
&\leq e^{a(m+1)} d(x_n, y_n),
\end{aligned}$$

so that (4.56) is satisfied and the result follows from Theorem 4.2. □

Acknowledgements. This work is part of the author's Ph.-D. thesis which has been achieved in collaboration with H.F. Münzner and L. Arnold. The author takes advantage of this publication to warmly thank H.F. Münzner for numerous inspiring, motivating and encouraging discussions and L. Arnold for drawing his attention to this field of mathematics and for constantly supporting him.

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