The Canonical Coherent States Associated With Quotients of the Affine Weyl-Heisenberg Group*

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Abstract

This paper is concerned with the uncertainty principle in the context of the affine-Weyl-Heisenberg group in one and two dimensions. As the representation of this group fails to be square integrable, we explore various admissible sections of this group, and calculate the minimizers with respect to these sections. Previous studies have shown that these sections give rise to mixed smoothness spaces. We demonstrate that the minimizers obtained for these sections actually interpolate between Gabor and wavelets functions.

Keywords: Affine Weyl-Heisenberg group, quotients, uncertainty principles, minimizing states, coorbit spaces.
AMS Subject Classification: 22D10, 47B25

1 Introduction

Several applications in the fields of signal and image processing involve the convolution of some filter bank with the signal to be processed. This convolution provides local features of the signal or image. Usually, we take some mother function and generate the whole filter bank using group operations, such as translations, rotations, scaling and modulations. Determining the mother function is usually done by some ad-hoc methods that account for the specific application involved. Special attention was given in the past to those functions that provide the maximal accuracy, and hence minimal uncertainty, for the values of the features involved, e.g. translation, scaling, rotation etc.

The classical example consists of the uncertainty relation associated with Short Time Fourier Transform (STFT). The STFT or so-called Gabor transform, see [6], is obtained by applying the action of the Weyl-Heisenberg group to a suitable window function and taking the inner product with the signal. Moreover, it can be shown that choosing the Gaussian function as the window function minimizes the uncertainty relation and therefore gives rise to canonical coherent states of the Weyl-Heisenberg group.

More recent studies considered the uncertainty principles which are related to the affine group in one dimension and the similitude as well as the affine group in two dimensions [1, 3, 8]. For the one dimensional affine group it was possible to find an analytical solution of the form:

$$\psi(x) = c(x-\eta)^{-\frac{1}{2}-i\eta\mu_2+i\mu_1}, \tag{1}$$

where c is some constant, η is purely imaginary and $\mu_1, \mu_2 \in \mathbb{R}$. However, for the two dimensional case, it was not possible to find solutions which simultaneously minimize the combined uncertainty with respect to all the parameters involved, and therefore solutions that accounted for various sub-groups were employed.

In this study we focus on the affine Weyl-Heisenberg group. There is a growing interest in this group as well as in the integral transform associated with it, and several studies have already dealt with it [4, 8, 11, 12, 13, 14]. We regard the studies of Dahlke et. al. [4] where some mixed forms of smoothness spaces that lie in between Besov and modulation spaces are constructed. These mixed smotthness spaces are the α-modulation spaces [4].

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Also, we consider the work of Torresani [12] where interpolating wavelet packets between the Gabor and wavelet transform are generated. Torresani has already shown [13] that the representation of this group fails to be square integrable. A possible remedy is to factor out a suitable closed subgroup and work with quotients. We calculate the minimizers for these subgroups and extend these results to the two dimensional affine Weyl-Heisenberg group.

This paper is organized as follows: First, we provide some background and related work. Next, we calculate the minimizers for the one dimensional affine Weyl-Heisenberg group, and address the issue of using admissible sections. Finally, we move to the two-dimensional affine Weyl-Heisenberg group and explore some possible subgroups for obtaining valid minimizers.

2 Background and Related Work

A general theorem which is well-known in quantum mechanics and harmonic analysis [5] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty relation. Before repeating this well-known result on uncertainties, let us fix some notations. Let A, B be two self-adjoint operators. Their commutator is defined by

$$[A, B] := AB - BA,$$

the expectation of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\mu(A) := \mu_A := \langle A\psi, \psi\rangle$$

and, finally, the variance of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\Delta A_\psi := \mu((A - \mu(A))^2).$$

Theorem 1 *Given two self-adjoint operators A and B, then for all $\psi \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$ they obey the uncertainty relation:*

$$\Delta A_\psi \Delta B_\psi \;\geq \frac{1}{2}|\langle [A, B]\rangle|. \tag{2}$$

A state ψ is said to have minimal uncertainty if the above inequality turns into an equality. This happens iff there exists an $\eta \in i\mathbb{R}$ such that

$$(A - \mu_A)\psi = \eta(B - \mu_B)\psi. \tag{3}$$

The Weyl-Heisenberg and the affine groups are both related to well known transforms in signal processing: the STFT and the wavelet transform respectively. Both can be derived from square integrable representations of these groups. The windowed Fourier transform is related to the Weyl-Heisenberg group and the wavelet transform is related to the affine group. The operation of the group close to the identity element can be described using the infinitesimal generators of the related Lie algebra. If the group representation is unitary then the infinitesimal generators are self-adjoint operators. Thus, the general uncertainty theorem stated above provides a tool for obtaining uncertainty principles using these infinitesimal generators. In the case of the Weyl-Heisenberg group, the canonical functions that minimize the corresponding uncertainty relation are Gaussian functions.

The canonical functions that minimize the uncertainty relations for the affine group in one dimension and for the similitude group in two dimensions, were the subject of previous studies [1, 3]. In these studies, it was shown that there is no non-trivial canonical function which minimizes the uncertainty equation associated with the similitude group of $\mathbb{R}^2$, $SIM(2)$. Thus, there is no non-zero solution for the set of differential equations obtained for these group generators. Rather than using the original generators of the $SIM(2)$ group, Dahlke and Maass [3] used a different set of operators that includes elements of the enveloping algebra, i.e. polynomials in the generators of the algebra, to obtain the $2D$ isotropic Mexican hat as a minimzer. Ali, Antoine and Gazeau [1] noted a symmetry in the set of commutators obtained for the $SIM(2)$ group and derived a possible minimizer in the frequency domain for some fixed direction. Their solution is a real wavelet which is confined to some convex cone in the positive half-plane of the frequency space and is exponentially decreasing inside.

The extension of these studies to the affine group in two dimensions resulted in two possible solutions [8]. The first accounted for the overall scaling and rotation, and then it was possible to use the results of [1, 3]. The second solution utilized a symmetry in the group of commutators which led to

$$\psi(x, y) = (\eta + x)^{-\frac{1}{2} - i\mu_{11} + i\eta\mu_{bx}} e^{i\mu_{b_y} y}. \tag{4}$$

It is square integrable with respect to the variable x if we select: $|\eta_3| \geq \frac{1}{2\mu_{b_x}}$, but not square integrable in terms of the variable y, although it is periodic.

The affine Weyl-Heisenberg (AWH) group has already been addressed in this context in the early $90's$. Torresani [13] considered wavelets associated with representations of the AWH group and has shown that the canonical representation of the AWH group is not square integrable, but can be regularized with some density function. This work was later extended to N-dimensional AWH wavelets [14]. Segman and Schempp [9] introduced ways to incorporate scales in the Heisenberg group with an intertwining operator and presented the resulting signal representations. More recently, Teschke [11] proposed a mechanism to construct generalized uncertainty principles and their minimizing wavelets in anisotropic Sobolev spaces. He derived a new set of uncertainties by weakening the two operator relations and by introducing a multi-dimensional operator setting. Recently, a study by Dahlke and coworkers [4] has considered generalizations of the coorbit space theory based on group representations modulo quotients. They have applied the general theory to the AWH group and obtained families of smoothness spaces that can be identified with the α-modulation spaces.

3 The $1D$ Affine Weyl–Heisenberg Group

The affine Weyl-Heisenberg group is generated by time translation $b \in \mathbb{R}$, frequency translations $\omega \in \mathbb{R}$, spatial dilations $a \in \mathbb{R}_+$, and a toral component $\phi \in \mathbb{R}$, and is equipped with the group law

$$(b, \omega, a, \phi) \circ (b'\omega', a', \phi') = (b + ab', \omega + \omega'/a, aa', \phi + \phi' + \omega ab').$$

The AWH group can be viewed as the extension of the affine group, incorporating frequency translations or, alternatively, as the extension of the Weyl-Heisenberg group incorporating dilations. The Stone-von-Neumann representation of G_{AWH} on $L_2(\mathbb{R})$ is given by:

$$[U(b, \omega, a, \phi)\psi](x) = a^{-\frac{1}{2}} e^{i\omega(x-b)} e^{\phi} \psi(\frac{x - b}{a}). \tag{5}$$

This representation, however, fails to be square integrable [12]. The AWH group raises a special interest as it "contains" both the affine group as well as the Weyl-Heisenberg group. If we consider cases where $a = 1$, we are in the Weyl-Heisenberg framework, and if we consider cases where $\omega = 0$ we are in affine framework. Two independent studies have regarded these attributes, and suggested a specific section of the AWH [4, 12], where the scale is represented as a function of the frequency. It was proven that this section is admissible and a mechanism that allows a smooth transition between the Weyl-Heisenberg and the affine cases has been provided.

In what follows, we regard this section, and calculate the appropriate minimizing functions with respect to the uncertainty principle related to it. Then, we study the section where the frequency is regarded as a function of the scale, consider its admissibility and calculate the appropriate minimizers.

3.1 The Sections Where the Scale is a Function of the Frequency

These kinds of sections appear in a quite natural way in the context of the α–modulation spaces. In the α-modulation spaces framework it is desired to construct mixed forms of smoothness spaces that lie in between Besov spaces (related to the affine group) and modulation spaces (related to the Weyl-Heisenberg group). For that, a group that contains all the components of Besov and modulation spaces is required, such as the affine Weyl-Heisenberg group. As the representation of this group is not square integrable, it is suggested to factor out a closed subgroup and work with the quotients.

The space G_{AWH}/H with

$$H := (0, 0, a, \phi) \in G_{AWH}$$

is considered, with Borel sections that are independent on b, namely:

$$\sigma(b, \omega) = (b, \omega, \beta(\omega), 0).$$

Further, a specific section is proposed, and is proved to be admissible:

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha}.$$

Next, we consider the effect of changing the value of α. If $\alpha = 0$ then we obtain:

$$\beta(\omega) = \eta_0(\omega)^{-1} = (1 + |\omega|)^0 = 1.$$

Thus, there are practically no dilations and we obtain Gabor analysis. For $\alpha \to 1$ we obtain:

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha} \longrightarrow_{\alpha \longrightarrow 1} \frac{1}{1 + |\omega|}.$$

Thus, the frequency translations and modulations are inversely proportional which resembles the wavelets analysis. The intermediate case for which $\alpha = \frac{1}{2}$ is known as the Fourier-Bros-Iagolnitzer transform.

The representation for the quotient as a function of α is then given by

$$[U(b, \omega, \eta_\alpha^{-1}(\omega))\psi](x) = (1 + |\omega|)^{\frac{\alpha}{2}} e^{i\omega(x-b)} \psi\left((1 + |\omega|)^\alpha (x - b)\right).$$

As can be seen, this representation is not C^1 for $\omega = 0$. Nevertheless, when calculating the infinitesimal generators, we may take the one-sided derivatives.

Lemma 2 *The infinitesimal operators T_b, T_ω associated with the one dimensional G_{AWH} are given by*

$$(T_b\psi) = -i\frac{\partial}{\partial x}\psi, \quad \textit{and} \quad (T_\omega\psi) = (i\frac{\alpha}{2} - x)\psi(x) + i\alpha x\frac{\partial}{\partial x}\psi(x). \tag{6}$$

The state ψ which is the minimizer of the associated uncertainty is of the form

$$\psi(x) = e^{\frac{-ix}{\alpha}}(\alpha\lambda x + 1)^{-\frac{1}{2} - \frac{i\mu_\omega}{\alpha} + \frac{i\mu_b}{\alpha\lambda} + \frac{i}{\alpha^2\lambda}}. \tag{7}$$

Proof; Taking the (one–sided) derivatives with respect to ω and b and evaluating them at $b = 0, \omega = 0$ leads us to

$$\frac{\partial}{\partial b}U(b, \omega, \eta_\alpha(\omega))\psi|_{b=0,\omega=0} = -\frac{\partial}{\partial x}\psi, \qquad \frac{\partial}{\partial \omega}U(b, \omega, \eta_\alpha(\omega))\psi|_{b=0,\omega=0} = (\frac{\alpha}{2} + ix)\psi(x) + \alpha x\frac{\partial}{\partial x}\psi(x).$$

These operators are not self–adjoint, but after a multiplication with the imaginary unit i we are in business. This proves(6).

The commutator between these two operators in non–zero. This means that we cannot know exactly the mean values of the spatial frequency and position simultaneously. By means of Theorem 1, we may calculate those states that minimize the corresponding uncertainty principle. Indeed, eq. (3) provides us with the differential equation

$$-i\frac{\partial}{\partial x}\psi(x) - \mu_b\psi(x) = \lambda\left((\frac{i\alpha}{2} - x)\psi(x) + \alpha x\frac{\partial}{\partial x}\psi(x) - \mu_\omega\psi(x)\right),$$

i.e.,

$$\frac{\partial}{\partial x}\psi(x) = i\psi(x)\left(\frac{-\lambda x + \frac{i\lambda\alpha}{2} - \lambda\mu_\omega + \mu_b}{\alpha\lambda x + 1}\right). \tag{8}$$

Now (8) can be solved by separation of variables which leads us to (7). □

In order for this solution to be square integrable, the following should be met: suppose that $\lambda = i\gamma$ where $\gamma \in \mathbb{R}$. Then, if $\gamma < 0$, then $\mu_b > -\frac{1}{\alpha}$. If $\gamma > 0$, then $\mu_b < -\frac{1}{\alpha}$. In figure 1 we may see the appearance of ψ, that is the minimizer for this section of the AWH group in $1D$.

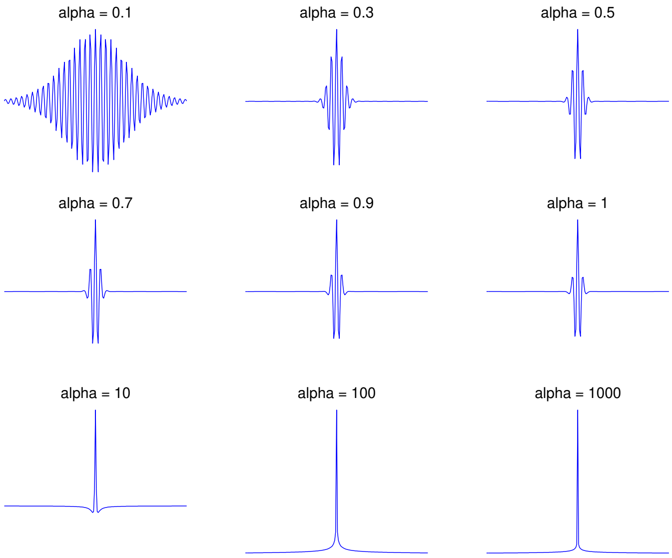

Figure 1: The behavior of the absolute value of the possible minimizing function of the AWH uncertainty, with respect to the parameter α. For this plot, we have selected: $\mu_b = -11, \mu_\omega = 0, \lambda = i$.

3.2 The Sections that Regard the Frequency as a Function of the Scale

If we consider the section

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha},$$

we note that it is not possible to obtain the affine framework using it. Let us explore the relationship $\omega = \zeta_\alpha(a) = a^{-\frac{1}{\alpha}} - 1$ that is explicitly derived as the inverse of $\beta(\omega) = (1 + |\omega|)^{-\alpha}$ for $\omega \geq 0$. Let us denote $\kappa = \frac{1}{\alpha}$, and restrict the discussion to values of κ ranging between 0 and 1 (corresponding to values of α ranging between 1 and ∞). Thus we obtain $\omega = \zeta(a) = a^{-\kappa} - 1$. If κ is selected to be zero, we then obtain no frequency modulation as then $\omega = 0$, and thus we are in the affine case. If κ is selected to be one, we again observe reciprocal relations between scale and frequency of the form: $|\omega| = a^{-1} - 1$, which is the same as $a = \frac{1}{1+|\omega|}$, and corresponds to the case of Gabor–like wavelets.

This concept has again an interpretation in the group theoretical setting. Once more we are working with the affine Weyl–Heisenberg group G_{AWH}, but this time we consider the subgroup

$$H := (0, \omega, 1, \phi) \in G_{AWH}$$

and the associated quotient group $X = G_{AWH}/H$. In order to make this setting well–defined, first of all it is necessary to establish square–integrability, see, e.g., [1] for a detailed discussion. In general, let a quasi-invariant measure μ on X and a section σ be given. Then a unitary representation U of G on a Hilbert space $\mathcal{H}$ is called *square-integrable modulo $(H;\sigma)$* if there exists a function $\psi \in \mathcal{H}$ such that the self-adjoint operator $A_\sigma : \mathcal{H} \to \mathcal{H}$ (dependent on σ and ψ) weakly defined by

$$A_\sigma f := \int_X \langle f, U(\sigma(x))\psi\rangle_{\mathcal{H}} U(\sigma(x))\psi d\mu(x) \tag{9}$$

is bounded and has a bounded inverse. The function ψ is then called *admissible*. If A_σ is a multiple of the identity then we are in the strictly admissible case.

Lemma 3 *For any $\psi \in L_2(\mathbb{R})$, the operator A_σ*

$$A_\sigma f(x) = \int \langle f, \psi_{a,b}\rangle \psi_{a,b} db \frac{da}{a} \tag{10}$$

with

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} e^{2\pi i \zeta(a) x} \psi\left(\frac{x-b}{a}\right), \tag{11}$$

can be written as a Fourier multiplier operator, i.e,

$$\widehat{A_\sigma f} = m_\zeta \hat{f} \tag{12}$$

with the symbol

$$m_\zeta(\xi) := \int_{\mathbb{R}} |\hat{\psi}(a(\xi - \zeta(a)))|^2 da. \tag{13}$$

Proof. For the sake of this calculation we use the approximation

$$A_\sigma^T f(x) = \int\int \langle f, \psi_{a,b}\rangle \psi_{a,b} \chi_{[-\frac{T}{2},\frac{T}{2}]}(b) db \frac{da}{a}. \tag{14}$$

In order to compute $\widehat{A_\sigma^T f(\gamma)}$, we first derive the Fourier transform of $\psi_{a,b}$.

$$\hat{\psi}_{a,b} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{2\pi i \sigma(a) x} e^{-2\pi i \omega x} \psi\left(\frac{x-b}{a}\right) dx. \tag{15}$$

If we apply the change of variables $y = \frac{x-b}{a}$ we obtain

$$\begin{aligned} \hat{\psi}_{a,b} &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{2\pi i \sigma(a)(ay+b)} e^{-2\pi i \omega(ay+b)} \psi(y) a dy \\ &= \sqrt{a} e^{2\pi i b(\sigma(a)-\omega)} \int_{-\infty}^{\infty} e^{-2\pi i y a(\omega - \sigma(a))} \psi(y) dy \qquad (16) \\ &= \sqrt{a} e^{2\pi i b(\sigma(a)-\omega)} \hat{\psi}(a(\omega - \sigma(a))). \qquad (17) \end{aligned}$$

Equipped with this, let us compute $\widehat{A_\sigma^T f(\gamma)}$. With the help of Plancherel's theorem we obtain

$$\langle f, \psi_{a,b}\rangle = \langle \hat{f}, \hat{\psi}_{a,b}\rangle = \int \sqrt{a} \hat{f}(\eta) e^{-2\pi i b(\zeta(a)-\eta)} \bar{\hat{\psi}}(a(\eta - \zeta(a))) d\eta \tag{18}$$

and thus

$$\begin{aligned} \widehat{A_\sigma^T f(\gamma)} &= \int\int \left[\sqrt{a} \hat{f}(\eta) e^{-2\pi i b(\zeta(a)-\eta)} \bar{\hat{\psi}}(a(\eta - \zeta(a))) d\eta\right] \sqrt{a} e^{2\pi i b(\zeta(a)-\gamma)} \hat{\psi}(a(\gamma - \zeta(a))) \chi_{[-\frac{T}{2},\frac{T}{2}](b)} \frac{da}{a} db \\ &= \int\int \hat{f}(\eta) \bar{\hat{\psi}}(a(\eta - \zeta(a))) \hat{\psi}(a(\gamma - \zeta(a))) \int e^{-2\pi i b(\gamma - \eta)} \chi_{[-\frac{T}{2},\frac{T}{2}](b)} db da d\eta \\ &= \int\int \hat{f}(\eta) \bar{\hat{\psi}}(a(\eta - \zeta(a))) \hat{\psi}(a(\gamma - \zeta(a))) T \frac{\sin(\pi(\gamma-\eta)T)}{\pi(\gamma-\eta)T} da d\eta \\ &= \int \hat{\psi}(a(\gamma - \zeta(a))) \int \hat{f}(\eta) \bar{\hat{\psi}}(a(\eta - \zeta(a))) T \frac{\sin(\pi(\gamma-\eta)T)}{\pi(\gamma-\eta)T} d\eta da. \end{aligned}$$

The term

$$\int f(\eta) T \frac{\sin(\pi(\gamma-\eta)T)}{\pi(\gamma-\eta)T} d\eta \tag{19}$$

can be seen as an approximation of a δ-function when T approaches infinity which yields $f(\gamma)$. Thus, we obtain

$$\widehat{A_\sigma(f)}(\gamma) = \hat{f}(\gamma) \int |\hat{\psi}(a(\gamma - \zeta(a)))|^2 da = \hat{f}(\gamma) m_\zeta(\gamma). \tag{20}$$

$\square$

We then can determine whether A_σ is bounded with a bounded inverse as this is true if and only if

$$C_1 \leq m_\zeta(\xi) \leq C_2 \tag{21}$$

holds true almost everywhere for constants $0 \leq C_1, C_2 \leq \infty$. If we check the admissibility of the section for the case $\alpha = 1$, i. e. $\big(\zeta_1(a) = \frac{1}{a} - 1.\big)$ Then

$$\begin{aligned} m_\zeta(\xi) &= \int_0^\infty |\hat{\psi}(a(\xi - \frac{1}{a} + 1))|^2 da \\ &= \int_0^\infty |\hat{\psi}(a(\xi + 1) - 1))|^2 da \\ &= \frac{1}{|\xi + 1|} \int_0^\infty |\hat{\psi}(x)|^2 dx \\ &\overset{|\xi| \to \infty}{\longrightarrow} 0. \end{aligned}$$

Thus, this shows, that ζ_1 is not admissible even for this easy case. More complicated calculations may show the same result for ζ_α for $0 < \alpha < 1$.

Nevertheless, a possible remedy for the non-admissibility of this section can be found in the quasi-coherent states framework [2]. In this case, although $m_\zeta(\xi)$ calculated here diverges, the corresponding integral with respect to a positive density $\iota(a, \xi) = \frac{1}{a}$ converges. This leads to quasi-coherent states that have the standard properties of the covariant coherent states: overcompleteness, resolution of a positive operator A_σ and having a reproducible kernel.

Under the choice of this section the representation is then given by

$$[U(b, a, \zeta_\alpha(a))\psi](x) = \frac{1}{\sqrt{a}} e^{i(a^{-\kappa} - 1)(x - b)} \psi\left(\frac{x - b}{a}\right).$$

Next, we calculate the two infinitesimal generators related to this representation.

Lemma 4 *The infinitesimal operators with respect to the representation above are*

$$T_a = (\kappa x - \frac{i}{2})\psi - ix\psi_x \quad \text{and} \quad T_b = -i\psi_x.$$

A minimizing state is then given by

$$\psi = (1 - \rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{\kappa}{\rho}} e^{-i\kappa x}. \tag{22}$$

In order for this solution to be square integrable, the following should be met: suppose that $\rho = i\gamma$ where $\gamma \in \mathbb{R}$. Then, if $\gamma < 0$, then $\mu_b < -\kappa$. If $\gamma > 0$, then $\mu_b > -\kappa$.

Proof: The proof can be performed by following the lines of the proof of Lemma 2. This time, the corresponding differential equation is given by

$$\psi_x = \frac{\left(\mu_b - \frac{1}{2} + \rho\kappa x - \rho\mu_a\right)}{i(\rho x - 1)} \psi, \tag{23}$$

where ρ is purely imaginary. Eq. (23) can again be solved by separation of variables which leads to (22). $\square$

In figure 2 we may see the appearance of ψ, that is the minimizer of this selection of section for the AWH group in $1D$.

It is interesting to compare this solution to the one obtained for the selection: $a = \beta(\omega)$ for $\alpha = 1$. At $\kappa = 1$ the current solution reduces to

$$\psi = (1 - \rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{1}{\rho}} e^{-ix}, \tag{24}$$

where the solution obtained in the previous section is, for $\alpha = 1$

$$\psi = (1 + \lambda x)^{i\frac{\mu_a}{\lambda} - \frac{1}{2} - i\mu_b + \frac{i}{\lambda}} e^{-ix} \tag{25}$$

The constraints for the two solutions to agree for $\alpha = 1$ are derived as follows

$$\lambda = -\alpha\rho, \quad \mu_a = 0, \quad \mu_\omega = 0.$$

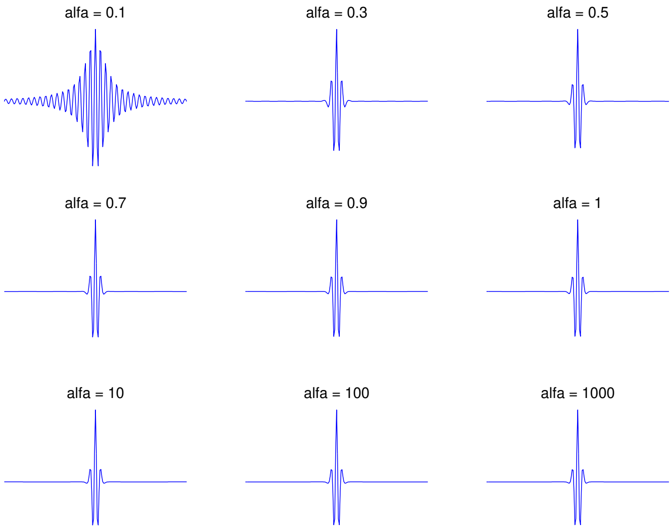

Figure 2: The behavior of the absolute value of the possible minimizing function of the AWH uncertainty, with respect to the parameter α. For this plot, we have selected: $\mu_b = -11, \mu_\omega = 0, \lambda = -i$.

4 The $2D$ Affine Weyl-Heisenberg Group

In this part we are interested in finding uncertainty minimizers when the two-dimensional affine Weyl-Heisenberg group with a generating element $(A, \omega, b, \phi), \omega, b \in \mathbb{R}^2, A \in Gl(2, \mathbb{R}), \phi \in \mathbb{R}$ and group law

$$(b, \omega, A, , \phi) \circ (b', \omega', A', \phi') = (b + Ab', \omega + A^{-1}\omega', AA', \phi + \phi' + \omega^T Ab').$$

The general representation of the AWH group in two-dimensions is given by:

$$[U(b, \omega, A, \phi)\psi](x, y) = \frac{1}{|\det(A)|} e^{i(\omega_x x + \omega_y y + \phi)} \psi\left(A^{-1}(x - b_x, y - b_y)\right) \tag{26}$$

with the unimodular Haar measure

$$\frac{1}{|\det(A)|^2} db d\omega dm(A) d\phi,$$

where $dm(A)$ denotes usual measure when parametrizing the matrix A. In our discussion we explore various sub-groups of the full $2D$ AWH group, starting from the $SIM(2)$ group, and moving back to the full group structure.

4.1 The $2D$ Similitude Weyl-Heisenberg Subgroup

Rather then the full $2D$ affine group, we start our discussion by considering the Similitude group. The general representation of the $2D$ Similitude Weyl-Heisenberg subgroup is given by

$$[U(b, \omega, a, \theta, 0)\psi](x, y) = \frac{1}{a} e^{i(\omega_x x + \omega_y y)} \psi\left(\tau_\theta\left(\frac{x - b_x}{a}, \frac{y - b_y}{a}\right)\right), \tag{27}$$

where $\tau_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. This representation fails to be square integrable [14], therefore, we are faced with the interesting question of selecting an appropriate section. To this end, let $H = (0, 0, a, 0, \phi)$ and consider $G_{AWH} \setminus H$ and take the section

$$\sigma(\theta, \omega, b) = (b, \omega, \Phi(\omega), \theta, 0).$$

We consider a coupling between some norm of the frequency ω and the scaling a by $a = \Phi(\omega)$. More specifically, in the spirit of the α-modulation spaces framework, we assume that the function Φ depends only on the p-norm of the frequency vector ω

$$\Phi(\omega) = g(\|\omega\|_p) = \frac{1}{(1+\|\omega\|_p)^\alpha}. \tag{28}$$

As is usual in our work, we would like to obtain the infinitesimal generators of this group by calculating the appropriate derivatives of the representation of this group at the identity element. Once this section is selected, the operator with respect to the frequency ω is given in terms of the infinitesimal operators of the representation where no specific section is selected

$$\tilde{T}_{\omega_k} = (T_{\omega_k} + \Phi_{\omega_k}(0)T_a), \tag{29}$$

for $k \in \{x, y\}$. Therefore, we have to estimate the derivatives of Φ at $\omega = 0$. Then, we have

$$\begin{aligned}
\Phi_{\omega_k} &= -\alpha(1+\|\omega\|_p)^{-\alpha-1}\frac{1}{p}\|\omega\|_p^{1-p}p\omega_k^{p-1}\mathrm{sign}(\omega_k) \\
&= \frac{-\alpha\|\omega\|_p^{1-p}\omega_k^{p-1}}{(1+\|\omega\|_p)^{\alpha+1}}\mathrm{sign}(\omega_k) \\
&= \frac{-\alpha}{(1+\|\omega\|_p)^{\alpha+1}}\left[\frac{\omega_k^p}{\|\omega\|_p^p}\right]^{\frac{p-1}{p}}\mathrm{sign}(\omega_k) \\
&= \frac{-\alpha}{(1+\|\omega\|_p)^{\alpha+1}}\left[\frac{\omega_k^p}{\sum_j |\omega_j|^p}\right]^{\frac{p-1}{p}}\mathrm{sign}(\omega_k).
\end{aligned} \tag{30}$$

Next, we have to evaluate this expression at $\omega_k = 0$ for all k. In two or more dimensions we do not have a single value for the $\frac{0}{0}$ situation we are faced with. This provides the motivation for selecting L_1-norm, as for that case, the infinitesimal generators can be calculated with no difficulty.

4.1.1 AWH Minimizers Using the L_1–Norm

In this case: $a = \Phi(\omega) = \frac{1}{(1+|\omega_x|+|\omega_y|)^\alpha}$, thus the representation becomes:

$$[U(b,\omega,,\Phi(\omega),\theta,0)\psi](x,y) = (1+|\omega_x|+\omega_y|)^\alpha e^{i(x\omega_x+y\omega_y)}\psi\left((1+|\omega_x|+\omega_y|)^\alpha\tau_\theta(x-b_x, y-b_y)\right). \tag{31}$$

The self-adjoint operators are given by:

$$\begin{aligned}
T_{\omega_x}\psi(x,y) &= (i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)), \\
T_{\omega_y}\psi(x,y) &= (i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)), \\
T_{b_x}\psi(x,y) &= -i\psi_x(x,y), \\
T_{b_y}\psi(x,y) &= -i\psi_y(x,y), \\
T_\theta\psi(x,y) &= i(y\psi_x(x,y) - x\psi_y(x,y)).
\end{aligned} \tag{32}$$

Out of the ten commutation relations, three vanish ($[T_{\omega_x}, T_{b_y}] = 0$, $[T_{\omega_y}, T_{b_x}] = 0$, $[T_{b_x}, T_{b_y}] = 0$) and we are left with seven partial differential equations.

1. $$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi = \lambda_1\left((i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi\right)$$

2. $$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi = \lambda_2\left(-i\psi_x(x,y) - \mu_{b_x}\psi\right)$$

3. $$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi = \lambda_3\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi\right)$$

4.
$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi = \lambda_4\left(-i\psi_y(x,y) - \mu_{b_y}\psi\right)$$

5.
$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi = \lambda_5\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi\right)$$

6.
$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi = \lambda_6\left(-i\psi_x(x,y) - \mu_{b_x}\psi\right)$$

7.
$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi = \lambda_6\left(-i\psi_y(x,y) - \mu_{b_y}\psi\right)$$

The only simultaneous solution to these equations is the trivial one ($\psi = 0$ everywhere). Therefore, we aim at finding a solution which involves operators from the enveloping algebra and is rotation invariant.

Suppose that the minimizer is of the form: $g(r)$ where $r = \sqrt{x^2+y^2}$. Then, we consider the following infinitesimal operators with respect to $g(r)$:

$$\begin{aligned} T_\theta(g(r)) &= 0, \\ T_b(g(r)) &= (T_{b_x}^2 + T_{b_y}^2)g(r) = -\frac{d^2g(r)}{dr^2} - \frac{1}{r}\frac{dg}{dr}. \end{aligned}$$

Moreover, the operators $T_{\omega_x}, T_{\omega_y}$ are commutating with respect to $g(r)$ (i.e., $[T_{\omega_x}, T_{\omega_y}](g(r)) = 0$. These observations lead to two possible solutions: the first involves defining a new operator: $T_\omega = T_{\omega_x}T_{\omega_y} - T_{\omega_y}T_{\omega_x}$ and consider it along with T_θ and T_b. Then, any function $g(r)$ that is rotation invariant is a valid minimizer of the uncertainties related to these operators. Another option is to consider T_{ω_x} and T_{ω_y} with respect to $g(r)$. The commutators of these operators with T_b is not equal to zero, and we obtain the differential equation

$$\psi_{rr} + \frac{1}{r}\psi_r + \mu_b\psi = 0 \tag{33}$$

whose solution is given by Bessel functions of the first and second kind:

$$\psi(r) = c_1 besselj(0, \sqrt{\mu_b}r) + c_2 bessely(0, \sqrt{\mu_b}r). \tag{34}$$

Nevertheless, this solution is not square integrable.

Another interesting effort is to find a solution for a single equation and thus obtain a selective minimal uncertainty with respect to two operators only. For example, let us consider equation 2. only:

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi = \lambda_2\left(-i\psi_x(x,y) - \mu_{b_x}\psi\right)$$

and select $\lambda_2 = 0$ to obtain:

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi = 0.$$

A possible solution is given by the expression:

$$\psi(x,y) = y^{-i\frac{\mu_{\omega_x}}{\alpha} - 1} e^{-i\frac{x}{\alpha}} \tau\left(\frac{x}{y}\right), \tag{35}$$

where τ is an arbitrary function of the variable $\frac{x}{y}$. Such a possible function is depicted in figure 3.

4.1.2 AWH Minimizers Using the L_2–Norm

As we have seen in the previous section, the selection:

$$a = \Phi(\omega) = \left(1 + \sqrt{\omega_x^2 + \omega_y^2}\right)^{-\alpha}$$

proves to be futile. It is interesting to explore another solution to the problem mentioned in 4.1, where the relationship between the scale and frequency is given by $a = \left(1 + \omega_x^2 + \omega_y^2\right)^{-\alpha}$. In this case, the problem of calculating the infinitesimal generators is avoided.

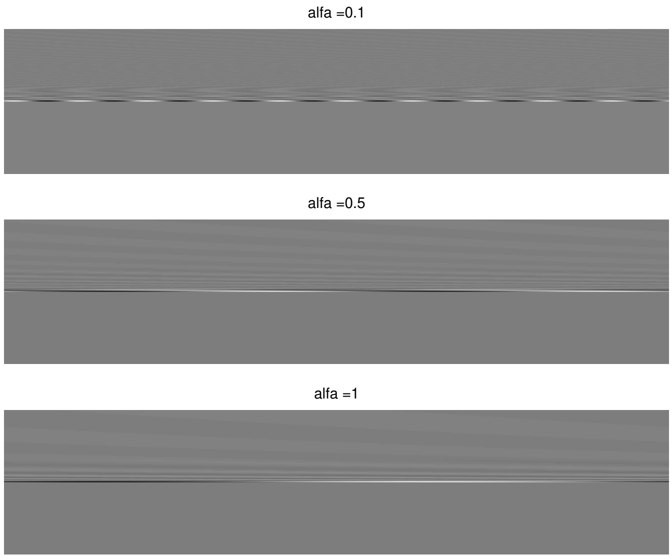

Figure 3: A possible minimizing function, where we select $\tau\begin{pmatrix} \frac{x}{y} \end{pmatrix} = 1$.

The unitary representation induced by this section of the similitude Weyl-Heisenberg group is then given by

$$[U(b, \omega, (\Phi(\omega), \theta, 0)\psi](x, y) = (1 + \omega_x^2 + \omega_y^2)^\alpha e^{i(x\omega_x + y\omega_y)}\psi\left((1 + \omega_x^2 + \omega_y^2)^\alpha \tau_\theta(x - b_x, y - b_y)\right), \tag{36}$$

and τ_θ is the same as already defined. The infinitesimal generators are then given by:

$$\begin{aligned} T_{\omega_x} &= -x\psi, \\ T_{\omega_y} &= -y\psi, \\ T_{b_x} &= -i\psi_x, \\ T_{b_y} &= -i\psi_y, \\ T_\theta &= i(y\psi_x - x\psi_y). \end{aligned} \tag{37}$$

It is interesting to note that the dependency on the parameter α has disappeared. This means that selecting this type of norm may provide a solution regardless of the smoothness space we are in.

The equations resulting from the non-commutating operators are:

$$\begin{aligned} -x\psi - \mu_{\omega_x}\psi &= \lambda_1(-i\psi_x - \mu_{b_x}\psi), \\ -x\psi - \mu_{\omega_x}\psi &= \lambda_2(i(y\psi_x - x\psi_y) - \mu_\theta\psi), \\ -y\psi - \mu_{\omega_y}\psi &= \lambda_3(-i\psi_y - \mu_{b_y}\psi), \\ -y\psi - \mu_{\omega_y}\psi &= \lambda_4(i(y\psi_x - x\psi_y) - \mu_\theta\psi), \\ -i\psi_x - \mu_{b_x}\psi &= \lambda_5(i(y\psi_x - x\psi_y) - \mu_\theta\psi), \\ -i\psi_y - \mu_{b_y}\psi &= \lambda_6(i(y\psi_x - x\psi_y) - \mu_\theta\psi). \end{aligned} \tag{38}$$

If we look for a solution which has a rotation invariance property, thus: $\psi(x, y) = g(r)$ we may satisfy all the equations that involve the operator T_θ. Moreover, applying restrictions to our parameters, e.g.: $\lambda_1 = \lambda_3 = \lambda, \mu_{b_x} = \mu_{b_y}, \omega_x = \omega_y$ we may obtain a rotationally invariant solution to the first and thirdequations as well, which is the Gaussian

$$\psi = e^{-\frac{i}{\lambda}\left(\frac{x^2+y^2}{2}\right)}. \tag{39}$$

4.2 AWH Minimizers When the Scale is Treated as a Vector

In the previous treatment of the two-dimensional case, we regard the frequency as a vector, but treat the scale as a scalar argument. Moreover, we use the $SIM(2)$ group rather than the full affine group. We are interested to add more degrees of freedom to our setting, and as a first step observe a relationship where the scale is a also a two dimensional vector and not a scalar

$$a_x = \beta_1(\omega_x), \qquad a_y = \beta_2(\omega_y).$$

As a first step, we explore a generalization to two dimensions of the affine Weyl-Heisenberg group in one dimension, and ignore for now rotation and shear. We thus consider the group with a generic element $g = (b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi,)$ where $b_x, b_y, \omega_x, \omega_y, \phi \in \mathbb{R}$ and $a_x, a_y \in \mathbb{R}_+$ equipped with a group law

$$(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi) \circ (b'_x, b'_y, \omega'_x, \omega'_y, a'_x, a'_y, \phi') =$$

$$(b_x + a_x b'_x, b_y + a_y b'_y, \omega_x + a_x^{-1}\omega'_x, \omega_y + a_y^{-1}\omega'_y, a_x a'_x, , \phi + \phi' + \omega_x a_x b'_x + \omega_y a_y b'_y).$$

This is a subgroup of the $2D$ AWH group. The inverse element of $g \in G$ is given by

$$g^{-1} = (-a_x^{-1}b_x, -a_y^{-1}b_y, -a_x\omega_x, -a_y\omega_y, a_x^{-1}, a_y^{-1}, -\phi + b_x\omega_x + b_y\omega_y) \tag{40}$$

Let us look at the following representation

$$[U(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi)\psi](x, y) = \frac{1}{\sqrt{a_x a_y}} e^{i((x-b_x)\omega_x + (y-b_y)\omega_y)+\phi} \psi\left(\frac{x - b_x}{a_x}, \frac{y - b_y}{a_y}\right) \tag{41}$$

which is the $2D$ extension of the Stone-von-Neumann representation of the $1D$ AWH group. This representation fails to be square integrable, and therefore we restrict ourselves to the homogeneous space G_{AWH}/H with

$$H := (0, 0, 0, 0, a_x, a_y, \phi) \in G_{AWH}. \tag{42}$$

Next, we consider the section $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0)$, and would like to prove that this section is admissible.

We define a self-adjoint operator $A_\sigma f$

$$\begin{aligned} A_\sigma f &:= \int_X \langle f, U(\sigma(x))\psi\rangle U(\sigma(x))\psi d\mu(x) \\ &= \int\int\int\int \langle f, \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y}\rangle \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y} db_x db_y d\omega_x d\omega_y. \end{aligned} \tag{43}$$

It can be written as a Fourier multiplier operator

$$(A_\sigma f)\hat{} = m_{\beta_x,\beta_y}\hat{f} \tag{44}$$

where

$$m_{\beta_x,\beta_y}(\xi_x, \xi_y) = \int\int |\hat{\psi}(\beta_x(u_x - \omega_x), \beta_y(u_y - \omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y. \tag{45}$$

Next, we follow the proofs of Dahlke et. al. [4]to show that m_{β_x,β_y} is bounded from above and below, i.e.

$$C_1 \leq m_{\beta_x,\beta_y} \leq C_2 \tag{46}$$

for constants $0 < C_1 < C_2 < \infty$. We start with the following lemma which is a straight forward generalization of Lemma 5.1 in [4] to the $2D$–case. Therefore we omit the details.

Lemma 5 *Consider the specific section σ given by the functions*

$$\begin{aligned} \beta_x(\omega_x) = \beta_{x,\alpha_x}(\omega_x) = (1 + |\omega_x|)^{-\alpha_x} \\ \beta_y(\omega_x) = \beta_{y,\alpha_y}(\omega_y) = (1 + |\omega_y|)^{-\alpha_y} \end{aligned}$$

Let us define

$$\begin{aligned} r_\xi(\omega_x) := \beta_x(\omega_x)(\xi - \omega_x) = (1 + |\omega_x|)^{-\alpha_x}(\xi - \omega_x) \\ r_\nu(\omega_y) := \beta_y(\omega_y)(\nu - \omega_y) = (1 + |\omega_y|)^{-\alpha_y}(\nu - \omega_y) \end{aligned}$$

Then, for any fixed $A>0$, there exist $\xi_A, \nu_A > 0$ such that for all $\xi \geq \xi_A, \nu \geq \nu_A$ the functions r_ξ, r_ν are invertible on

$$A_{\omega_x} = \{\omega_x : r_\xi(\omega_x) \in [-A, A]\} \quad \textit{and} \quad A_{\omega_y} = \{\omega_y : r_\nu(\omega_y) \in [-A, A]\}$$

respectively. The inverse functions r_ξ^{-1}, r_ν^{-1} of r_ξ, r_ν on $[-A, A]$ have the form

$$r_\xi^{-1} = -x_1 g_1(\xi, x_1) + \xi, \qquad r_\nu^{-1} = -x_2 g_2(\nu, x_2) + \nu,$$

with some functions $g_1(x_1,\xi), g_2(x_2,\nu)$ satisfying

$$x_1 g_1(\xi, x_1) + g_1(\xi, x_1)^{\frac{1}{\alpha_x}} = 1 + \xi \quad \textit{and} \quad x_2 g_2(\nu, x_2) + g_2(\nu, x_2)^{\frac{1}{\alpha_y}} = 1 + \nu.$$

Furthermore, g_1, g_2 fulfills

$$\lim_{\xi\to\infty} \xi^{-\alpha_x} g_1(\xi, x_1) = 1, \qquad \lim_{\nu\to\infty} \nu^{-\alpha_y} g_2(\nu, x_2) = 1$$

uniformly for $x_1, x_2 \in [-A, A]$.

Theorem 6 *Let the Borel section σ be given by $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0, 0)$ with $\beta_x(\omega_x) = (1+|\omega_x|)^{-\alpha_x}, \beta_y(\omega_y) = (1+|\omega_y|)^{-\alpha_y}$. Let ψ be a non zero L_2 function whose Fourier transform is compactly supported. Then, ψ is admissible, i.e., the condition*

$$C_1 \leq m_{\beta_x,\beta_y}(\xi, \nu) \leq C_2$$

is satisfied for $0 < C_1 \leq C_2 < \infty$.

Proof. The proof can be performed by following the lines of the proof of Theorem 5.2 in [4]. For reader's convenience, we briefly sketch the arguments. We consider the case where either ξ or ν tend to $+\infty$. Let us assume that $\mathrm{supp}(\hat{\psi}) \subset [-A, A] \times [-A, A]$. We substitute $x_1 = r_\xi(\omega_x), x_2 = r_\nu(\omega_y)$ for $\xi \geq \xi_A > 0, \nu \geq \nu_A > 0$ in the expression for $m_{\beta_x,\beta_y}(\xi,\nu)$ to obtain

$$\begin{aligned} m_{\beta_x,\beta_y}(\xi,\nu) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{\psi}(r_\xi(\omega_x), r_\nu(\omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\psi(x_1\hat{,} x_2)|^2 \beta_x(r_\xi^{-1}(x_1))\beta_y(r_\nu^{-1}(x_2))(r_\xi^{-1})'(r_\nu^{-1})' dx_1 dx_2. \end{aligned} \tag{47}$$

Next, we calculate the values of the derivatives of the inverse functions r_ξ^{-1}, r_ν^{-1} using

$$r'_\xi(\omega_x) = \beta'_x(\omega_x)(\xi - \omega_x) - \beta_x(\omega_x) = -\beta_x(\omega_x)\left(\alpha_x \frac{\xi - \omega_x}{1+\omega_x} + 1\right),$$
$$r'_\nu(\omega_y) = \beta'_y(\omega_y)(\nu - \omega_y) - \beta_y(\omega_y) = -\beta_y(\omega_y)\left(\alpha_y \frac{\nu - \omega_y}{1+\omega_y} + 1\right)$$

to obtain

$$(r_\xi^{-1})'(x_1) = \frac{1}{r'_\xi(r_\xi^{-1}(x_1))} = -\frac{1}{\beta_x(r_\xi^{-1}(x_1)\left(1 + \alpha_x \frac{\xi - r_\xi^{-1}(x_1)}{1 + r_\xi^{-1}(x_1)}\right)},$$
$$(r_\nu^{-1})'(x_2) = \frac{1}{r'_\nu(r_\nu^{-1}(x_2))} = -\frac{1}{\beta_y(r_\nu^{-1}(x_2)\left(1 + \alpha_y \frac{\nu - r_\nu^{-1}(x_2)}{1 + r_\nu^{-1}(x_2)}\right)}.$$

Thus, for values of $\xi \geq \xi_A > 0, \nu \geq \nu_A > 0$ we have

$$m_{\beta_x,\beta_y}(\xi,\nu) = \int_{-A}^{A}\int_{-A}^{A} |\hat{\psi}(x_1, x_2)|^2 G(\xi, \nu, x_1, x_2) dx_1 dx_2 \tag{48}$$

where

$$G(\xi,\nu,x_1,x_2) = \frac{1}{\left(1+\alpha_x\frac{\xi-r_\xi^{-1}(x_1)}{1+r_\xi^{-1}(x_1)}\right)}\frac{1}{\left(1+\alpha_y\frac{\nu-r_\nu^{-1}(x_2)}{1+r_\nu^{-1}(x_2)}\right)} =$$

$$\frac{1}{1+\alpha_x x_1 g_1(\xi,x_1)^{1-\frac{1}{\alpha_x}}}\frac{1}{1+\alpha_y x_2 g_2(\nu,x_2)^{1-\frac{1}{\alpha_y}}},$$

where we have used the definitions in the previous lemma. According to this lemma, we may substitute ξ^{α_x} for $g_1(\xi,x_1)$ when ξ goes to infinity, and the same for $g_2(\nu,x_2)$ when $\nu\to\infty$

$$\begin{aligned}
\lim_{\xi\to\infty,\nu\to\infty} G(\xi,\nu,x_1,x_2) &= \lim_{\xi\to\infty,\nu\to\infty}\frac{1}{1+\alpha_x x_1 g_1(\xi,x_1)^{1-\frac{1}{\alpha_x}}}\frac{1}{1+\alpha_y x_2 g_2(\nu,x_2)^{1-\frac{1}{\alpha_y}}}\\
&= \lim_{\xi\to\infty,\nu\to\infty}\frac{1}{1+\alpha_x x_1 \xi^{\alpha_x(1-\frac{1}{\alpha_x})}}\frac{1}{1+\alpha_x x_2 \nu^{\alpha_y(1-\frac{1}{\alpha_y})}}\\
&= 1,
\end{aligned}$$

and therefore we finally have

$$\lim_{\xi\to\infty,\nu\to\infty} m_{\beta_x,\beta_y}(\xi,\nu) = \int_{-A}^{A}\int_{-A}^{A}|\hat{\psi}(x)|^2 dx_1 dx_2 \tag{49}$$

for any L_2-function with compact support in the Fourier domain, and thus we obtain that m_{β_x,β_y} is bounded from below and above. $\square$

Now, that this section is proven to be admissible, we would like to explore the uncertainty principle minimizers associated with this representation. We assume that it should be a two-dimensional extension of the one-dimensional solution obtained earlier. The representation for the quotient as a function of α_x,α_y is then given by:

$$[U(b_x,b_y,\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),0)\psi](x,y) =$$

$$(1+\omega_x|)^{\frac{\alpha_x}{2}}(1+\omega_y|)^{\frac{\alpha_y}{2}}e^{i(\omega_x(x-b_x)+\omega_y(y-b_y))}\psi\left((1+|\omega_x|)^{\alpha_x}(x-b_x),(1+|\omega_y|)^{\alpha_y}(y-b_y)\right).$$

From this representation we may see that the x and y axes are not correlated, and thus we obtain the following infinitesimal generators

$$\begin{aligned}
(T_{b_x}\psi) &= -\frac{\partial}{\partial x}\psi,\\
(T_{b_y}\psi) &= -\frac{\partial}{\partial y}\psi,\\
(T_{\omega_x}\psi) &= (\frac{\alpha_x}{2}+i)\psi(x,y)+\alpha_x x\frac{\partial}{\partial x}\psi(x,y),\\
(T_{\omega_y}\psi) &= (\frac{\alpha_y}{2}+i)\psi(x,y)+\alpha_y y\frac{\partial}{\partial y}\psi(x,y).
\end{aligned}\tag{50}$$

In order to make these operators self-adjoint, we multiply them by i. The commutators between the x and y operators vanish, and we have to solve two independent one-dimensional problems, with the following solutions

$$\psi(x,y) = (\alpha_x\lambda_x x+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_x}}{\alpha_x}+\frac{i\mu_{b_x}}{\alpha_x\lambda_x}+\frac{i}{\alpha_x^2\lambda_x}}e^{\frac{-ix}{\alpha_x}}(\alpha_y\lambda_y y+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_y}}{\alpha_y}+\frac{i\mu_{b_y}}{\alpha_y\lambda_y}+\frac{i}{\alpha_y^2\lambda_y}}e^{\frac{-iy}{\alpha_y}}. \tag{51}$$

In order for this solution to be square integrable, the following should be met: we denote $\lambda_x = i\gamma_x, \lambda_y = i\gamma_y$ where $\gamma_x,\gamma_y\in\mathbb{R}$. Then, if $\gamma_x,\gamma_y<0$, then $\mu_{b_x} > -\frac{1}{\alpha_x}, \mu_{b_y} > -\frac{1}{\alpha_y}$. If $\gamma_x,\gamma_y>0$, then $\mu_{b_x} < -\frac{1}{\alpha_x}, \mu_{b_y} < -\frac{1}{\alpha_y}$.

5 Discussion and Conclusions

The STFT and wavelet transforms can both be viewed as the integral transforms related to the Weyl-Heisenberg and affine groups respectively. From a signal processing viewpoint, it is interesting to ask whether there is a combined integral transform that smoothly interpolates between these two. In a recent publication [4] an answer to this question was given from a harmonic analysis viewpoint. This work deals with coorbit spaces that are associated with some group. These spaces contain all functions for which the associated integral transform is contained in some weighted L_p-space. The coorbit spaces associated with the Weyl-Heisenberg group are the modulation spaces [7] and those associated with the affine group are the Besov spaces. Dahlke et. al. [4] proposed to construct some mixed smoothness spaces that lie in between the Besov and modulation spaces. They have suggested to use the affine Weyl-Heisenberg group as it contains both groups. Since this group is known to have no representation which is square integrable, they offered to factor out a suitable closed subgroup and work with quotients. Specifically, they offer to work with suitable sections.

We adapted their setting, and analyzed in this work several possible sections in the framework of the affine Weyl-Heisenberg setting. We explored the uncertainty relations related to various selections of sections, and aimed at providing their minimizers.

The sections we have explored in this manuscript are those that provide inverse relations between the scale and frequency attributes. This is a natural framework, as we expect high frequency phenomena to manifest themselves within a small scale and vise-versa. Moreover, using the α-modulation spaces, we can smoothly move from the Gabor (Weyl-Heisenberg) transform, via the Gabor wavelet transform to the wavelet (affine) transform. In the one dimensional case, we have considered the section which regards the scale as a function of frequency. This section is known to be admissible, and we have explicitly calculated the minimizer with respect to both time and frequency localization. However, using this section, we can only treat the Gabor and Gabor wavelets transform. Therefore, the section in which the frequency is regarded as a function of the scale was considered. It turns out that this section is not admissible. Still, as it can be treated in the framework of quasi-coherent states, we have also calculated the minimizers for this case. We have also found that for the Gabor-wavelets case, the two sections may provide the same solution under some constraints.

In the two dimensional case there are several possible sections that can be selected. We have presented in this manuscript the similitude Weyl-Heisenberg group and treated it with respect to the L_1-norm and the squared L_2-norm. We also regarded the case where the scale is also considered as a vector. As it is not possible to find a minimizer for all the uncertainty relations, we may consider a single such relation (e.g. the minimizer with respect to the frequency in the x direction and rotation), or we could find a rotation invariant solution when we consider elements of the enveloping algebra.

To summarize, we have considered possible uncertainty minimizers for the AWH group, and have especially regarded the attribute of interpolating between the Gabor and wavelet transforms. The applicative significance of these minimizers is still not well established. However, based on the importance that the Gabor functions have in signal and image processing, we believe that this significance awaits discovering.

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