INTERPOLATING REFINABLE FUNCTIONS AND WAVELETS FOR GENERAL SCALING MATRICES

Stephan Dahlke *
Institut für Geometrie
und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany

Peter Maass †
Fachbereich Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam
Germany

Abstract

This paper introduces a general procedure for constructing interpolating refinable functions for arbitrary dilation matrices. The key ideas are based on the construction presented in [24]. Several families of interpolating refinable functions are computed exlicitly. They originate from a convolution product of some simple function, either generalized B–splines or the Laplace scheme. A suitable correction is added to obtain interpolating solutions.

Key Words: interpolating scaling functions, spline functions, wavelets, expanding scaling matrices

AMS Subject classification: Primary 26B05, 41A05, 41A15, Secondary 41A30, 41A63

*This work has been supported by the Volkswagen–Stiftung (RiP–Programm Oberwolfach)
†This work has been supported by the Volkswagen–Stiftung (RiP–Programm Oberwolfach)

1 INTRODUCTION

The objective of this paper is the construction of specific kinds of multivariate scaling functions and wavelets. In general, a system $\{\psi^\rho\}_{\rho\in R}$ of functions in $L_2(\mathbb{R}^d)$, where R is some finite set, is called a system of (mother) **wavelets**, if all its dilated, translated and scaled versions, i.e., the set

$$\psi^\rho_{j,k}(x) := |\det M|^{j/2}\psi(M^j x - k), \qquad \rho \in R,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d, \tag{1.1}$$

forms a basis of $L_2(\mathbb{R}^d)$. Here M denotes an **expanding** integer scaling matrix, i.e., all its eigenvalues have modulus larger than one. In general, a wavelet basis is constructed by means of a **multiresolution analysis** i.e., by a nested sequence $\{V_j\}_{j\in\mathbb{Z}}$ of shift–invariant closed subspaces of $L_2(\mathbb{R}^d)$ whose union is dense in $L_2(\mathbb{R}^d)$ while their intersection is zero. As usual, we shall assume that each of the V_j's is spanned by the translates of a dilated version of one single function called the **generator** (or the scaling function) of the multiresolution analysis. It is well–known that almost all interesting properties of the wavelet basis, e.g., its regularity, are already determined by the characteristics of the generator. Consequently, the first step in the construction of a wavelet basis with specific properties is to find a suitable scaling function. The primary aim of this paper is the construction of (multivariate) **interpolating** scaling functions φ, i.e., we impose the condition

$$\varphi(k) = \delta_{0,k} \qquad \text{for all} \quad k \in \mathbb{Z}^d. \tag{1.2}$$

A corresponding property can be transmitted to the resulting wavelet basis, see Section 5. For several reasons, scaling functions satisfying (1.2) have become of increasing interest in the last few years. For instance, by using interpolating scaling functions and wavelets, it becomes particularly simple to compute the coefficients of an associated wavelet expansion. This is especially important for the treatment of nonlinear terms that may arise when dealing with wavelet Galerkin methods, see e.g. [2]. Moreover, the interpolatory setting simplifies the incorporation of boundary conditions [1].

For practical reasons, we are in particular interested in compactly supported and sufficiently smooth generators. Several examples of scaling functions satisfying these requirements have been constructed in the last years, see e.g. [6], [12], [15], [16] [17], [24] (this list is clearly not complete). The applications to wavelet decompositions have been clarified in [13] and [14], see also [4]. However, most of these approaches are based on dyadic scalings, i.e., they deal with the specific case $M = 2I$. In contrary to this, our aim was to find an approach which works for an arbitrary scaling matrix. This is important since the number of wavelets that is needed is equal to $|\det M| - 1$, i.e., the costs grow exponentially with respect to the spatial dimension in the dyadic case. We try to avoid this problem by using scaling matrices satisfying $|\det M| = 2$. Our work was inspired by and is closely related to the work of Riemenschneider and Shen [24]. Their approach is based on box splines. Therefore, the theory presented in [24] is restricted to a very small class of scaling matrices, see Section 2. Consequently, it was one of our goals to find objects that can play the role of the box spline in the general case.

This paper is organized as follows. In Section 2, we explain the general setting of interpolatory scaling functions. We derive some necessary conditions on the symbol and give an outline how scaling functions satisfying these conditions can be found. The idea is to convolve some "canonical" generators, which are available for almost all scaling matrices, by some suitable distribution. In Section 3, we briefly recall a technique to estimate the Hölder– and Sobolev–regularity of the resulting function, respectively. In Section 4, we discuss some examples, and finally, in Section 5, we present the construction of an interpolatory wavelet basis.

2 THE GENERAL SETTING

We shall be concerned with compactly supported functions $\phi \in L_2(\mathbb{R}^d)$ which satisfy a **two–scale–relation**

$$\phi(x) = \sum_{k \in \mathbb{Z}^d} a_k \phi(Mx - k), \quad \mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d), \tag{2.1}$$

where M is an expanding integer scaling matrix. It is well–known that under some natural conditions the generator ϕ of a multiresolution analysis satisfies (2.1) with some suitable sequence $\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d}$. In the sequel, we will always assume that supp $\mathbf{a} := \{k \in \mathbb{Z}^d \mid a_k \neq 0\}$ is finite and that

$$\sum_{k \in \mathbb{Z}^d} a_k = m, \quad m := |\det M|, \tag{2.2}$$

holds. A function ϕ satisfying (2.1) is often called a **refinable** function. To indicate the dependency on $\mathbf{a}$ and M, we will also use the term $(\mathbf{a}, M)$–refinable. Applying Fourier transform to (2.1) yields

$$\hat{\phi}(\xi) = \frac{1}{m} a(e^{-iM^{-T}\xi}) \hat{\phi}(M^{-T}\xi), \quad \xi \in \mathbb{R}^d, \tag{2.3}$$

where the **symbol** $a(z)$ is the Laurent polynomial

$$a(z) := \sum_{k \in \mathbb{Z}^d} a_k z^k, \qquad z \in \mathcal{T}^d, \tag{2.4}$$

and $\mathcal{T}^d$ clearly denotes the d–dimensional torus, $\mathcal{T}^d := \{z \in \mathbb{C}^d \mid |z_i| = 1,\ 1 = 1, \ldots, d\}$.

The aim of this paper is to construct families of **fundamental** refinable functions φ, i.e., we require that φ is an interpolating function in the sense that

$$\varphi(k) = \delta_{0,k}, \quad k \in \mathbb{Z}^d, \tag{2.5}$$

holds. We start this project by deriving some necessary conditions on the symbol $a(z)$ of φ which are implied by (2.5).

By the Poisson summation formula, it is easy to check that (2.5) is equivalent with

$$\sum_{k\in \mathbb{Z}^d} \hat{\varphi}(\xi + 2\pi k) = 1, \quad \xi \in \mathbb{R}^d. \tag{2.6}$$

Inserting (2.3) into (2.6) leads to

$$m = \sum_{\tilde{\rho}\in R^T} a(\zeta_{\tilde{\rho}} e^{-iM^{-T}\xi}), \quad \xi \in \mathbb{R}^d, \tag{2.7}$$

where $R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{m-1}\}$ denotes a complete set of representatives of $\mathbb{Z}^d/M^T\mathbb{Z}^d$ and $\zeta_{\tilde{\rho}}$ is defined by

$$\zeta_{\tilde{\rho}} := e^{-i2\pi M^{-T}\tilde{\rho}}. \tag{2.8}$$

The necessary condition (2.7) has some important consequences for the **subsymbols**

$$a_\rho(z) := \sum_{k\in \mathbb{Z}^d} a_{\rho+Mk} z^k, \quad \rho \in R, \tag{2.9}$$

where now R is a complete set of representatives of $\mathbb{Z}^d/M\mathbb{Z}^d$. (Without loss of generality, we will always assume that $\rho_0 = \tilde{\rho}_0 = 0$ in the sequel. Furthermore, we shall use the abbreviation $z^M := (z^{M^{(1)}}, \cdots, z^{M^{(d)}})$, where $M^{(j)}$ denotes the j–th column of M).

Lemma 2.1 *Let φ be a compactly supported fundamental refinable function with symbol $a(z)$. Then*

$$1 = a_0(z^M). \tag{2.10}$$

Proof: Inserting the relation

$$a(z) = \sum_{\rho\in R} z^\rho a_\rho(z^M) \tag{2.11}$$

into eq. (2.7) we find with $z = e^{-i\xi}$

$$\begin{aligned} m &= \sum_{\tilde{\rho}\in R^T} a(\zeta_{\tilde{\rho}} e^{-i\xi}) \\ &= \sum_{\tilde{\rho}\in R^T} \sum_{\rho\in R} \zeta_{\tilde{\rho}}^\rho e^{-i\xi\rho} \sum_{k\in \mathbb{Z}^d} a_{\rho+Mk}(e^{-i2\pi M^{-T}\tilde{\rho}\cdot Mk} e^{-i\xi Mk}) \\ &= \sum_{\tilde{\rho}\in R^T} \sum_{\rho\in R} \zeta_{\tilde{\rho}}^\rho e^{-i\xi\rho} \sum_{k\in \mathbb{Z}^d} a_{\rho+Mk}(e^{-i2\pi\tilde{\rho}k} e^{-i\xi Mk}) \\ &= \sum_{\rho\in R} \left(\sum_{\tilde{\rho}\in R^T} \zeta_{\tilde{\rho}}^\rho \right) e^{-i\xi\rho} a_\rho(z^M). \end{aligned} \tag{2.12}$$

The above expression can be simplified by employing the following fundamental lemma proved by Chui and Li [3].

Lemma 2.2 *Let $\zeta_{\tilde\rho}$ be defined by (2.8) Then*

$$\sum_{\tilde\rho\in R^T} \zeta_{\tilde\rho}^{\rho'} \zeta_{\tilde\rho}^{-\rho''} = m \cdot \delta_{\rho',\rho''}, \qquad \rho', \rho'' \in R. \tag{2.13}$$

The result now follows by using (2.13) with $\rho'' = 0$. □

The next claim is to find a procedure to construct symbols $a(z)$ such that the corresponding subsymbols $a_0(z^M)$ satisfy (2.10). The idea is to start with a 'nice', i.e., sufficiently smooth refinable and compactly supported function ϕ with symbol $b(z)$. In practice, ϕ could be a box spline or a generalized cardinal B–spline as described below. Then one way to find solutions of (2.10) is to convolve ϕ with some suitable refinable and compactly supported distribution η, i.e., we define φ by

$$\varphi := \phi * \eta. \tag{2.14}$$

Since refinability is preserved under convolution, φ satisfies a two–scale–relation and its symbol can be easily computed as

$$a(z) = \frac{1}{m} b(z) q(z), \tag{2.15}$$

where $q(z)$ clearly denotes the symbol of η. We have to find η in such a way that $a_0(z^M)$ satisfies (2.10). As we shall now explain, this is possible if the translates of ϕ are linearly independent, i.e.,

$$\sum_{k\in\mathbb{Z}^d} \lambda_k \phi(x-k) = 0 \quad \text{implies} \quad \lambda_k = 0 \quad \text{for all} \quad k \in \mathbb{Z}^d, \quad \lambda \in \ell(\mathbb{Z}^d). \tag{2.16}$$

Lemma 2.3 *Suppose that ϕ has linearly independent translates. For some $\alpha \in \mathbb{Z}^d$, let $\tilde q_\rho(z^M)$ denote the solutions of*

$$z^{M\alpha} = \sum_{\rho\in R} b_\rho(z^M) \tilde q_\rho(z^M). \tag{2.17}$$

and let $q(z)$ be defined by

$$q(z) := \sum_{\rho\in R} z^{-\rho} \tilde q_\rho(z^M). \tag{2.18}$$

Then the subsymbol $a_0(z)$ of

$$a(z) := z^{-M\alpha} q(z) b(z) \tag{2.19}$$

satisfies (2.10).

Proof: By Bezout's theorem, there exist solutions of (2.18) if the subsymbols $b_\rho(z)$, $\rho \in R$ have no common zeros in $(\mathbb{C}\backslash\{0\})^d$. It was shown by Jia and Micchelli [20] that a compactly supported function ϕ has linear independent translates if and only if

$$\sup_{k\in\mathbb{Z}^d} |\hat\phi(\xi + 2\pi k)| > 0 \qquad \text{for all} \quad \xi \in \mathbb{C}^d, \tag{2.20}$$

where $\hat{\phi}$ denotes the Fourier–Laplace–Transform of ϕ. Employing (2.3) and the relation (2.11) yields

$$0 < \sup_{k\in \mathbb{Z}^d} \frac{1}{m} |\sum_{\rho\in R} e^{-iM^{-T}(\xi+2\pi k)\rho} b_\rho(e^{-i\xi})\hat{\phi}(M^{-T}(\xi+2\pi k))|,$$

so that the subsymbols $b_\rho(z),\ \rho\in R$ have indeed no common zeros in $(\mathbb{C}\setminus\{0\})^d$. It remains to show that $a(z)$ defined by (2.19) satisfies (2.10). To this end, we use the relation

$$z^\rho a_\rho(z^M) = m^{-1} \sum_{\tilde{\rho}\in R^T} \zeta_{\tilde{\rho}}^{-\rho} a(\zeta_{\tilde{\rho}} z) \tag{2.21}$$

for $\rho = 0$ and obtain by (2.19),(2.18),(2.11) and Lemma 2.2

$$\begin{aligned}
a_0(z^M) &= m^{-1} \sum_{\tilde{\rho}\in R^T} a(\zeta_{\tilde{\rho}} z) \\
&= m^{-1} \sum_{\tilde{\rho}\in R^T} (\zeta_{\tilde{\rho}} z)^{-M\alpha} q(\zeta_{\tilde{\rho}} z) b(\zeta_{\tilde{\rho}} z) \\
&= m^{-1} \sum_{\tilde{\rho}\in R^T} (\zeta_{\tilde{\rho}} z)^{-M\alpha} \left(\sum_{\rho'\in R} \zeta_{\tilde{\rho}}^{-\rho'} z^{-\rho'} \tilde{q}_{\rho'}((\zeta_{\tilde{\rho}} z)^M)\right) \cdot \left(\sum_{\rho''\in R} \zeta_{\tilde{\rho}}^{\rho''} z^{\rho''} b_{\rho''}((\zeta_{\tilde{\rho}} z)^M)\right) \\
&= m^{-1} z^{-M\alpha} \sum_{\rho',\rho''\in R} \left(\sum_{\tilde{\rho}\in R^T} \zeta_{\tilde{\rho}}^{-\rho'} \zeta_{\tilde{\rho}}^{\rho''}\right) z^{-\rho'} z^{\rho''} \tilde{q}_{\rho'}(z^M) b_{\rho''}(z^M) \\
&= z^{-M\alpha} \sum_{\rho'\in R} \tilde{q}_{\rho'}(z^M) b_{\rho'}(z^M) = 1.
\end{aligned}$$

□

Once a suitable distribution is found, one still has to check that the resulting refinable function is indeed fundamental (observe that condition (2.10) is only necessary), and one has to estimate its regularity. For the first task we use the following theorem proved in [23].

Theorem 2.1 *Suppose that* $\mathbf{a}$ *is a finitely supported sequence satisfying a(1)=m. A necessary and sufficient condition for a continuous* $(\mathbf{a}, M)$*–refinable function to be interpolatory is that the sequence* δ *is the unique eigenvector of the operator*

$$(W_a b)(l) = \sum_{k\in\mathbb{Z}^d} a_{Ml-k} b_k, \qquad \{b_k\}_{k\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d). \tag{2.22}$$

A method to estimate the regularity will be discussed later.

Before we can construct examples for our approach, we have to clarify how a suitable sufficiently smooth and compactly supported refinable function ϕ can be found. A good starting mask is essential for the success of our purpose. For the case $M = 2I$, one natural choice would be to use a box spline $B(\ \cdot\ |X_\nu),\ X_\nu = (x^1,\ldots,x^\nu),\ x^l \in \mathbb{Z}^d\setminus\{0\},\ \nu \geq d$, which is defined by

$$\widehat{B(\ \cdot\ |X_\nu)}(\xi) = \prod_{x^l\in X_\nu} \left(\frac{1-e^{-ix^l\cdot\xi}}{ix^l\cdot\xi}\right). \tag{2.23}$$

It is well–known that $B(\;\cdot\;|X_\nu)$ is refinable with respect to $M = 2I$. The resulting symbol is

$$a(z) = 2^{d-\nu}\prod_{l=1}^{\nu}(1 + z^{x^l}). \tag{2.24}$$

This approach was in detail discussed in [24]. However, for a more complicated scaling matrix, it is in general not possible to find an associated refinable box spline. One has to restrict oneself to scaling matrices satisfying

$$M^d = 2I, \tag{2.25}$$

see e.g. [7] for details. Matrices satisfying (2.25) will be called **box spline matrices** in the sequel. One way to handle the general case is to replace the box spline by a so–called **generalized cardinal B–spline** as, e.g., studied in [7]. This approach is based on self–affine lattice tilings. We say that a set Q gives rise to a **self–affine lattice tiling** if it satisfies

$$|Q| = 1, \quad I\!R^d \simeq \bigcup_{k\in Z\!\!Z^d}(Q+k), \quad Q\cap(Q+k)\simeq\emptyset, \quad k\neq 0\ , \tag{2.26}$$

$$Q = \bigcup_{i=0}^{m-1} M^{-1}(Q+\rho_i) \quad , \tag{2.27}$$

where the union in (2.27) is assumed to be disjoint. Clearly, "$\simeq$" means equality up to sets of measure zero. Once a set of representatives is chosen, a corresponding self–affine set can be constructed by a generalized iterated function system. More precisely, one can use the following lemma, proved by Gröchenig and Madych [19].

Lemma 2.4 *Let $\{\rho_0,\ldots\rho_{m-1}\}$ be some enumeration of cosets in $Z\!\!Z^d/MZ\!\!Z^d$. If $\tilde{Q}_0$ is any compact set, then the sequence $\tilde{Q}_1,\tilde{Q}_2,\ldots$ defined by*

$$\tilde{Q}_{N+1} := \bigcup_{i=0}^{m-1} M^{-1}(\rho_i + \tilde{Q}_N) \tag{2.28}$$

converges in the metric $\tilde{\varrho}$, defined by

$$\tilde{\varrho}(P,Q) := max\{\varrho(P,Q),\varrho(Q,P)\} \tag{2.29}$$

where

$$\varrho(P,Q) := \sup_{x\in P}\inf_{y\in Q}|x-y| \quad .$$

It is easy to check that the limit set Q satisfies the self–similarity relation (2.27), and that the union in (2.27) is disjoint. Furthermore, it can be shown that if the limit set Q satisfies

$$|Q| = 1, \tag{2.30}$$

then it gives rise to a self–affine lattice tiling, i.e., (2.26) holds, see [19] for details. It may happen that the limit set has a larger measure. In general, one only knows that Q

has integer measure and that it tiles $\mathbb{R}^d$ with respect to a subset of the lattice $\mathbb{Z}^d$, see Lagarias and Wang [22] and Gröchenig and Haas [18] for details. We want to present a motivating two–dimensional specimen of a self–affine tile here. It is the so–called **twin–dragon–set** which is obtained by employing the scaling matrix $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. This case was already studied in [19] and [5]. The scaling matrix satisfies $|\det M| = 2$ and therefore $\mathbb{Z}^2/M\mathbb{Z}^2$ consists of exactly two elements. It can be checked that the set $M\mathbb{Z}^2$ is the so–called **quincunx grid** Υ, i.e.,

$$k = (k_1, k_2) \in \Upsilon \quad \text{if and only if} \quad k_1 + k_2 \text{ is even.} \tag{2.31}$$

As mentioned above we always choose $\rho_0 = 0$. Some possible choices $\rho_1^0, \ldots, \rho_1^3$ for the second representatives are given by

$$\rho_1^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \rho_1^1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \rho_1^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \rho_1^3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.32}$$

Figure 1 shows the resulting self–affine tile obtained by taking ρ_1^0 as the second representative.

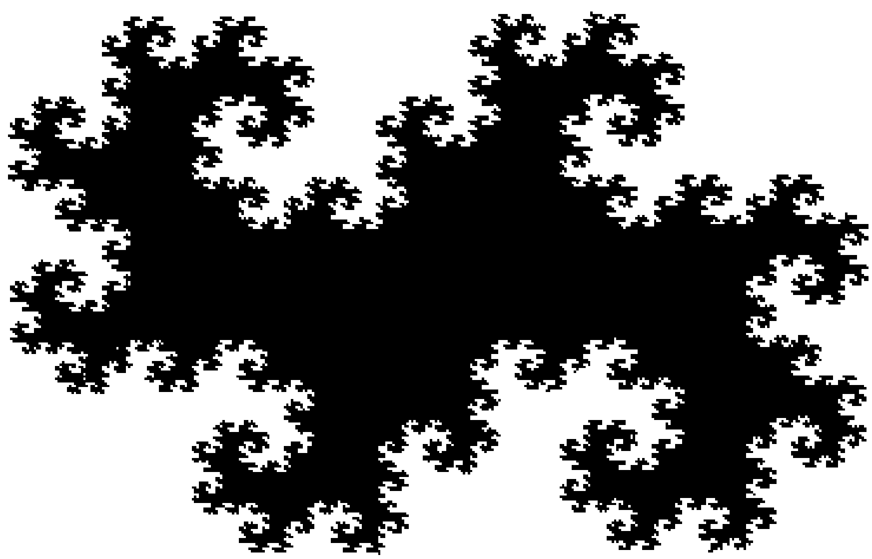

Figure 1

Using the characteristic function χ_Q of a limit set Q obtained by employing Lemma 2.4, we now want to define a generalized spline function. First of all, we infer from (2.27) that

$$\chi_Q(x) = \sum_{j=0}^{m-1} \chi_Q(Mx - \rho_j). \tag{2.33}$$

Now suppose we have $K+1$ sets of representatives $R_i = \{\rho_0^i, \ldots, \rho_{m-1}^i\}$, $i = 0, \ldots, K$, and let Q_i denote the limit set associated, in view of Lemma 2.4, with R_i. Then, for

$N_0, \ldots, N_K \in I\!N$ let the **generalized cardinal B–spline** $\phi_{N_0,\ldots,N_K}$ be defined by

$$\phi_{N_0,\ldots,N_K} := \underbrace{\chi_{Q_0} * \cdots \chi_{Q_0}}_{N_0-times} * \underbrace{\chi_{Q_1} * \cdots \chi_{Q_1}}_{N_1-times} * \cdots * \underbrace{\chi_{Q_K} * \cdots \chi_{Q_K}}_{N_K-times}. \tag{2.34}$$

Employing the fact that, according to (2.33), each χ_{Q_i} is a refinable function, it is easy to check that $\phi_{N_0,\ldots,N_K}$ satisfies a two–scale–relation similar to (2.33),

$$\phi_{N_0,\ldots,N_K}(x) = \sum_{k \in Z\!\!Z^d} a_k \phi_{N_0,\ldots,N_K}(Mx - k) \tag{2.35}$$

with

$$a(z) = m^{-(N_0+\ldots+N_K-1)} a_{Q_0}(z)^{N_0} a_{Q_1}(z)^{N_1} \ldots a_{Q_K}(z)^{N_K} \quad . \tag{2.36}$$

Here $a_{Q_i}(z)$ denotes the symbol of χ_{Q_i} which, according to (2.33), is given by

$$a_{Q_i}(z) = \sum_{j=0}^{m-1} z^{\rho_j^i} \quad , \ z := e^{-i\xi} \quad .$$

Since all the functions $\chi_{Q_i}, \ i = 0, \ldots, K$, are compactly supported, $\phi_{N_0,\ldots,N_K}$ will also be compactly supported. Taking more convolutors clearly increases the smoothness of the resulting spline. For a qualitative study, the reader is referred to [7]. In order to obtain optimal smoothness, it is useful to work with **different** sets of representatives. However, then one has to deal with some stability problems. In general, the translates of a function $\phi \in L_2(I\!R^d)$ are said to be **ℓ_2–stable** if there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|\lambda\|_{\ell_2} \leq \| \sum_{k \in Z\!\!Z^d} \lambda_k \phi(\cdot - k)\|_{L_2(I\!R^d)} \leq c_2 \|\lambda\|_{\ell_2} \quad \text{for all} \quad \lambda \in \ell_2(Z\!\!Z^d). \tag{2.37}$$

If Q gives rise to a tiling, then the translates of χ_Q are orthonormal and therefore stable. (Moreover, they are also linearly independent, see [8] for details). This property is preserved if we take only convolutions of χ_Q with itself, but it is a difficult task to check stability and linear independence for convolutions of different functions. For recent results on this topic, the reader is referred to [9]. One should furthermore observe that even with convolutions of χ_Q with itself the smoothness can be increased arbitrarily. The reason is that $\phi := \chi_Q * \chi_Q$ is already continuous and therefore also Hölder continuous. (This is a consequence of the refinability of ϕ, see [10] for details). Consequently, taking a sufficiently large number of convolutions increases the decay of the corresponding Fourier transform arbitrarily high. (This is the reason why we concentrate on convolutions of one single function with itself in the examples in Section 4).

We also want to mention an interesting property of characteristic functions of lattice tilings. Since the translates of of χ_Q are orthonormal, it is easy to check that the autocorrelation function

$$\phi_{aut} := \chi_Q(\cdot) * \chi_Q(-\cdot) \tag{2.38}$$

is an interpolating refinable function. Therefore the tiling approach provides us with a family of continuous fundamental functions. Taking more convolutions adds to the smoothness of the refinable function but destroys the interpolation property, hence a correction term has to be constructed.

Although the tiling approach seems to be somewhat canonical, there are clearly other possible ways to find smooth refinable functions which can play the role of a box spline. In Section 4, we shall also be concerned with refinable functions stemming from the so–called **Laplacian scheme** associated with the symbol

$$b(z) = (4 + z_2 + z_1 + z_2^{-1} + z_1^{-1})/4. \tag{2.39}$$

Taking iterates of this mask produces a family of refinable functions with arbitrarily high regularity. Again these functions can be converted to an interpolating family by convolving them with a suitable distribution, see Section 4 for details.

3 REGULARITY

Once we have determined a symbol $q(z)$ such that $a(z)$ satisfies (2.10), we have to compute the corresponding refinable function and to estimate its Hölder– and Sobolev–regularity, respectively. This can be done by a version of the usual Payley–Littlewood–technique as discussed, e.g., in [6], [24], [25], [26], [27]. For reader's convenience, we briefly state the main ideas. By iterating (2.3) we obtain the following expression for $\hat{\varphi}(\xi)$:

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m^{-1} a(e^{-i(M^{-T})^j \xi}) = \prod_{j=1}^{\infty} m^{-1} b(e^{-i(M^{-T})^j \xi}) \prod_{j=1}^{\infty} q(e^{-i(M^{-T})^j \xi}). \tag{3.1}$$

To estimate the regularity of φ, we have to determine the decay of $\hat{\varphi}$, i.e., the decay of the products at the right–hand side of (3.1). Indeed, let

$$\kappa_p := \sup\{\kappa \ : \ \int_{\mathbb{R}^d} (1 + |\xi|^p)^\kappa |\hat{\varphi}(\xi)|^p d\xi < \infty\}. \tag{3.2}$$

Then $\varphi \in C^{\kappa_1 - \varepsilon}(\mathbb{R}^d)$ and $\varphi \in H^{\kappa_2}(\mathbb{R}^d)$. To determine κ_1 and κ_2, respectively, for φ defined by (3.1), we divide $\mathbb{R}^d$ into disjoint pieces $C_n := (M^T)^n T^d \backslash (M^T)^{n-1} T^d$, $T^d := [-\pi, \pi]^d$. We always assume that $M^l = rI$ for some $l, r \in \mathbb{N}$ and $T^d \subset M^T T^d$ which is true for most of the interesting examples of scaling matrices. Let us for the moment assume that we have already found a number $\lambda \in \mathbb{R}$ such that

$$\int_{C_n} |\hat{\varphi}(\xi)|^p d\xi \leq C \lambda^n. \tag{3.3}$$

Then one easily checks that

$$\int_{\mathbb{R}^d} (1 + |\xi|^p)^\kappa d\xi \leq C \sum_{n=1}^{\infty} r^{n(p\kappa + \log_r(\lambda) l)},$$

and the series on the right–hand side is finite if and only if

$$\kappa < -\frac{\log_r(\lambda)l}{p}. \tag{3.4}$$

For simplicity, we shall denote all constants by C in the sequel, although the value of C may change. It turns out that in our case λ is determined by the spectral radius of the transition operator corresponding to $a(z)$. In general, let $g(z)$ be a Laurent polynomial satisfying $g(1) = 1$. We define the operator

$$T_g f := \sum_{\tilde{\rho} \in R^T} g(M^{-T}(\cdot + 2\pi\tilde{\rho}))f(M^{-T}(\cdot + 2\pi\tilde{\rho})). \tag{3.5}$$

Following the lines of Riemenschneider and Shen, one can check by induction that

$$T_g^n(f) = \sum_{k \in \mathbb{Z}^d} f(M^{-T}(\cdot + 2\pi k))\hat{\varphi}_n(\cdot + 2\pi k), \tag{3.6}$$

where

$$\hat{\varphi}_n(\xi) := \prod_{j=1}^{n} g((M^{-T})^j \cdot)\chi_{(M^T)^n T^d}, \qquad \hat{\varphi}_0 = \chi_{T^d}. \tag{3.7}$$

It is easy to verify that (3.6) implies that

$$\int_{T^d} |T_g^n f(\xi)| d\xi \leq \int_{\mathbb{R}^d} |f((M^{-T})^n \xi)||\hat{\varphi}_n(\xi)| d\xi, \tag{3.8}$$

with equality if f and g are positive. Let us now furthermore suppose that g satisfies

$$D^\beta g(M^{-T} 2\pi\tilde{\rho}) = 0, \qquad |\beta| \leq \mu. \tag{3.9}$$

We define the space of trigonometric polynomials

$$V_\mu := \{f = \sum_{|k| \leq N} a_k e^{-ik\xi}, \quad D^\beta f(0) = 0, \ |\beta| \leq \mu\}. \tag{3.10}$$

For N large enough (depending on M), eq. (3.9) implies that V_μ is an invariant subspace of T_g. It turns out that the spectral radius of $T_g|_{V_\mu}$ can be used to determine the regularity of the associated refinable function g.

Proposition 3.1 *Suppose that g satisfies* (3.9) *and $g \geq 0$. Furthermore, let $\tilde{\lambda}$ denote the spectral radius of $T_g|_{V_\mu}$. For φ_g defined by*

$$\hat{\varphi}_g(\cdot) := \prod_{j=1}^{\infty} g((M^{-T})^j \cdot) \tag{3.11}$$

there exists for any given $\varepsilon > 0$ a constant C such that

$$\int_{C_n} |\hat{\varphi}_g(\xi)| d\xi \leq C(\tilde{\lambda} + \varepsilon)^n. \tag{3.12}$$

Proof: The proof can be performed by following the lines of Riemenschneider and Shen, so that we only sketch the basic steps. It can be checked that

$$\int_{T^d} |T_g^n f(\xi)| d\xi \le C(\tilde{\lambda} + \varepsilon)^n ||f||$$

for some suitable norm $\|\cdot\|$ on V_μ and

$$\int_{C_n} |\hat{\varphi}_g(\xi)| d\xi \le C \int_{C_n} |\hat{\varphi}_n(\xi)| d\xi.$$

The function $f(\xi) = \sum_{n=1}^{d}(1 - \cos(\xi_n))^\mu$ is contained in V_μ and satisfies $f \ge C$ for $\xi \in T^d \backslash M^{-T}T^d$ so that, by employing (3.8), we obtain

$$\begin{aligned}\int_{C_n} |\hat{\varphi}_n(\xi)| d\xi &\le C^{-1}\int_{C_n} f((M^{-T})^n\xi)|\hat{\varphi}_n(\xi)| d\xi \le C^{-1}\int_{(M^T)^n T^d} f((M^{-T})^n\xi)|\hat{\varphi}_n(\xi) d\xi \\ &= \int_{\mathbb{R}^d} f((M^{-T})^n\xi)|\hat{\varphi}_n(\xi)| d\xi = \int_{T^d} T_g^n f(\xi) d\xi \le C(\tilde{\lambda} + \varepsilon)^n.\end{aligned}$$

□

Proposition 3.1 can now be applied to our interpolating refinable functions with $g = m^{-1}a(z)$, provided that $a(z) \ge 0$ which gives an estimate for the Hölder regularity of φ. If $a(z) \ge 0$ does not hold, one can apply the proposition to the autocorrelation function $\phi = \varphi(\cdot) * \varphi(-\cdot)$ which gives an estimate for the Sobolev regularity of φ. By using Sobolev embeddings, we also obtain an estimate for the Hölder exponent.

4 EXPLICIT CONSTRUCTIONS

In this section we will explicitly construct different families of interpolating refinable functions. The procedure described in the previous sections is valid for arbitrary dimensions and dilation matrices. However we will concentrate on two well–known dilation matrices with determinant ±2. We start with a construction of a C^1 example for the box–spline matrix

$$M_b = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

However out main goal is to construct smooth examples for the true quincunx case

$$M_q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Families of interpolating refinable functions for these type of matrices can already be found in [6], we will return to their construction later.

The subgrid generated by both matrices is the quincunx grid Υ as defined in (2.31). Hence we can always choose the representatives of $\mathbb{Z}^2 \backslash M\mathbb{Z}^2$ as

$$R = \left\{ \rho_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \rho_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

We will describe two different construction procedures, the first one is based on the theory of self–affine tilings, the second one is based on the techniques described in [5], it uses iterations of the Laplace scheme. The regularity of the resulting refinable functions is estimated.

4.1 The q-Term Construction

The construction of interpolating refinable functions in this section is very much based on the same ideas as used in constructing orthogonal wavelets: one starts with a promising symbol $b(z)$, which leads to a smooth refinable functions but does not fulfill the interpolation condition (2.10). Then one tries to construct a correction term $q(z)$ such that $a(z) = \frac{1}{m}b(z)q(z)$ satisfies this condition. Finally one has to check some regularity conditions, ensuring that $a(z)$ actually leads to an interpolating function with some smoothness.

The procedure described in the previous sections proceeds as follows:

- Choose a particular mask. We always start with the mask of a B–spline of odd degree ($N = 2l + 1$)

  $$b(z) = \frac{1}{2^{N-1}}(1 + z_1)^N$$

  or with a symmetrized B-spline of even degree

  $$b(z) = \frac{1}{2^{N-1}} z_1^{-N/2}(1 + z_1)^N \quad .$$

  We shall always interpret these masks in a multivariate setting, i.e., we are interested in multivariate functions which are $(\mathbf{b}, M)$–refinable with respect to the scaling matrices introduced above. From this point of view, it is easy to check that our specific choice corresponds to a refinable function

  $$\phi \;=\; \chi_Q * \ldots * \chi_Q$$

  which stems from N convolutions of the generalized Haar function χ_Q or to a shifted and therefore symmetrized version of it. For M_b the set Q is simply a parallelepiped
  $Q = \{(x_1, x_2) \mid 0 \leq x_2 < 1 \;,\; 0 \leq x_1 - x_2 < 1\;\}$, for M_q we obtain the twin–dragon–set as shown in Figure 1. The convolutions increase the smoothness of the refinable functions, see again [7] for details.

- Compute the decomposition

  $$b(z) \;=\; b_0(z^M) \;+\; z^{\rho_1} b_1(z^M) \;,$$

  compare with (2.11). Determine correction terms $\tilde{q}_0(z)$, $\tilde{q}_1(z)$ by choosing an appropriate α and solving

  $$z^{M\alpha} \;=\; b_0(z^M)\tilde{q}_0(z^M) + b_1(z^M)\tilde{q}_1(z^M) \;,$$

  compare with (2.17). We either choose $M\alpha = (0, 0)^T$ or $M\alpha = (2, 0)^T$.

- Compute the correction term $q(z)$ by

$$q(z) \;=\; \tilde{q}_0(z^M) \;+\; z^{-\rho_1}\tilde{q}_1(z^M)$$

and put

$$a(z) \;=\; z^{-M\alpha}b(z)q(z)\;,$$

compare with (2.18) and (2.19). According to Section 2, this mask fulfills the necessary condition (2.10) for interpolation.

- Check the regularity of the resulting refinable function φ. If φ is continuous and if the conditions of Theorem 2.1 are satisfied, then φ is indeed a interpolating refinable function. For the box–spline matrix M_b we can use the lazy check, which asks to estimate the Hoelder– or Sobolev–regularity by computing

$$B_j \;=\; \sup_{|z|=1}\;|\prod_{l=1}^{j}\;q(e^{-i(M^{-T})^n\xi})|\;.$$

For M_q we estimated the regularity by the eigenvalue technique described in Section 3.

Since M_b and M_q generate the same grid we can compute the correction terms $\tilde{q}_0(z)$ and $\tilde{q}_1(z)$ simultaneously for both matrices.

For $N = 2$ we simply have $b(z) = \frac{1}{2}z_1^{-1}(1+z_1)^2$, hence

$$b_0(z^M) = 1 \quad\text{and}\quad b_1(z^M) = \frac{1}{2}(z_1^{-2}+1)\quad.$$

Choosing $\alpha = (0,0)^T$ allows the solution

$$\tilde{q}_0(z^M) = 1 \quad\text{and}\quad \tilde{q}_1(z^M) = 0.$$

(Different solutions are possible, e.g. $\tilde{q}_0(z^M) = -z_1^{-2}$, $q_1(z^M) = 2$.) Putting both parts together yields $q(z) = 1$ and $a(z) = \frac{1}{2}z_1^{-1}(1+z_1)^2$. This is not surprising since this choice for $b(z)$ corresponds to the symbol of the autocorrelation function of the characteristic function of the self–affine tile Q which is automatically interpolating, compare with Section 2.

For $N = 4$ we obtain

$$b_0(z^M) = \frac{1}{8}(z_1^{-2}+6+z_1^2) \quad\text{and}\quad b_1(z^M) = \frac{1}{2}(z_1^{-2}+1).$$

Choosing e.g. α such that $M\alpha = (2,0)^T$ we have

$$\tilde{q}_0(z^M) = -2 \quad\text{and}\quad \tilde{q}_1(z^M) = \frac{1}{2}+\frac{5}{2}z_1^2,$$

which leads to

$$q(z) = \frac{1}{2}z_1^{-1} - 2 + \frac{5}{2}z_1.$$

Again a more convenient choice is $\alpha = (0,0)^T$, which leads to

$$\tilde{q}_0(z^M) = 2\ ,\ \tilde{q}_1(z^M) = -\frac{1}{2}(1+z_1^2) \quad \text{and} \quad q(z) = 2 - \frac{1}{2}(z_1^{-1} + z_1),$$

hence

$$a(z) = b(z)q(z) = \frac{-1}{16}z_1^{-3} + \frac{9}{16}z_1^{-1} + 1 + \frac{9}{16}z_1 - \frac{1}{16}z_1^3\ .$$

	$N=2\ ,\ M\alpha = (0,0)^T$	$N=4\ ,\ M\alpha = (2,0)^T$	$N=4\ ,\ M\alpha = (0,0)^T$
$b_0(z^M)$	1	$\frac{1}{8}(z_1^{-2} + 6 + z_1^2)$	
$b_1(z^M)$	$\frac{1}{2}(z_1^{-2}+1)$	$\frac{1}{2}(z_1^{-2}+1)$	
$\tilde{q}_0(z^M)$	1 or $-z_1^{-2}$	-2	2
$\tilde{q}_1(z^M)$	0 or 2	$0.5 + 2.5z_1^2$	$-0.5(1+z_1^2)$
$z^{-M\alpha}q(z)$	1	$0.5z_1^{-3} - 2.0z_1^{-2} + 2.5z_1^{-1}$	$2 - 0.5(z_1^{-1} + z_1)$
$a(z)$	$\frac{1}{2}z_1^{-1}(1+z_1)^2$		$\frac{-1}{16}z_1^{-3} + \frac{9}{16}z_1^{-1} + 1 + \frac{9}{16}z_1 - \frac{1}{16}z_1^3$

	$N=3\ ,\ M\alpha = (0,0)^T$	$N=6\ ,\ M\alpha = (0,0)^T$
$b_0(z^M)$	1	$\frac{1}{32}(6z_1^{-2} + 20 + 6z_1^2)$
$b_1(z^M)$	$\frac{1}{2}(z_1^{-2}+1)$	$\frac{1}{32}(z_1^{-4} + 15z_1^{-2} + 15 + z_1^2)$
$\tilde{q}_0(z^M)$	1 or $-z_1^{-2}$	$\frac{1}{8}(3z_1^{-2} + 38 + 3z_1^2)$
$\tilde{q}_1(z^M)$	0 or 2	$\frac{-9}{4}(1+z_1^2)$
$z^{-M\alpha}q(z)$	1	$\frac{1}{8}(3z_1^{-2}18z_1^{-1} + 38 - 18z_1 + 3z_1^2)$
$a(z)$	$\frac{1}{2}z_1^{-1}(1+z_1)^2$	$\frac{1}{256}(3z_1^{-5} - 25z_1^{-3} + 150z_1^{-1} + 256 + 150z_1 - 25z_1^3 + 3z_1^5)$

4.1.1 The Box–Spline Matrix

The regularity estimates differ for the matrices M_b and M_q. For the matrix M_b we used the representation of $\hat{\varphi}(\xi)$ as an infinite product

$$\hat{\varphi}(\xi) \ = \ \prod_{j=1}^{\infty} \frac{1}{m} a(e^{-i(M^{-T})^j \xi})\ .$$

Putting $z = e^{i\xi}$ and expressing $q(z)$ as a trigonometric polynomial in $\xi = (\xi_1, \xi_2)^T$ allows to estimate the decay rate of this infinite product from

$$B_j \ = \ \sup_{\xi \in \mathbb{R}} \left| \prod_{n=1}^{j} q(e^{-i2^{-n}\xi}) \right|\ .$$

Combining these values with the known decay rate for the B–spline product of order N we obtain that

$$\varphi \in C^\beta \quad \text{for} \quad \beta < N - 2 - (\ln(B_j)/(j \ln(2.0)) \ .$$

The following values were computed with Maple.

$\ln(B_j)/(j\ln(2.0))$	N=3	N=4	N=6
j=1		1.5849	3.2927
j=2	0.9251	1.3612	2.7925
j=3	0.8747	1.3614	2.8071
j=4	0.8634	1.2481	2.7758
β	0.1366	0.7519	1.224

The β–values are lower bounds for the Hoelder exponents of the related interpolating refinable functions. In particular starting with a B–spline of order six and choosing the correction term as stated above yields a continuously differentiable function.

4.1.2 The Quincunx Matrix

In this case the regularity estimates were obtained by the matrix–based techniques described in Section 3. We present the results for $N = 3, 4$. For $N = 3$ the masks does not have a positive symbol, hence the matrix corresponding to $T_g|_{V_\mu}$ has to be computed for the autocorrelation function, i.e., with respect to $g(z) = m^{-2}|a(z)|^2$. The largest eigenvalue whose corresponding eigenvector lies in the invariant subspace V_μ is $\lambda = 0.5858$ (computed with Maple). Since we used the autocorrelation function the Hoelder regularity is estimated by $\beta < -\ln(\lambda)/\ln(2.0) - 1 = -0.2285$, hence we cannot prove that in this case the refinement equation has a continuous solution, which is interpolating. All we can prove is that the solution lies in all Sobolev spaces of order $s < 0.7715$.

Due to symmetrization we obtain a positive symbol for $N = 4$. Hence we do not need the autocorrelation function. The four largest eigenvalues were $1.0, 0.79869, 0.5 + 0.5i, 0.5 - 0.5i$. Unfortunately, already the second eigenvalue corresponds to an eigenvector in V_μ, which yields $\beta < -2\ln(0.7987)/\ln(2.0) = 0.6485$: we obtain a continuous and interpolating refinable function for the quincunx matrix. The reader should observe that this function is already smoother than the autocorrelation function $\chi_Q(\cdot) * \chi_Q(-\cdot)$ for which $\beta < 0.4764$ holds.

4.2 Bezout Constructions

For the completeness of this exposition we include a well–known short–cut for constructing interpolating refinable functions for our specific choice of matrices, see [5]. The interpolation condition $a_0(z^M) = \sum_{k \in M\mathbb{Z}^d} a_k z^k = 1$ states, that the symbol $a(z)$ may have no coefficients on $M\mathbb{Z}^d$ except for $k = (0,0)^T$. The grid $M\mathbb{Z}^d$ contains gridpoints (i, j), where $i + j$ even, hence the interpolation condition read as

$$a_0(z^M) \ = \ \frac{1}{2}(\ a(z) + a(-z)\ ) \ = \ 1 \ ,$$

compare with (2.21). Let us consider masks who are symmetric in the sense that $a(z)$ is a polynomial in $\cos^2(\xi_1)$, resp. in $(\cos^2(\xi_1/2)+\cos^2(\xi_2/2))/2$ $(z=e^{-i\xi})$:

$$a(z) = 2p(\cos^2(\xi_1/2)) \text{ , resp. } a(z) = 2p\left(\frac{\cos^2(\xi_1/2)+\cos^2(\xi_2/2)}{2}\right) .$$

Then the necessary condition can be rephrased as ($x=\cos^2(\xi_1)$, resp. $x=(\cos^2(\xi_1/2)+\cos^2(\xi_2/2))/2$)

$$p(x)+p(1-x)=1 \ .$$

The construction now follows the first steps in constructing orthogonal wavelets. If p is chosen as a pure power of x multiplied by a shifted correction polynomial q, i.e. $p(x)=x^N q(1-x)$, then

$$q(x) = \sum_{l=0}^{N-1} \binom{N+l-1}{l} x^l$$

is the unique polynomial of degree $N-1$, such that $p(x)+p(1-x)=1$ holds.

To be more specific let us reconsider the symmetrized B–spline of order $2N$, the corresponding symbol obeys ($z=e^{-i\xi}, x=\cos^2(\xi_1/2)$)

$$b(z) \ = \ \frac{1}{2^{2N-1}}\left(z_1^{-1}(1+z_1)^2\right)^N \ = \ 2\cos^{2N}(\xi/2) \ = \ 2\ x^N \ .$$

The above procedure for $N=2$, i.e. the B–spline of order 4, yields the correction term

$$q(1-x) \ = \ 1+2(1-x) \ = \ 3-2\cos^2(\xi_1/2) \ = \ 2-\cos(\xi_1) \ = \ -0.5\ z_1^{-1}+2-0.5\ z_1 \ .$$

Hence we regain the examples of the previous section for $\alpha=(0,0)^T$. For the dilation matrix M_b we thus obtain a family with arbitrarily high regularity as N tends to infinity.

In order to construct a family with arbitrary high regularity for the quincunx dilation matrix M_q one has to start with the basic Laplace mask ($x=(\cos^2(\xi_1/2)+\cos^2(\xi_2/2))/2$)

$$b(z) \ = \ (4+z_2+z_1+z_2^{-1}+z_1^{-1})/4 \ = \ \cos^2(\xi_1/2)+\cos^2(\xi_2/2) \ = 2x.$$

The interpolation property of this mask was already observed in [11], its regularity was estimated in [6]. It yields a refinable function with Hoelder exponent $\alpha>0.61$. For higher regularity one iterates the Laplace scheme:

$$b^N(z) \ = \ \frac{1}{2^{N-1}}\left((4+z_2+z_1+z_2^{-1}+z_1^{-1})/4\right)^N \ = \ 2x^N \ .$$

Let us choose e.g. $N=3$, then $q(x)=1+3y+6y^2$. This yields

$$p(x)=x^3q(1-x)=x^3(10-15x+6x^2)$$

or equivalently $a(z)=2p(x)$ with $x=(\cos^2(\xi_1/2)+\cos^2(\xi_2/2))/2$.

The regularity of the related refinable functions was estimated in [6]. We thus obtain a family with arbitrarily high regularity as N tends to infinity for the dilation matrix M_q.

Remark 4.1 *The Bezout techniques do not directly extend to other dilation matrices, hence the q-term construction is more general.*

5 Wavelets

Once we have found an interpolating refinable function, a corresponding wavelet basis can be easily constructed. There are actually two possibilities. One way is to construct a classical pre–wavelet basis. This can be done by performing a general procedure as e.g. explained in [21]. One has to find an extension of the row $(a_{\rho_0}(z),\ldots,a_{\rho_{m-1}}(z))$ over $\mathcal{T}^d$ which is always possible if the underlying refinable functions has linear independent translates. (Observe that this condition is satisfied in our case since φ is fundamental). However, in the interpolating case, there is another very natural method to construct a generalized wavelet basis by using a different scalar product. We only briefly explain the basic ideas, for the general setting, the reader is referred to Chui and Li [4]. If φ is fundamental, it can be written as

$$\varphi(x) = \varphi(Mx) + \sum_{\rho\in R\backslash\{0\}} \sum_{k\in\mathbb{Z}^d} \varphi(k+M^{-1}\rho)\varphi(M(x-k)-\rho). \tag{5.1}$$

Therefore, by defining

$$\psi^\rho(x) := \varphi(Mx-\rho), \qquad \rho\in R\backslash\{0\}, \tag{5.2}$$

we obtain

$$\varphi(Mx) = \varphi(x) - \sum_{\rho\in R\backslash\{0\}} \sum_{k\in\mathbb{Z}^d} \varphi(k+M^{-1}\rho)\psi^\rho(x-k). \tag{5.3}$$

Now we set

$$V_j := \left\{ \sum_{k\in\mathbb{Z}^d} c_k\varphi(M^j\cdot -k): \quad \{c_k\}_{k\in\mathbb{Z}^d}\in\ell_\infty(\mathbb{Z}^d)\right\}, \tag{5.4}$$

$$W_j^\rho := \left\{ \sum_{k\in\mathbb{Z}^d} d_k^\rho\psi^\rho(M^j\cdot -k): \quad \{d_k^\rho\}_{k\in\mathbb{Z}^d}\in\ell_\infty(\mathbb{Z}^d)\right\}, \tag{5.5}$$

and obtain an direct sum decomposition

$$V_{j+1} = V_j \dot{+} W_j^{\rho_1} \dot{+} \cdots \dot{+} W_j^{\rho_{m-1}}. \tag{5.6}$$

Therefore, the system $\{\psi^\rho\}_{\rho\in R\backslash\{0\}}$ can be interpreted as a generalized wavelet basis.

From the definition (5.2), it is clear that the interpolating property of the generator φ carries over to the wavelet basis in the sense that

$$\psi^\rho(k+M^{-1}\rho) = \delta_{k,0}, \tag{5.7}$$

i.e., each wavelet interpolates on a shifted grid. On the other hand, we have to pay for this very convenient property by the loss of vanishing moments.

References

[1] S. Bertoluzza and G. Naldi, A wavelet collocation method for the numerical solution of partial differential equations, *Appl. Comput. Harmonic Anal.* 3 (1996), 1–9.

[2] G. Beylkin and J.M. Keiser, An adaptive pseudo–wavelet approach for solving nonlinear partial differential equations, Preprint.

[3] C.K. Chui and C. Li, A general framework of multivariate wavelets with duals, *Appl. Comput. Harmonic Anal.* 1 (1994), 368–390.

[4] C.K. Chui and C. Li, Multivariate interpolating wavelets, *in:* Approximation Theory VIII, Vol 2: "Wavelets and Multilevel Approximation", (C.K. Chui, L.L. Schumaker, Eds.), World Scientific, 1995, 9–16.

[5] A. Cohen and I. Daubechies, Non–separable bidimensional wavelet bases, *Rev. Mat. Iberoam.* 9 (1993), 51–137.

[6] A. Cohen and I. Daubechies, A new technique to estimate the regularity of refinable functions, Preprint Nr. 95/12, CEREMADE.

[7] S. Dahlke, W. Dahmen, and V. Latour, Smooth refinable functions and wavelets obtained by convolution products, *Appl. Comput. Harmonic Anal.* 2 (1995), 68–84.

[8] S. Dahlke and V. Latour, A note on the linear independence of characteristic functions of self–similar sets, *Arch. Math.* 66 (1996), 80–88.

[9] S. Dahlke, V. Latour, and M. Neeb, Generalized cardinal B–splines: stability, linear independence, and appropriate scaling matrices, *Constr. Approx.* 13 (1997), 29–56.

[10] W. Dahmen and A. Kunoth, Multilevel preconditioning, *Numer. Math.* 63 (1992), 315–344.

[11] G. Deslaurier and S. Dubuc, Interpolation dyadique, *in:* "Fractals, Dimensions non Entières et Applications", (G. Cherbit, Ed.), Masson, Paris, 1987, 44–45.

[12] G. Deslaurier, J. Dubois, and S. Dubuc, Multidimensional iterative interpolation, *Can. J. Math* 43 (1991), 297–312.

[13] R. DeVore, B. Jawerth, and B. Lucier, Image compression through wavelet transform coding, *IEEE Trans. Inform. Th.* 38, 719–746.

[14] D. Donoho, Interpolating wavelet transforms, Preprint.

[15] N. Dyn, J. Gregory, and D. Levin, A butterfly subdivision scheme for surface interpolation with tension control, *ACM Trans. on Graphics*, 9 (1990), 160–169.

[16] N. Dyn and D. Levin, Interpolatory subdivision schemes for the generation of curves and surfaces, *in:* "Multivariate Approximation and Interpolation", (W. Haussmann and K. Jetter, Eds.), Birkhauser Verlag, Basel, 1990, 91–106.

[17] N. Dyn, D. Levin, and C. Micchelli, Using parameters to increase smoothness of curves and surfaces generated by subdivision, *Comp. Aided Geometric Design* 7 (1990), 91–106.

[18] K. Gröchenig and A. Haas, Self–similar lattice tilings, to appear in: *J. Fourier Anal. Appl.*

[19] K. Gröchenig and W.R. Madych, Multiresolution analysis, Haar bases and self-similar tilings of $\mathbb{R}^n$, *IEEE Trans. Inform. Th.* 38 (2) (1992), 556–568.

[20] R.Q. Jia and C.A. Micchelli, On the linear independence of integer translates of a finite number of functions, Preprint.

[21] R.Q. Jia and C.A. Micchelli, Using the refinement equations for the construction of Pre-wavelets II: Powers of two, *in:* "Curves and Surfaces", (P.J. Laurent, A. Le Méhauté and L.L. Schumaker, Eds.), Academic Press, New York, (1991).

[22] J.C. Lagarias and Y. Wang, Integral self–affine tiles in $\mathbb{R}^n$ I. Standard and nonstandard digit sets, to appear.

[23] W. Lawton, S.L. Lee, and Z. Shen, Stability and orthonormality of multivariate refinable functions, Preprint.

[24] S.D. Riemenschneider and Z. Shen, Multidimensional interpolatory subdivision schemes, Preprint, University of Alberta, 1996.

[25] L. Villemoes, Continuity of nonseparable quincunx wavelets, MAT- Report Nr. 1992-13, (1992).

[26] L. Villemoes, Sobolev regularity of wavelets and stability of iterated filter banks, to appear.

[27] L. Villemoes, Wavelet analysis of two–scale difference equations, to appear.