A New Approach to Interpolating Scaling Functions

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Abstract

We present a new method to construct interpolating refinable functions in higher dimensions. The approach is based on the solutions to specific Lagrange interpolation problems by polynomials and applies to a large class of scaling matrices. The resulting scaling functions automatically satisfy certain Strang–Fix conditions. Several examples are discussed.

Key Words: Interpolating scaling functions, Strang–Fix conditions, polynomial interpolation, expanding scaling matrices.

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1 Introduction

In recent years, the construction of interpolating scaling functions has become a field of increasing importance. In general, a function $\phi \in L_2(\mathbb{R}^d)$ is called a *scaling* function or a *refinable* function if it satisfies a *two–scale–relation*

$$\phi(x) = \sum_{k \in \mathbb{Z}^d} a_k \phi(Ax - k), \qquad \mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d), \tag{1.1}$$

where A is an *expanding* scaling matrix on $\mathbb{Z}^d$. This means that A has integer entries and all its eigenvalues have modulus larger than one. Refinable functions play an important role for the construction of a multiresolution analysis and an associated wavelet basis, see, e.g., the books of Chui [3], Daubechies [10], and Meyer [17]. They are also frequently used in computer aided geometric design in connection with subdivision algorithms, see Cavaretta, Dahmen, and Micchelli [2]. For several practical reasons, it is often convenient to work with *interpolating* scaling functions, i.e., in addition to (1.1) one requires that ϕ is at least continuous and satisfies

$$\phi(k) = \delta_{0,k}, \qquad k \in \mathbb{Z}^d. \tag{1.2}$$

Furthermore, the function ϕ should be sufficiently smooth and well–located. In recent studies, several examples of refinable functions satisfying these conditions have been constructed, see e.g. [9], [11], [12] and [18]. In this paper, we present a new approach which is based on the usual Lagrange interpolation by polynomials. Our method yields compactly supported functions and has the advantage that Strang–Fix conditions of a certain order automatically hold. This is important since the Strang–Fix conditions always serve as indicators for a certain smoothness. The construction can be interpreted as *one* natural generalization of one–dimensional concepts, see Section 2 for details.

This paper is organized as follows. In Section 2, we briefly recall the setting of interpolating scaling functions. Moreover, we describe different approaches to find interpolating refinable functions in $L_2(\mathbb{R})$ for scalings by powers of two and discuss their possible generalizations to arbitrary scaling matrices. In Section 3, we introduce generalized Strang–Fix conditions which are suitable for our purposes. Furthermore, we describe an algorithm to find symbols satisfying these conditions. Section 4 is devoted to the construction of the associated scaling functions, and, finally, in Section 5 we discuss some examples to explain the applicability of our approach.

For later use, let us fix some notation. Let $q = |\det A|$. Furthermore, let

$$R = \{\rho_0, \ldots, \rho_{q-1}\}, \;\; R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{q-1}\} \tag{1.3}$$

denote complete sets of representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$ and $\mathbb{Z}^d / B\mathbb{Z}^d$, $B = A^T$, respectively. Without loss of generality, we shall always assume that $\rho_0 = \tilde{\rho}_0 = 0$.

2 The General Construction of Scaling Functions

In the sequel, we shall only consider compactly supported scaling functions. Moreover, we shall always assume that $\operatorname{supp} \mathbf{a} := \{k \in \mathbb{Z}^d \mid a_k \neq 0\}$ is finite. We write $B = A^T$

and apply the Fourier transform to (1.1) to obtain

$$\hat{\phi}(\omega) = \frac{1}{q} \sum_{k \in \mathbb{Z}^d} a_k e^{-2\pi i \langle k, B^{-1}\omega \rangle} \hat{\phi}(B^{-1}\omega). \tag{2.4}$$

By iterating (2.4) we obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega), \tag{2.5}$$

where the *symbol* $m(\omega)$ is defined by

$$m(\omega) := \frac{1}{q} \sum_{k \in \mathbb{Z}^d} a_k e^{-2\pi i \langle k, \omega \rangle}. \tag{2.6}$$

Equation (2.5) means that instead of trying to construct a refinable function directly we may also start with a symbol $m(\omega)$. Then the question arises which conditions imply that $\hat{\phi}$ in (2.5) is well–defined in $L_2(\mathbb{R}^d)$ and has some additional desirable properties such as sufficient smoothness. Moreover, for our purposes, we have to clarify how the interpolating property (1.2) can be guaranteed. Some sufficient conditions are summarized in the following theorem which is due to Lemarié in dimension 1 [15, 16] and extends to higher dimensions without change.

Theorem 2.1 *Let $m(\omega)$ be a trigonometric polynomial which satisfies*

(C1) $m(0) = 1$;

(C2) $m(\omega) \geq 0$;

(C3) $\sum_{\tilde{\rho} \in R^T} m(\omega + B^{-1}\tilde{\rho}) = 1$, *where R^T is defined in* (1.3);

(C4) $m(\omega)$ *satisfies Cohens's condition.*

Then $m(\omega)$ is a symbol of an interpolating refinable function ϕ.

Remark 2.1 *i) Cohens's condition is a technical condition which requires the existence of a compact set K which contains a neighborhood of the origin and satisfies*

- $\bigcup_{l \in \mathbb{Z}^d} (l + K) = \mathbb{R}^d$
- $K \bigcap (l + K) \simeq \emptyset$, *whenever* $l \neq 0$,

such that

$$m(B^{-j}\omega) \neq 0 \quad \textit{for all} \quad \omega \in K \quad \textit{and} \quad j \geq 1, \ j \in \mathbb{Z}^d.$$

ii) (C1) *is clearly necessary for the pointwise convergence of the right–hand side in* (2.5).

iii) One may actually dispense with condition (C2)*. However,* (C2) *is convenient since it implies that $\hat{\phi} \in L_1(I\!R^d)$ by an argument similar to Theorem 2 in* [5]*, in particular ϕ is continuous. Moreover,* (C2) *makes it sometimes much simpler to check the inconvenient condition* (C4)*. Furthermore,* (C2) *has to be imposed if one wants to use $m(\omega)$ for the construction of orthonormal wavelets, see* [10] *for details.*

iv) Condition (C3) *follows easily from* (1.2) *by combining the Poisson summation formula with* (2.4)*, see, e.g.,* [9, 16, 18] *for details.*

In general, one wants to find scaling functions with a certain smoothness. To this end, one often requires that the *Strang–Fix conditions* are satisfied, i.e.,

$$\text{(C5)} \qquad \left(\frac{\partial}{\partial\omega}\right)^l m(B^{-1}\tilde{\rho}) = 0 \qquad \text{for all} \quad |l| \leq L-1 \quad \text{and all} \quad \tilde{\rho} \in R^T\backslash\{0\}.$$

In dimension 1, (C5) implies a factorization

$$m(\omega) = \left(\frac{1+e^{-2\pi i\omega}}{2}\right)^L b(\omega), \tag{2.7}$$

and in this case such a factorization is a necessary condition for $\phi \in H^s,\ s > L$. In the multivariate case, the relations between Strang–Fix conditions and regularity are not so easy, but (C5) always serves as an indicator for regularity. For a further discussion of this topic, the reader is referred to [8].

In this paper we derive an algorithm for the construction of symbols satisfying (C1)–(C5). To understand the underlying idea, we briefly recall the univariate situation. There one has to find a non–negative trigonometric polynomial $m(\omega)$ satisfying

$$m(\omega) + m(\omega+1/2) = 1, \tag{2.8}$$
$$m^{(l)}(1/2) = 0, \quad 0 \leq l \leq 2L-1\ . \tag{2.9}$$

To our knowledge, there exist five different approaches to solve this problem.

I. A substitution $y = \sin^2(\pi\omega)$ leads to the Bezout problem

$$(1-y)^L P(y) + y^L P(1-y) = 1,$$

which can be solved explicitly. The solutions are of the form

$$P(y) = P_L(y) + y^L R(1/2-y),$$

where R is an odd polynomial and

$$P_L(y) = \sum_{k=0}^{L-1} \binom{L-1+k}{k} y^k.$$

This method was derived by I. Daubechies [10].

II. Make the ansatz

$$m(\omega) = 1 - c_L \int_0^{\omega} \sin^{2L-1}(2\pi\omega) d\omega$$

and choose c_L such that $m(1/2) = 0$ [16, 17].

III. The conditions (2.8) and (2.9) give $2L$ linear equations for the coefficients of m, which can be solved directly.

IV. The simplest solution is

$$\cos^2(\pi\omega) + \sin^2(\pi\omega) = 1.$$

Take the $(2L-1)$–th power of this equation and rearrange the terms in a clever way. This yields the solution P_L described in I., see [19, 16].

V. Let $\ell_n(x), n = -L, -L+1, \ldots, L-1$, denote the fundamental Lagrange interpolation polynomials on the nodes $-L, \ldots, L-1$, i.e., $\ell_n(k) = \delta_{k,n}$. Then

$$m(\omega) = \frac{1}{2} + \frac{1}{2} \sum_{n=-L}^{L-1} \ell_n(-1/2) e^{-2\pi i(2n+1)\omega}$$

is a solution of (2.8) and (2.9) [14].

In dimension $d = 1$ these approaches are equivalent, because the trigonometric polynomial of minimal degree satisfying the conditions (2.8,2.9) is unique. The result of these methods are the Daubechies polynomials, compare with [10].

It seems quite natural to ask if one of these approaches can be generalized to higher dimensions and to arbitrary scaling matrices. The first method is not suitable since in higher dimensions one does not necessarily have a canonical factorization. For the third method, the technical difficulties increase alarmingly. There has already been a successful attempt to generalize the fourth method, see [11] for details. In this paper, we focus on the last approach. We show that this method can indeed be generalized in a natural way to a large class of scaling matrices. The second method clearly also has some potential since it avoids the factorization problem and will be studied in a forthcoming paper.

In contrast to the one-dimensional situation, in higher dimensions the approaches IV. and V. do not seem to be equivalent and produce different interpolating scaling functions.

3 The Strang–Fix Conditions

We first establish the connection between Lagrange interpolation and the Strang–Fix conditions in higher dimensions.

We say that a symbol $m(\omega)$ satisfies the *Strang–Fix conditions* with respect to a set of polynomials Π, if $(D = \frac{\partial}{\partial\omega})$

$$(p(D)m)(B^{-1}\tilde{\rho}) = 0 \quad \text{for all} \quad p \in \Pi, \ \tilde{\rho} \in R^T \backslash \{0\}. \tag{3.10}$$

First we explain a general method for the construction of symbols satisfying general Strang–Fix conditions, which is based on the usual Lagrange interpolation by polynomials. This is a natural generalization of the fifth method described in the previous section. For any subset $\mathcal{T} \subseteq \mathbb{Z}^d$, $\Pi_{\mathcal{T}}$ will always denote a finite–dimensional subspace of polynomials such that the Lagrange interpolation problem with respect to $\mathcal{T}$ is uniquely solvable. This means that for all $k \in \mathcal{T}$ there exists a unique $p_k \in \Pi_{\mathcal{T}}$ such that

$$p_k(j) = \delta_{j,k}, \qquad \text{for all} \quad j \in \mathcal{T}. \tag{3.11}$$

Consequently every $p \in \Pi_{\mathcal{T}}$ can be written as

$$p(x) = \sum_{k\in\mathcal{T}} p(k)p_k(x). \tag{3.12}$$

Under this standing hypothesis we show the following theorem.

Theorem 3.1 *Let $\mathcal{P}$ be a subspace of $\Pi_{\mathcal{T}}$ satisfying*

(1) If $p \in \mathcal{P}$, then $p\,(c(Ax+\rho)) \in \Pi_{\mathcal{T}}$ for $c \in \mathbb{C}$, $\rho \in R$;

(2) $p(0) = 0$ for all $p \in \mathcal{P}$.

Then the symbol $m(\omega)$ defined by

$$m(\omega) = \frac{1}{q} + \frac{1}{q}\sum_{k\in\mathcal{T}}\sum_{\rho\in R\setminus\{0\}} p_k(-A^{-1}\rho)e^{-2\pi i\langle Ak+\rho,\omega\rangle} \tag{3.13}$$

satisfies (C1), (C3)*, and the Strang–Fix conditions* (3.10) *with respect to $\mathcal{P}$.*

Proof: Since $\tilde\rho \to e^{-2\pi i\langle\rho,B^{-1}\tilde\rho\rangle}$ is a character of the Abelian group $\mathbb{Z}^d/B\mathbb{Z}^d$, a well–known lemma about character sums says that

$$\frac{1}{q}\sum_{\tilde\rho\in R^T} e^{-2\pi i\langle\rho,B^{-1}\tilde\rho\rangle} = 0 \quad \text{if } \rho \in R\setminus\{0\}. \tag{3.14}$$

Conditions (C1) and (C3) follow easily. Let us first verify (C3):

$$\begin{aligned}
\sum_{\tilde\rho\in R^T} m(\omega+B^{-1}\tilde\rho) &= 1 + \frac{1}{q}\sum_{\tilde\rho\in R^T}\sum_{k\in\mathcal{T}}\sum_{\rho\in R\setminus\{0\}} p_k(-A^{-1}\rho)e^{-2\pi i\langle Ak+\rho,\omega+B^{-1}\tilde\rho\rangle}\\
&= 1 + \frac{1}{q}\sum_{k\in\mathcal{T}}\sum_{\rho\in R\setminus\{0\}}\left(\sum_{\tilde\rho\in R^T} e^{-2\pi i\langle\rho,B^{-1}\tilde\rho\rangle}\right) p_k(-A^{-1}\rho)e^{-2\pi i\langle Ak+\rho,\omega\rangle}\\
&= 1 + \sum_{k\in\mathcal{T}}\sum_{\rho\in R\setminus\{0\}} \delta_{\rho,0}p_k(-A^{-1}\rho)e^{-2\pi i\langle Ak+\rho,\omega\rangle} = 1\,.
\end{aligned}$$

Furthermore, for $\tilde\rho \neq 0$ we obtain the Strang–Fix conditions of order 1 as follows

$$m(B^{-1}\tilde\rho) = \frac{1}{q} + \frac{1}{q}\sum_{\rho\in R\setminus\{0\}} e^{-2\pi i\langle\rho,B^{-1}\tilde\rho\rangle}\sum_{k\in\mathcal{T}} p_k(-A^{-1}\rho) = \frac{1}{q} + \frac{1}{q}\sum_{\rho\in R\setminus\{0\}} e^{-2\pi i\langle\rho,B^{-1}\tilde\rho\rangle}.$$

Applying (3.14) with B replaced by A gives for $\tilde{\rho} \neq 0$

$$m(B^{-1}\tilde{\rho}) \;=\; \frac{1}{q} + \frac{1}{q} \sum_{\rho \in R \setminus \{0\}} e^{-2\pi i \langle \rho, B^{-1}\tilde{\rho} \rangle} = 0.$$

Consequently from (C3) with $\omega = 0$ we have $m(0) = 1$, and (C1) is shown.

It remains to check the Strang–Fix conditions of higher order. We obtain

$$\begin{aligned}(p(D)m)(\omega) &= \frac{1}{q} \sum_{k \in \mathcal{T}} \sum_{\rho \in R \setminus \{0\}} p_k(-A^{-1}\rho)(p(D)e^{-2\pi i \langle Ak+\rho, \cdot \rangle})(\omega) \\ &= \frac{1}{q} \sum_{\rho \in R \setminus \{0\}} \sum_{k \in \mathcal{T}} p_k(-A^{-1}\rho) p(-2\pi i(Ak+\rho)) e^{-2\pi i \langle Ak+\rho, \omega \rangle},\end{aligned}$$

and therefore

$$\begin{aligned}(p(D)m)(B^{-1}\tilde{\rho}) &= \frac{1}{q} \sum_{\rho \in R \setminus \{0\}} \sum_{k \in \mathcal{T}} p_k(-A^{-1}\rho) p(-2\pi i(Ak+\rho)) e^{-2\pi i \langle Ak+\rho, B^{-1}\tilde{\rho} \rangle} \\ &= \frac{1}{q} \sum_{\rho \in R \setminus \{0\}} e^{-2\pi i \langle \rho, B^{-1}\tilde{\rho} \rangle} \sum_{k \in \mathcal{T}} p_k(-A^{-1}\rho) p(-2\pi i(Ak+\rho)).\end{aligned}$$

By hypothesis (1), $p(-2\pi i(A \cdot + \rho)) \in \Pi_{\mathcal{T}}$, therefore the interpolation property (3.12) and hypothesis (2) imply that

$$\sum_{k \in \mathcal{T}} p_k(-A^{-1}\rho) p(-2\pi i(Ak+\rho)) = p(-2\pi i(A \cdot + \rho))|_{-A^{-1}\rho} = p(0) = 0.$$

$\square$

Remark 3.1 *If $\mathcal{P}$ is spanned by a set of monomials, then condition (1) may be replaced by*

(1') whenever $p \in \mathcal{P}$, then $p(Ax+\rho) \in \Pi_{\mathcal{T}}$, $\rho \in R$,

and we may only consider spaces of real polynomials.

Since Lagrange interpolation on general sets of nodes in $\mathbb{R}^d$ is far from understood, see [1, 4] for contributions, we restrict ourselves to very simple sets with additional symmetry. Let $\mathcal{T}$ consist of all lattice points in a cube in $\mathbb{R}^d$, i.e., for $L \in \mathbb{N}$ and $a \in \mathbb{Z}^d$ we set

$$\mathcal{T} = \mathcal{T}_{L,a} := \{k \in \mathbb{Z}^d : \; a_i \leq k_i \leq L + a_i, \quad i = 1, \ldots, d\} = (a + [0,L]^d) \cap \mathbb{Z}^d. \quad (3.15)$$

The Lagrange interpolation problem is always unisolvable on $\mathcal{T}$ by the polynomial subspace

$$\Pi_{\mathcal{T}} = \operatorname{span}\{x^k, \; k \in \mathbb{Z}^d, \|k\|_\infty \leq L\}. \quad (3.16)$$

The fundamental Lagrange interpolants are simply tensor products of the univariate Lagrange polynomials and can be written explicitly as

$$p_k(x) = \ell_{k_1}(x_1)\ell_{k_2}(x_2)\cdots\ell_{k_d}(x_d), \qquad \ell_{k_i}(x_i) := \prod_{n=a_i, n \neq k_i}^{L+a_i} \frac{x_i - n}{k_i - n}. \quad (3.17)$$

This leads to the following corollary.

Corollary 3.1 *Let $\mathcal{T}$ and $\Pi_{\mathcal{T}}$ be defined by* (3.15) *and* (3.16)*, respectively. Then $m(\omega)$ defined by* (3.13) *satisfies the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}}$. In particular, the usual Strang–Fix conditions of order $L+1$ are satisfied.*

Proof: We apply Theorem 3.1 to the subspace

$$\tilde{\Pi}_{\mathcal{T}} := \operatorname{span}\{x^k \mid k \in \mathbb{Z}^d,\ 1 \le \|k\|_\infty \le L\}. \tag{3.18}$$

□

4 Interpolating Scaling Functions

In this section, we show that under certain conditions $m(\omega)$ defined by (3.13), (3.15) and (3.16) is indeed a natural candidate for a symbol of an interpolating refinable function.

Theorem 4.1 *Assume that the scaling matrix A satisfies one of the following conditions:*

i) $|\det A|$ *is odd;*

ii) $|\det A|$ *is even and $\mathbb{Z}^d/A\mathbb{Z}^d$ is a cyclic group of order* $|\det A|$.

Then $\mathcal{T}_{a,L}$ defined by (3.15) *can be chosen such that the symbol $m(\omega)$ in* (3.13) *and* (3.16) *is real–valued and satisfies* (C1), (C3), *and the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}_{a,L}}$.*

Proof: It was already shown in Theorem 3.1 and Corollary 3.1 that $m(\omega)$ satisfies (C1), (C3), and the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}_{a,L}}$. To obtain a *real* symbol, $\mathcal{T}$ has to satisfy additional symmetry conditions. This is where the form of the dilation matrix comes into play. The trigonometric polynomial $m(\omega)$ is real–valued if and only if the coefficients a_k, see (2.6), satisfy

$$a_{Ak+\rho} = \overline{a}_{-Ak-\rho}, \qquad k \in \mathbb{Z}^d,\ \rho \in R.$$

If we write the representative $-\rho$ as $-\rho = \rho' + Ak_\rho$ for $\rho, \rho' \in R$ and suitable $k_\rho \in \mathbb{Z}^d$, then this condition reduces to

$$p_k(-A^{-1}\rho) = a_{Ak+\rho} = \overline{a}_{-Ak-\rho} = p_{-k+k_\rho}(-A^{-1}\rho') \tag{4.19}$$

for $k \in \mathcal{T}$ and $\rho \in R$. Thus $k \in \mathcal{T}$ if and only if $-k + k_\rho \in \mathcal{T}$, or in other words

$$\mathcal{T} = -\mathcal{T} + k_\rho \quad \text{for all } \rho \in R. \tag{4.20}$$

If some k_ρ's were distinct, then $\mathcal{T}$ would have to be symmetric about two points, which is impossible for a finite set. To avoid this, we choose an appropriate set of representatives. By [13], Lemma 15.4, there exists a basis $\{e_i, i = 1, \ldots, d\}$ for $\mathbb{Z}^d$ and integers q_i, such that $\{q_i e_i, i = 1, \ldots, d\}$ is a basis for $A\mathbb{Z}^d$ and $|\prod_{i=1}^d q_i| = |\det A|$.

If $|\det A|$ is odd, then all q_i's are odd. Thus we can choose R to be symmetric about the origin as

$$R = \{\rho = \sum_{i=1}^{d} l_i e_i, -\left[\frac{q_i}{2}\right] \leq l_i \leq \left[\frac{q_i}{2}\right]\}.$$

If $\rho \in R$, then $-\rho \in R$ and thus $k_\rho = 0$ for all $\rho \in R$. Choosing $\mathcal{T} = [-L, L]^d \cap \mathbb{Z}^d$, we find that $p_k(x) = p_{-k}(-x)$ and therefore condition (4.19) is satisfied. By construction the associated trigonometric polynomial is real-valued.

If $|\det A|$ is even, then in general we cannot choose a symmetric R and the above construction does not work. However, if q is even and if $\mathbb{Z}^d/A\mathbb{Z}^d$ is a cyclic group, then the construction goes through. For in this case $q_1 = q$ and $q_i = 1$ for $i = 2, \ldots, d$, and we chose

$$R = \{\rho = je_1, j = 0, \ldots, q-1\}.$$

Then $-je_1 = (q-j)e_1 - qe_1$ and $\kappa = k_\rho = -qA^{-1}e_1 \in \mathbb{Z}^d$ is independent of ρ. The symmetry $\mathcal{T} = -\mathcal{T} + \kappa$ can be achieved by the choice

$$\mathcal{T} = \prod_{i=1}^{d}[-L + \frac{\kappa_i - \epsilon_i}{2}, L + \frac{\kappa_i + \epsilon_i}{2}] \cap \mathbb{Z}^d,$$

where $\epsilon_i = 0$, if κ_i is even, and $\epsilon_i = 1$, if κ_i is odd. Then the trigonometric polynomial associated to $\mathcal{T}$ by (3.13) is real-valued, satisfies (C1), (C3), and the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}}$.

$\square$

5 Examples

So far, we have shown that the symbols defined by (3.13) are good candidates for symbols of interpolating refinable functions. They satisfy the Strang–Fix conditions and (C1) and (C3) hold. Moreover, under certain restrictions, the symbols are also real. In this section we construct some symbols explicitly for two important dilation matrices in dimension 2. For these examples we show that all necessary conditions are satisfied and obtain a smoothness estimate for the scaling function.

Example 5.1 Let us first consider $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $q = 2$ and a set of representatives is given by $\rho_0 = 0, \rho_1 = \binom{1}{0}$. Obviously $-A^{-1}\binom{1}{0} = \binom{-1/2}{1/2}$ and $\mathcal{T}$ needs to be symmetric about $(-1/2, 1/2)$. This is the case for $\mathcal{T} = [-L, L-1] \times [-L+1, L] \cap \mathbb{Z}^2$. Let ℓ_n denote the basic Lagrange interpolating polynomial for $n \in \{-L, -L+1, .., L-1\}$ and $\tilde{\ell}_n$ is the basic Lagrange interpolating polynomial for $n \in \{-L+1, -L+2, .., L\}$, then $\tilde{\ell}_{n+1}(1/2) = \ell_n(-1/2)$. With

$$q_L(x) := \sum_{n=-L}^{L-1} \ell_n(-1/2) e^{-2\pi i n x}, \tag{5.21}$$

we obtain for m corresponding to (3.13)

$$\begin{aligned}
m(\omega) &= \frac{1}{2} + \frac{1}{2}\sum_{k\in\mathcal{T}} p_k(-A^{-1}\rho_1)e^{-2\pi i\langle Ak+\rho_1,\omega\rangle} \\
&= \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega_1}\sum_{k\in\mathcal{T}} p_k(-1/2,1/2)e^{-2\pi i\langle k,B\omega\rangle} \\
&= \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega_1}\sum_{k_1=-L}^{L-1}\sum_{k_2=-L+1}^{L} \ell_{k_1}(-1/2)\tilde{\ell}_{k_2}(1/2)e^{-2\pi i k_1(\omega_1+\omega_2)}e^{-2\pi i k_2(\omega_2-\omega_1)} \\
&= \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega_1}\left(\sum_{k_1=-L}^{L-1}\ell_{k_1}(-1/2)e^{-2\pi i k_1(\omega_1+\omega_2)}\right)\cdot \\
&\qquad\qquad\left(e^{-2\pi i(\omega_2-\omega_1)}\sum_{k_2=-L}^{L-1}\tilde{\ell}_{k_2+1}(1/2)e^{-2\pi i k_2(\omega_2-\omega_1)}\right) \\
&= \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega_2}q_L(\omega_1+\omega_2)q_L(\omega_2-\omega_1) \\
&= \frac{1}{2} + \frac{1}{2}e^{-2\pi i(\omega_1+\omega_2)/2}q_L(\omega_1+\omega_2)e^{-2\pi i(\omega_2-\omega_1)/2}q_L(\omega_2-\omega_1). \qquad (5.22)
\end{aligned}$$

It follows from Corollary 3.1 that m satisfies the Strang–Fix conditions with respect to $\Pi_{\mathcal{T}}$, and in particular, the usual Strang–Fix conditions of order $2L$ are satisfied.

To show that m is always non–negative, we note that

$$\mu(\omega) = \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega}\sum_{n=-L}^{L-1}\ell_n(-1/2)e^{-2\pi i 2n\omega} = \frac{1}{2} + \frac{1}{2}e^{-2\pi i\omega}q_L(2\omega)$$

is a trigonometric polynomial of order $2L-1$ which satisfies (2.8) and (2.9). Since this polynomial is uniquely determined, see for instance [10], it coincides with Daubechies' solution $\mu(\omega) = \cos^{2L}\pi\omega\sum_{k=0}^{L-1}\binom{L-1+k}{k}\sin^{2k}(\pi\omega)$. This implies that $|q_L(\omega)| \leq 1$ and $e^{-\pi i\omega}q_L(\omega) = 1$ if and only if $\omega \in 2\mathbb{Z}$ and $e^{-\pi i\omega}q_L(\omega) = -1$, if and only if $\omega \in 1+2\mathbb{Z}$. Consequently, $m(\omega) \geq \frac{1}{2} - \frac{1}{2}\sup_{\omega\in R}|q_L(\omega)|^2 \geq 0$ is non-negative.

To check Cohen's condition, we need to know the zeros of m. $m(\omega) = 0$, if and only if $e^{-\pi i(\omega_1+\omega_2)}q_L(\omega_1+\omega_2) = \pm 1$ and $e^{-\pi i(\omega_2-\omega_1)}q_L(\omega_2-\omega_1) = \mp 1$, which is the case, if and only if $(\omega_1,\omega_2) \in (\pm 1/2, 1/2) + A\mathbb{Z}^2$. It is well-known that a trigonometric polynomial with such a zero set satisfies Cohen's condition, see e.g. [7].

For $L = 2$ we obtain explicitly

$$q_2(x) = \frac{1}{16}\left(-e^{4\pi i x} + 9e^{2\pi i x} + 9 - e^{-2\pi i x}\right), \qquad (5.23)$$

and the nonvanishing coefficients of the resulting mask, see Figure 1, can be computed as follows.

$$a_{(0,0)} = \frac{1}{2}; \qquad (5.24)$$

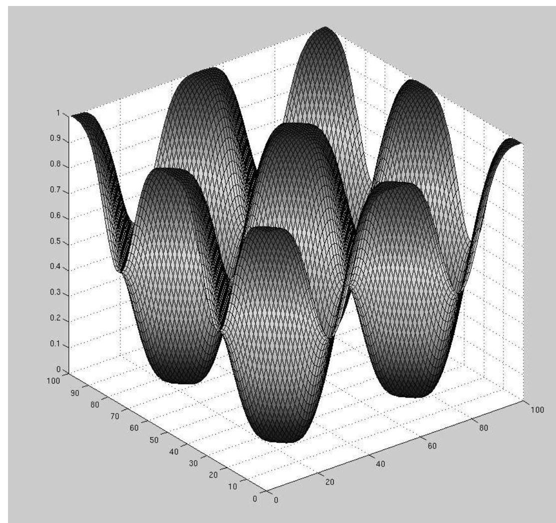
$$
\begin{aligned}
a_{(1,0)} &= a_{(0,1)} = a_{(-1,0)} = a_{(0,-1)} = \frac{81}{512};\\
a_{(3,0)} &= a_{(0,3)} = a_{(-3,0)} = a_{(0,-3)} = \frac{1}{512};\\
a_{(2,1)} &= a_{(1,2)} = a_{(-1,2)} = a_{(-2,1)} = a_{(-2,-1)} = a_{(-1,-2)} = a_{(1,-2)} = a_{(2,-1)} = -\frac{9}{512}.
\end{aligned}
$$

The corresponding symbol $m(\omega_1, \omega_2)$ is depicted in Figure 2.

Figure 1: coefficient mask ($L = 2$):

			$\frac{1}{512}$			
		$\frac{-9}{512}$		$\frac{-9}{512}$		
	$\frac{-9}{512}$		$\frac{81}{512}$		$\frac{-9}{512}$	
$\frac{1}{512}$		$\frac{81}{512}$	$\frac{1}{2}$	$\frac{81}{512}$		$\frac{1}{512}$
	$\frac{-9}{512}$		$\frac{81}{512}$		$\frac{-9}{512}$	
		$\frac{-9}{512}$		$\frac{-9}{512}$		
			$\frac{1}{512}$			

Figure 2: symbol $m(\omega_1, \omega_2)$



It remains to estimate the smoothness of the resulting refinable function. For $\phi \in H^{s-1}$, it is sufficient to establish an estimate of the form

$$
\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega) \leq C(1 + \|\omega\|)^{-s-\epsilon} \qquad \text{for some} \quad \epsilon > 0. \tag{5.25}
$$

To compute the infinite product in (5.25), we want to use a suitable factorization, i.e., we try to find a symbol $b(\omega)$ such that

$$
m(\omega) = b(\omega)c(\omega), \tag{5.26}
$$

where $c(\omega)$ is some 'nice' trigonometric series which tends to increase the decay. We choose

$$
c(\omega) = \left(\frac{\sin^2(\pi(\omega_1 + \omega_2)) + \sin^2(\pi(\omega_2 - \omega_1))}{2(\sin^2(\pi\omega_1) + \sin^2(\pi\omega_2))} \right)^2. \tag{5.27}
$$

The decay of the corresponding infinite product has been computed by Cohen and Daubechies [6]:

$$
\prod_{j=1}^{\infty} c(B^{-j}\omega) \leq M(1 + \|\omega\|)^{-4}. \tag{5.28}
$$

It remains to estimate

$$\prod_{j=1}^{\infty} b(B^{-j}\omega) = \prod_{j=1}^{\infty} \frac{m(B^{-j}\omega)}{c(B^{-j}\omega)}. \tag{5.29}$$

Setting

$$\begin{aligned} b_n(\omega) &:= b(\omega)b(B\omega)\cdots b(B^{n-1}\omega) \\ \beta_n &:= \sup_{\omega} b_n(\omega), \end{aligned}$$

standard computations show that

$$\prod_{j=1}^{\infty} b(B^{-j}\omega) \leq N(1+\|\omega\|)^{\frac{2\log\beta_n}{n\log 2}},$$

and therefore

$$\hat{\phi}(\omega) \leq C(1+\|\xi\|)^{-4+\frac{2\log\beta_n}{n\log 2}}.$$

The value β_n can be estimated numerically by plotting the regularity function b_n. As an example, Figure 3 displays b_3. It turns out that $\beta_3 \leq 5$, and therefore

$$\phi \in H^s, \quad s < 3 - \frac{2\log\beta_3}{3\log 2} \sim 1.452.$$

Figure 4 shows the resulting interpolating refinable function.

Figure 3: regularity function $b_3(\omega)$

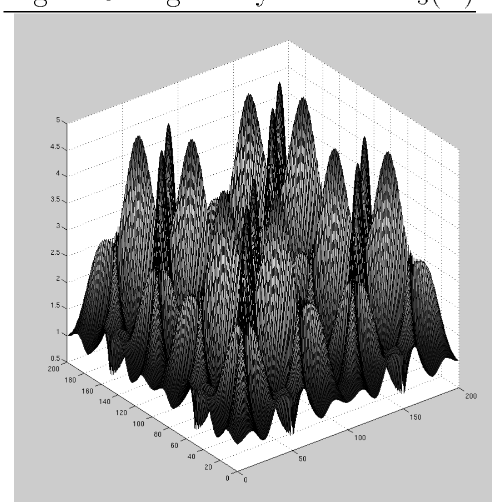
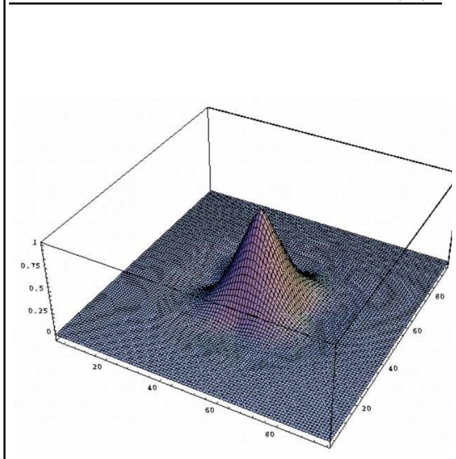


Figure 4: scaling function $\varphi(x)$:



Remark 5.1 *If we perform a similar calculation as in Example 1 for* $L=1$*, we obtain the mask*

$$a_{(0,1)} = a_{(1,0)} = a_{(-1,0)} = a_{(0,-1)} = \frac{1}{8}, \quad a_{(0,0)} = \frac{1}{2}.$$

This mask has also been studied in [6].

Example 5.2 For the second example, let us consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$. This matrix arises from the similarity matrix $\begin{pmatrix} 3/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{pmatrix}$ on the hexagonal lattice spanned by $(1,0)$ and $(1/2, \sqrt{3}/2)$ by a coordinate transform to $\mathbb{Z}^2$.

In this case, $q = 3$ and a canonical symmetric set of representatives according to Theorem 4.1 is given by $R = \{\binom{0}{0}, \binom{0}{1}, \binom{0}{-1}\}$. To apply the construction from above, we have to choose the set $\mathcal{T}$ symmetric with respect to the origin. We choose $L = 2$. Then

$$\begin{aligned} m(\omega) &= \frac{1}{3} + \frac{1}{3} \sum_{\rho \in R \setminus \{0\}} e^{-2\pi i \langle \rho, \omega \rangle} \sum_{k \in \mathcal{T}} p_k(-A^{-1}\rho) e^{-2\pi i \langle k, B\omega \rangle} \\ &= \frac{1}{3} + \frac{1}{3} e^{-2\pi i \omega_2} P_{\binom{0}{1}}(B\omega) + \frac{1}{3} e^{2\pi i \omega_2} P_{\binom{0}{-1}}(B\omega). \end{aligned}$$

With $-A^{-1}\binom{0}{1} = -\binom{1/3}{1/3}$, $-A^{-1}\binom{0}{-1} = \binom{1/3}{1/3}$, the polynomials $P_{\binom{0}{1}}$ and $P_{\binom{0}{-1}}$ can be easily computed. We obtain

$$\begin{aligned} P_{\binom{0}{1}}(x,y) &= \left(\frac{2}{9} e^{2\pi i x} + \frac{8}{9} - \frac{1}{9} e^{-2\pi i x}\right) \left(\frac{2}{9} e^{2\pi i y} + \frac{8}{9} - \frac{1}{9} e^{-2\pi i y}\right), \\ P_{\binom{0}{-1}}(x,y) &= \left(-\frac{1}{9} e^{2\pi i x} + \frac{8}{9} + \frac{2}{9} e^{-2\pi i x}\right) \left(-\frac{1}{9} e^{2\pi i y} + \frac{8}{9} + \frac{2}{9} e^{-2\pi i y}\right), \end{aligned} \tag{5.30}$$

and therefore the nonvanishing coefficients of $m(\omega)$ are given by

$$\begin{aligned} a_{(0,0)} &= \frac{1}{3}; \\ a_{(0,4)} &= a_{(0,-4)} = \frac{1}{243}; \\ a_{(-2,2)} &= a_{(2,0)} = a_{(2,-2)} = a_{(-2,0)} = -\frac{2}{243}; \\ a_{(0,-2)} &= a_{(0,2)} = \frac{4}{243}; \\ a_{(-1,3)} &= a_{(1,2)} = a_{(-1,-2)} = a_{(1,-3)} = -\frac{8}{243}; \\ a_{(-1,0)} &= a_{(1,-1)} = a_{(-1,1)} = a_{(1,0)} = \frac{16}{243}; \\ a_{(0,1)} &= a_{(0,-1)} = \frac{64}{243}. \end{aligned}$$

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References

[1] C. de Boor and A. Ron, The least solution of the polynomial interpolation problem, *Math. Z.* **210** (1992), 347–378.

[2] A.S. Cavaretta, W. Dahmen, and C.A. Micchelli, Stationary Subdivision, *Mem. Amer. Math. Soc.* **93** (1991), 1–186.

[3] C.K. Chui, *An Introduction to Wavelets*, Academic Press, Boston, (1992).

[4] K.C. Chung and T.H. Yao, On lattices admitting unique Lagrange interpolations, *SIAM J. Numer. Anal.* **14** (1977), 735–743.

[5] A. Cohen, Ondelettes, analyses multirésolutions et filtres miroirs en quadrature, *Ann. Inst. Poincaré* **7** (1990), 439–459.

[6] A. Cohen and I. Daubechies, Non–separable bidimensional wavelet bases, *Rev. Mat. Iberoamericana* **9** (1993), 51–137.

[7] A. Cohen and I. Daubechies, A new technique to estimate the regularity of refinable functions, *Rev. Mat. Iberoamericana* **12** (1996), 527–591.

[8] A. Cohen, K. Gröchenig, and L. Villemoes, Regularity of multivariate refinable functions, to appear in: *Constr. Approx.*

[9] S. Dahlke and P. Maass, Interpolating refinable functions and wavelets for general scalings, *Numer. Funct. Anal. Optim.* **18(5&6)** (1997), 521–539.

[10] I. Daubechies, *Ten Lectures on Wavelets*, CBMS–NSF Regional Conference Series in Applied Math. **61**, SIAM, Philadelphia, (1992).

[11] J. Derado, Multivariate refinable interpolating functions, Preprint, University of Connecticut, (1997).

[12] G. Deslaurier, J. Dubois, and S. Dubuc, Multidimensional iterative interpolation, *Can. J. Math.* **43** (1991), 297–312.

[13] L. Fuchs, *Abelian Groups*, Academic Press, (1970).

[14] M. Holschneider, *Wavelets: An Analysis Tool*, Oxford Univ. Press, Oxford, (1995).

[15] J.–P. Kahane and P.–G. Lemarié–Rieusset, *Fourier Series and Wavelets*, Gordon and Breach Science Publishers, Luxembourg, (1995).

[16] P.–G. Lemarié–Rieusset, Polynômes de Bernstein en théorie des ondelettes, *C. R. Acad. Sci. Paris* **319** (1994), 21–24.

[17] Y. Meyer, *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics, **vol. 37**, Cambridge, (1992).

[18] S.D. Riemenschneider and Z. Shen, Multidimensional interpolatory subdivision schemes, Preprint, University of Alberta, (1996).

[19] R. Strichartz, Construction of orthonormal wavelets, In "Wavelets: Mathematics and Applications", (J.J. Benedetto and M.W. Frazier, eds.), CRC Press, Boca Raton, (1994), 23–50.