Besov Regularity for Interface Problems

Stephan Dahlke*
Institut für Geometrie
und Praktische Mathematik
RWTH Aachen
Templergraben 55
52056 Aachen
Germany

Abstract

This paper is concerned with the Besov regularity of the solutions to interface problems in a segment S of the unit disk in $\mathbf{R}^2$. We investigate the smoothness of the solutions as measured in the specific scale $B^s_\tau(L_\tau(S))$, $1/\tau = s/2+1/p$, of Besov spaces which determines the order of approximation that can be achieved by adaptive and nonlinear numerical schemes. The proofs are based on representations of the solution spaces which were derived by Kellogg [15] and on characterizations of Besov spaces by wavelet expansions.

Key Words: Interface problems, adaptive methods, nonlinear approximation, Besov spaces, wavelets.

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1 Introduction

In recent years, the use of adaptive schemes has become a widespread strategy in numerical analysis. In particular, adaptive algorithms have been successfully implemented for the numerical treatment of boundary value problems of the form

$$\begin{aligned} Au &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^d, \\ Bu &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

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where A is a second order elliptic differential operator and B reflects the boundary conditions, see, e.g., [1, 2, 3, 21]. As usual, we study (1.1) in the weak formulation

$$a(u,v) = (f,v), \qquad v \in H^1_B(\Omega), \tag{1.2}$$

where $a(\cdot,\cdot)$ is the bilinear form induced by (1.1) and $H^1_B(\Omega)$ is a suitable subspace of $H^1(\Omega)$ which depends on the boundary conditions. On the one hand, computational studies indicate that in many cases adaptive schemes for (1.2) are indeed superior when compared with nonadaptive methods. On the other hand, from a theoretical point of view, a rigorous foundation for the use of adaptive algorithms is still in its infancy. However, this seems to be an important issue, especially since, in the realm of complexity theory, theoretical results tend to be pessimistic [19]. In this note, we try to provide some arguments that justify the use of adaptive schemes for (1.2), at least for the special case of the interface problem in two independent variables. The setting can be described as follows. Let (r,θ) be polar coordinates in $\mathbf{R}^2$ and let S be a segment of the disc of radius r_0 in $\mathbf{R}^2$ given by $0 < \theta < \theta_M$. Furthermore, let p be a function on S which is a function of θ alone. We assume that S is divided into sectors $S_i\,:\ \theta_{i-1} < \theta < \theta_i,\ 1 \leq i \leq M$, in each of which p takes a positive constant value. Moreover, let there be given on $\overline{S}$ a continuously differentiable, symmetric, positive definite second order matrix function $(a_{l,m}(x))_{l,m=1,2}$, and a bounded nonnegative function $a_0(x)$. Then, the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_S (\sum_{l,m=1}^{2} p a_{l,m} u_l v_m + a_0 uv)dx, \quad u,v \in H^1_0(S),$$

and, given $f \in L_2(S)$, we study the problem

$$\int_S (\sum_{l,m=1}^{2} p a_{l,m} u_l v_m + a_0 uv)dx = \int_S fv dx, \quad \text{for all} \quad v \in H^1_0(S). \tag{1.3}$$

It is well–known that (1.3) has a unique solution $u \in H^1_0(S)$. Equations of this form are clearly important in practice since a lot of problems in physics and mechanics are modelled by (1.3); u may represent for instance a temperature or the displacement of a membrane. According to the setting introduced above, we include here also the case that the data, e.g., the heat conductivity, have discontinuities.

In general, the numerical treatment of (1.2) (and therefore also of (1.3)) is performed by means of a Galerkin approach, i.e., the problem is projected onto an increasing sequence of linear approximation spaces such as usual finite element spaces based on uniform grid refinement, where we assume that the union of the linear spaces is dense in $H^1_B(\Omega)$. Since the approximation comes from linear spaces, the usual Galerkin approach can be interpreted as some kind of *linear approximation*. It is well–known that the order of approximation for linear methods to recover the solution u of (1.2) is determined by the regularity of u in the usual Sobolev scale $H^s(\Omega),\ s \geq 1$, see, e.g., [7, 20] for a further discussion. If the domain Ω, the right–hand side f and the coefficients of the operator A are smooth, then this Sobolev regularity is sufficiently high so that linear methods are appropriate, see [14, 17]. The situation changes completely in the nonsmooth case, for then the Sobolev regularity decreases significantly, see, e.g., [13], and the order of convergence for linear methods drops down. One possible remedy is to use adaptive schemes. Then an underlying grid is refined only in regions where the approximation is still 'far away' from the exact solution u. Therefore one does not use the whole linear approximation spaces but only suitable parts, so that an adaptive scheme can be interpreted as some kind of *nonlinear approximation.* In general, the order of convergence that can be achieved by nonlinear methods is not determined by the Sobolev but by the Besov regularity as we shall now explain. In nonlinear approximation, a function $F \in L_p(\mathbf{R}^d)$ is approximated by the elements of a nonlinear manifold $\mathcal{M}_n$ of dimension n. We consider the error

$$\sigma_n(F)_{L_p(\mathbf{R}^d)} := \inf_{G \in \mathcal{M}_n} \|F - G\|_{L_p(\mathbf{R}^d)}. \tag{1.4}$$

In many cases, the following characterization holds:

$$\sum_{n=1}^{\infty} [n^{s/d} \sigma_n(F)_{L_p(\mathbf{R}^d)}]^{\tau} \frac{1}{n} < \infty \Longleftrightarrow F \in B^s_{\tau}(L_{\tau}(\mathbf{R}^d)),\ \tau = (s/d + 1/p)^{-1}, \tag{1.5}$$

where $B^s_{\tau}(L_{\tau}(\mathbf{R}^d))$ are the *Besov* spaces (see Section 2 for the definition of Besov spaces). In particular, (1.5) holds in the case of nonlinear *wavelet* approximation, see [10] for details. Let $\{\eta_I,\ I \in D, \eta \in \Psi\}$ be an orthonormal wavelet basis of $L_2(\mathbf{R}^d)$ as described in Section 2. Then, in nonlinear wavelet approximation, the nonlinear manifold $\mathcal{M}_n$ consists of all functions

$$G = \sum_{(I,\eta) \in \Lambda} a_{I,\eta} \eta_I$$

with $\Lambda \subset D \times \Psi$ of cardinality n.

Having the characterization (1.5) in mind, it is now natural to ask the following question: what is the regularity of the solution u to (1.2) as measured in the specific scale of Besov spaces $B^s_{\tau}(L_{\tau}(\Omega)),\ \tau = (s/d + 1/p)^{-1}$, and does it have a higher smoothness order s when compared to the usual Sobolev scale $H^s(\Omega),\ s \geq 1$? For then, adaptive schemes can indeed perform better than nonadaptive methods, in principle. (In this paper, we are primarily interested in the order of approximation with respect to L_2. Therefore we shall mainly be concerned with the specific scale $B^s_{\tau}(L_{\tau}(\Omega)),\ 1/\tau = s/d + 1/2$.) In [4, 5, 6, 8] these questions have been studied for operators with smooth coefficients in Lipschitz domains. It has turned out that in many cases the smoothness index s for the Besov scale $B^s_{\tau}(L_{\tau}(\Omega)),\ \tau = (s/d + 1/p)^{-1}$, is indeed much higher than the one for the usual Sobolev scale $H^s(\Omega)$ so that adaptive methods are justified. In this paper, we establish a similar result for the interface problem (1.3). Obviously, the smoothness index for the solution u to (1.3) in the usual Sobolev scale is at least one. However, since in our case the coefficients are discontinuous at the interfaces $\theta = \theta_i$, there is no hope to obtain a much high smoothness order, even for smooth right–hand sides, and linear methods cannot perform satisfactorily. Therefore adaptive schemes based, e.g., on local grid refinement in the vicinity of the interfaces seem to be appropriate. As explained above, such an approach is justified if the regularity in the specific Besov scale is higher than the Sobolev regularity. As we shall see in Section 3, Theorem 3.1, this is indeed the case. It turns out that the solution u is in fact contained in the Besov spaces $B^s_\tau(L_\tau(S)), \tau = (s/2 + 1/2)^{-1}$, for all $s < 2$. The proof of Theorem 3.1 is based on wavelet analysis, i.e., we use the fact that smoothness spaces can be characterized by wavelet expansions. Therefore, in the next section, we briefly summarize some of the basic concepts used in wavelet analysis, and we focus on the characterizations of Besov spaces.

2 Wavelets and Besov Spaces

In this section, we first recall some facts from wavelet analysis. Then we define the Besov spaces and give their characterization in terms of wavelet decompositions.

In general, a function $\psi \in L_2(\mathbf{R})$ is called an orthonormal *wavelet* if all its scaled, dilated and integer translated versions,

$$\psi_{j,k}(\cdot) := 2^{j/2}\psi(2^j \cdot -k), \qquad j,k \in \mathbf{Z}, \tag{2.1}$$

form an orthonormal basis of $L_2(\mathbf{R})$. We shall only need the univariate family $D_N,\ N = 1, 2, \ldots$ of compactly supported wavelets as constructed by I. Daubechies [9]. The smoothness of D_N increases without bound as N tends to infinity, as does the support of D_N. The wavelet D_N has N vanishing moments, i.e.,

$$\int_{\mathbf{R}} x^\beta D_N(x)dx = 0, \qquad \beta = 0, \ldots, N-1. \tag{2.2}$$

With the aid of the univariate wavelets D_N, multivariate orthonormal bases can be constructed as follows. We fix an arbitrary value of N and let $\phi = \phi_N$ be the univariate scaling function which generates the wavelet $\psi = D_N$. We define $\psi^0 := \phi$ and $\psi^1 := \psi$. Further, let E denote the nontrivial vertices of the square $[0,1]^d$. Then, the set Ψ of the $2^d - 1$ functions

$$\psi^e(x_1, \ldots, x_d) := \prod_{j=1}^d \psi^{e_j}(x_j), \quad e \in E, \tag{2.3}$$

generate by shifts and dilates an orthonormal (wavelet) basis for $L_2(\mathbf{R}^d)$. Namely, let $D := D(\mathbf{R}^d)$ denote the set of dyadic cubes in $\mathbf{R}^d$. Each cube $I \in D$ is of the form $I = 2^{-j}k + 2^{-j}[0,1]^d$ with $k \in \mathbf{Z}^d$, $j \in \mathbf{Z}$. The functions

$$\eta_I := \eta_{j,k} := 2^{jd/2}\eta(2^j \cdot -k),\ I = 2^{-j}k + 2^{-j}[0,1]^d,\ k \in \mathbf{Z}^d, j \in \mathbf{Z}, \eta \in \Psi, \tag{2.4}$$

form an orthonormal basis for $L_2(\mathbf{R}^d)$. Therefore, each $F \in L_2(\mathbf{R}^d)$ has the wavelet decomposition

$$F = \sum_{I \in D} \sum_{\eta \in \Psi} \langle F, \eta_I \rangle \eta_I. \tag{2.5}$$

We can also restrict the wavelet expansion (2.5) to those η_I with $|I| \le 1$. For this, we define V_0 to be the closure in $L_2(\mathbf{R}^d)$ of the finite linear combinations of the integer shifts of the function $\phi(x_1)\cdots\phi(x_d)$ and let P_0 be the orthogonal projector which maps $L_2(\mathbf{R}^d)$ onto V_0. Then, for each $F \in L_2(\mathbf{R}^d)$, we have

$$F = P_0(f) + \sum_{I\in D^+}\sum_{\eta\in\Psi}\langle F, \eta_I\rangle \eta_I \tag{2.6}$$

with D^+ the set of dyadic cubes with measure ≤ 1.

It is one of the most important features of wavelet analysis that wavelet expansions can be used to characterize function spaces such as Besov spaces. Before we state a characterization result which is suitable for our purposes, let us briefly recall the definition of Besov spaces. If $h \in \mathbf{R}^d$, we denote by Ω_h the set of all $x \in \Omega$ such that the line segment $[x, x+h]$ is contained in Ω. The modulus of smoothness $\omega_r(F,t)_{L_p(\Omega)}$ of a function $F \in L_p(\Omega)$, $0 < p \le \infty$, is defined by

$$\omega_r(F,t)_{L_p(\Omega)} := \sup_{|h|\le t}\|\Delta_h^r(F,\cdot)\|_{L_p(\Omega_{rh})}, \quad t > 0,$$

with Δ_h^r the r–th difference with step h. For $s > 0$ and $0 < q, p \le \infty$, the Besov space $B_q^s(L_p(\Omega))$ is defined as the space of all functions F for which

$$|F|_{B_q^s(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty [t^{-s}\omega_r(F,t)_{L_p(\Omega)}]^q dt/t\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t\ge 0} t^{-s}\omega_r(F,t)_{L_p(\Omega)}, & q = \infty\ , \end{cases} \tag{2.7}$$

is finite with $r := [s]+1$. Then, (2.7) is a semi–(quasi)norm for $B_q^s(L_p(\Omega))$. If we add $\|f\|_{L_p(\Omega)}$ to (2.7), we obtain a (quasi)norm for $B_q^s(L_p(\Omega))$.

At least on all of $\mathbf{R}^d$, the Besov spaces $B_q^s(L_p(\mathbf{R}^d))$ can be characterized by wavelet coefficients, provided the parameters s and p satisfy certain restrictions. As already stated above, we are mainly interested in the spaces $B_\tau^s(L_\tau(\mathbf{R}^d))$, $\tau = (s/d + 1/2)^{-1}$. For this specific scale, the following result holds, see [12, 16, 18] for details.

Proposition 2.1 *Let ϕ and ψ be in $C^r(\mathbf{R})$, $r > s$. Then a function F is in the Besov space $B_\tau^s(L_\tau(\mathbf{R}^d))$, $\tau = (s/d+1/2)^{-1}$, if and only if,*

$$F = P_0(F) + \sum_{I\in D^+}\sum_{\eta\in\Psi}\langle F, \eta_I\rangle \eta_I \tag{2.8}$$

with

$$\|P_0(F)\|_{L_\tau(\mathbf{R}^d)} + \left(\sum_{I\in D^+}\sum_{\eta\in\Psi}|\langle F,\eta_I\rangle|^\tau\right)^{1/\tau} < \infty \tag{2.9}$$

and (2.9) *provides an equivalent (quasi)norm for $B_\tau^s(L_\tau(\mathbf{R}^d))$.*

3 Regularity in Besov Spaces

As explained above, the Sobolev regularity of the solution u to (1.3) will in general not be very high. This difficulty is caused by the fact that we are dealing with nonsmooth coefficients. In this section, we prove the main result of this paper which says that, in contrary to the Sobolev regularity, the disontinuities of p do not diminish the regularity of u in the Besov scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/2 + 1/2$. As we shall see below, this is a consequence of the fact that the discontinuities of p only occur on a 'thin' set.

Theorem 3.1 *The solution u to* (1.3) *satisfies*

$$u \in B^s_\tau(L_\tau(S)), \quad 0 < s < 2, \quad \frac{1}{\tau} = \frac{s}{2} + \frac{1}{2}. \tag{3.1}$$

Proof: We want to prove the theorem by using Proposition 2.1. To this end, we have to estimate the wavelet coefficients of u. The first step is to extend u to all of $\mathbf{R}^2$. This is clearly possible since the domain is minimally smooth. We denote this extended version also by u. Since the functions $\eta \in \Psi$ are compactly supported, supp η_I is contained in a cube $Q(I)$ satisfying $|Q(I)| \lesssim |I|$. (In this paper, ' $\lesssim$ ' indicates inequality up to constant factors). Let Γ denote the set of all $(I, \eta) \in D^+$ such that $Q(I)$ has a nontrivial intersection with S. Then, on S, u has an expression

$$u = P_0 u + \sum_{(I,\eta)\in\Gamma} \langle u, \eta_I \rangle \eta_I. \tag{3.2}$$

Consequently, we have to show that

$$\sum_{(I,\eta)\in\Gamma} |\langle u, \eta_I \rangle|^\tau < \infty. \tag{3.3}$$

Let Σ denote the skeleton produced by the interfaces, that is,

$$\Sigma := \partial S \bigcup_{i=1}^{N} J_i, \quad J_i := \{(r, \theta) \in \mathbf{R}^2,\ \theta = \theta_i,\ 0 \leq r \leq r_0\}. \tag{3.4}$$

We fix a refinement level by defining the sets

$$\Lambda_j := \{(I, \eta) \in \Gamma \mid |I| = 2^{-2j}\}. \tag{3.5}$$

Then we collect all indices in Λ_j for which the support of the corresponding wavelet intersects the skeleton,

$$\Lambda_j^{skel} := \{(I, \eta) \in \Lambda_j \mid Q(I) \cap \Sigma \neq \emptyset\} \tag{3.6}$$

and set

$$\Lambda_j^{int} := \Lambda_j \backslash \Lambda_j^{skel}. \tag{3.7}$$

We treat the families $\{\Lambda_j^{skel}, j \geq 1\}$ and $\{\Lambda_j^{int}, j \geq 1\}$ separately and start with showing that

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^{skel}} |\langle u, \eta_I\rangle|^\tau < \infty \tag{3.8}$$

for all τ as in the statement of the theorem. This can be performed by following the lines of the proof of Theorem 3.2 in [8]. We want to exploit the fact that not only the Besov spaces but also the Sobolev spaces can be characterized by wavelet expansions. Especially, a function F is contained in $H^s(\mathbf{R}^d)$ if and only if

$$\|P_0(F)\|_{L_2(\mathbf{R}^d)} + \left(\sum_{I\in D^+}\sum_{\eta\in\Psi} 2^{2sj}|\langle F, \eta_I\rangle|^2\right)^{1/2} < \infty. \tag{3.9}$$

For the proof of (3.9) the reader is, e.g., referred to the book of Meyer [18]. If we use Hölders's inequality and the fact that

$$|\Lambda_j^{skel}| \lesssim 2^j,$$

we obtain

$$\begin{aligned}\sum_{(I,\eta)\in\Lambda_j^{skel}} |\langle u, \eta_I\rangle|^\tau &\lesssim 2^{j(1-\tau/2)}\left(\sum_{(I,\eta)\in\Lambda_j^{skel}} |\langle u, \eta_I\rangle|^2\right)^{\tau/2} \\ &\lesssim 2^{j(1-\tau/2)}2^{-j\tau}\left(\sum_{(I,\eta)\in\Lambda_j^{skel}} 2^{2j}|\langle u, \eta_I\rangle|^2\right)^{\tau/2}.\end{aligned}$$

Therefore, using Hölder's inequality for another time yields

$$\sum_{j=0}^{\infty}\sum_{(I,\eta)\in\Lambda_j^{skel}} |\langle u, \eta_I\rangle|^\tau \lesssim \left(\sum_{j=0}^{\infty}\sum_{(I,\eta)\in\Lambda_j^{skel}} 2^{2j}|\langle u, \eta_I\rangle|^2\right)^{\frac{\tau}{2}} \left(\sum_{j=0}^{\infty} 2^{j-\frac{2j\tau}{2-\tau}}\right)^{\frac{2-\tau}{2}}.$$

Since u is contained in H^1, the first sum is finite by (3.9), and the second sum is finite if and only if

$$1 - \frac{2\tau}{2-\tau} < 0, \quad \text{i.e.,} \quad \tau > \frac{3}{2},$$

which corresponds to $s < 2$.

The treatment of the sets Λ_j^{int} is a little bit more involved. First of all, we have to introduce the following function spaces. Let $\mathcal{W}_0(S,p)$ be the set of all functions $v \in H_0^1(S)$ which are continuous in S, which have all derivatives of order up to and including the third uniformly continuous in each region $\varepsilon < r < r_0$, $\theta_{i-1} \leq \theta \leq \theta_i$, for each $\varepsilon > 0$, which satisfy $rD^2v \longrightarrow 0$ as $r \longrightarrow 0$ for each second derivative D^2v of v, and which satisfy the interface conditions

$$\begin{aligned} u(r,\theta_i - 0) &= u(r,\theta_i + 0), \qquad 1 \leq i \leq M, \\ p(\theta - 0)u_\theta(r,\theta_i - 0) &= p(\theta + 0), u_\theta(r,\theta_i + 0), \quad 1 \leq i \leq M, \end{aligned} \tag{3.10}$$

for $u = v$ and $u = v_r$. We denote by $\mathcal{W}(S,p)$ the closure of $\mathcal{W}_0(S,p)$ in $H^1_0(S)$. In addition, let us consider the Sturm–Liouville problem

$$\begin{aligned} p\zeta'' + p\lambda\zeta &= 0, \quad \theta \neq \theta_i, \\ \zeta(0) &= \zeta(\theta_M) = 0, \\ \zeta(\theta_i - 0) &= \zeta(\theta_i + 0), \quad 1 \leq i \leq M, \\ p(\theta_i - 0)\zeta'(\theta_i - 0) &= p(\theta_i + 0)\zeta'(\theta_i + 0), \quad 1 \leq i \leq M. \end{aligned} \tag{3.11}$$

It is well–known that (3.11) has a countable number of eigenvalues $\lambda_1 \leq \lambda_2 \leq ..., \lambda_l > 0$. We denote the corresponding eigenfunctions by ζ_n. Moreover, we set $\gamma_n = \lambda_n^{1/2}$, $v_n = r^{\gamma_n}\zeta_n(\theta)$, and we denote by $\vartheta(r)$ a suitable C^∞ truncation function. Furthermore, let $\mathcal{H}(S,p)$ be the linear span of the funtion ϑv_n with $\gamma_n < 1$. The following theorem was shown in [15]:

Theorem 3.2 *Let $\mathcal{D}(S,p)$ denote the set of all solutions u to* (1.3) *as f varies over $L_2(S)$. Then*

$$\mathcal{D}(S,p) = \mathcal{H}(S,p) + \mathcal{W}(S,p).$$

According to Theorem 3.2, u can be decomposed as

$$u = u_1 + u_2, \qquad u_1 \in \mathcal{H}(S,p),\ u_2 \in \mathcal{W}(S,p). \tag{3.12}$$

We are left with showing

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^{int}} |\langle u_1, \eta_I\rangle|^\tau < \infty \tag{3.13}$$

and

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^{int}} |\langle u_2, \eta_I\rangle|^\tau < \infty. \tag{3.14}$$

We start with establishing (3.14). Let $\Lambda_j^{int,i}$ denote the set of all indices for which the support of the corresponding wavelet is contained in the sector S_i,

$$\Lambda_j^{int,i} := \{(I,\eta) \in \Lambda_j^{int} \mid Q(I) \subset S_i\},\ i = 1,\ldots,M. \tag{3.15}$$

We may treat each of the sets $\Lambda_j^{int,i}$ separately. Since $u_2 \in \mathcal{W}(S,p)$ we have

$$u_2|_{S_i} \in H^2(S_i), \quad i = 1,\ldots,M, \tag{3.16}$$

see again [15] for details. On S_i, we have the embedding $H^2(S_i) \hookrightarrow B^s_\tau(L_\tau(S_i))$, $1/\tau = s/2+1/2$, $s < 2$. Moreover, there exists a (nonlinear) extension operator $\mathcal{E}_{S_i} :\ B^s_\tau(L_\tau(S_i)) \longrightarrow B^s_\tau(L_\tau(\mathbf{R}^2))$, see [11]. Hence, if we apply Proposition 2.1 to the extended version of $u_2|_{S_i}$, it follows that indeed

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^{int,i}} |\langle u_2, \eta_I\rangle|^\tau < \infty,$$

proving (3.14).

It remains to study the part u_1. Since $\mathcal{H}(S,p)$ is finite dimensional, it is sufficient to establish Besov regularity for each of the functions v_n, i.e., we have to show that

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^{int,i}} |\langle v_n, \eta_I\rangle|^\tau < \infty \tag{3.17}$$

holds for all $v_n \in \mathcal{H}(S,p)$ and all τ as in the statement of the theorem. The main difficulty is caused by the fact that a typical function v_n is not contained in $H^2(S_i)$ due to singularities at the origin so that more subtle estimations are necessary. One possible way is to use similar arguments as stated in [4]. Let

$$\delta_I := \inf_{x\in Q(I)} r(x) \tag{3.18}$$

denote the distance of the cube $Q(I)$ to the origin. We cover the subset of the domain S_i corresponding to $\Lambda_j^{int,i}$ by layers of squares by defining the sets

$$\Lambda_{j,k}^{int,i} := \{(I,\eta)\in\Lambda_j^{int,i} \mid k2^{-j} \le \delta_I < (k+1)2^{-j}\}. \tag{3.19}$$

Then one has

$$|\Lambda_{j,k}^{int,i}| \lesssim k. \tag{3.20}$$

Furthermore, by using polar coordinates, it is easy to check that

$$|v_n|_{W^m(L_\infty(Q(I)))} \lesssim \delta_I^{\gamma_n - m}, \tag{3.21}$$

so that a classical Whitney-type estimate yields

$$\begin{aligned} |\langle v_n, \eta_I\rangle| &\lesssim 2^{-j(m+1)}|v_n|_{W^m(L_\infty(Q(I)))} \\ &\lesssim 2^{-j(m+1)}\delta_I^{\gamma_n-m}. \end{aligned} \tag{3.22}$$

Hence, by combining (3.20) and (3.22) we obtain

$$\begin{aligned} \sum_{(I,\eta)\in\Lambda_j^{int,i}} |\langle v_n,\eta_I\rangle|^\tau &\lesssim \sum_{k=1}^{\infty}\sum_{(I,\eta)\in\Lambda_{j,k}^{int,i}} 2^{-j(m+1)\tau}\delta_I^{(\gamma_n-m)\tau} \\ &\lesssim \sum_{k=1}^{\infty} k\cdot 2^{-j(m+1)\tau}(k\cdot 2^{-j})^{(\gamma_n-m)\tau} \\ &\lesssim 2^{-j(1+\gamma_n)\tau}\sum_{k=1}^{\infty} k^{1+(\gamma_n-m)\tau}. \end{aligned}$$

Choosing m large enough, the series involving k is finite, therefore we are left with a geometric series which is clearly convergent for all τ as in the statement of the theorem. □

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