Besov Regularity for the Stokes Problem

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Abstract

This paper is concerned with regularity estimates for the solutions to the Stokes problem in polygonal domains in $\mathbf{R}^2$. Especially, we derive regularity results in specific scales of Besov spaces which arise in connection with adaptive numerical schemes. The proofs of the main results are based on representations of the solution spaces which were given by Osborn [20] and on characterizations of Besov spaces by wavelet expansions.

Key Words: Stokes problem, adaptive methods, nonlinear approximation, Besov spaces, wavelets.

AMS Subject classification: Primary 35B65, secondary 41A46, 46E35, 65N30.

1 Introduction

In recent years, much effort has been spent to design and to analyze adaptive schemes for the numerical treatment of elliptic boundary value problems of the form

$$\begin{aligned} Lu &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^d, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where L denotes a second order elliptic differential operator and Ω is a Lipschitz domain. Especially, the investigation of adaptive algorithms based on *wavelet expansions*

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has become a field of increasing importance [1, 7]. The general idea of adaptive schemes is to improve the performance of the numerical algorithm by using nonuniform grid or space refinements, respectively, i.e., the underlying approximation space is refined only in regions where the current approximation is still 'far away' from the exact solution u to (1.1). Although this strategy seems to be very plausible at first glance, the principal question arises if an adaptive scheme indeed provides some gain of efficiency when compared with uniform (nonadaptive) methods. It turns out that the answer to this question is related with the regularity properties of the solution u in (1.1) as we shall now explain very briefly. In general, an adaptive scheme can be interpreted as some kind of *nonlinear approximation*. It can be shown that for a function F in $L_2(\Omega)$ the order of approximation that can be achieved by a nonlinear wavelet method is determined by its regularity in the specifc scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$, of *Besov* spaces (see, e.g., [12, 22] for the definition and the main properties of Besov spaces). For a detailed description of these fundamental relationships and of its consequences for numerical schemes, the reader is referred, e.g., to [6, 10, 11]. In contrary to this, the efficiency of *uniform* methods is determined by the regularity of F in the usual Sobolev scale $H^s(\Omega)$. Therefore the following question arises: Does the solution u to (1.1) have a higher regularity in the scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$, of Besov spaces compared to the corresponding Sobolev scale? For then, adaptive algorithms can indeed perform better than uniform schemes, at least in principle.

Quite recently, several results in this direction have been shown [2, 3, 4, 5, 8, 16]. It has turned out that for many important cases the Besov regularity of the solution u is high enough to justify the use of adaptive schemes. The deepest results were obtained for problems on general Lipschitz domains where the Sobolev regularity decreases significantly due to singularities near the boundary [8]. This note can be interpreted as a continuation of the above studies. We shall be concerned with an important special case, i.e., with the *2D–Stokes problem*. Let Ω be a bounded, simply connected, polygonal domain in $\mathbf{R}^2$. Then, given a vector field $f \in H^{-1}(\Omega)^2$ and a function $g \in L_{2,0}(\Omega) := \{q \in L_2(\Omega) \ : \ \int_\Omega q(x)dx = 0\}$, one has to determine the velocity $u \in H^1_0(\Omega)^2$ and the pressure $p \in L_{2,0}(\Omega)$ such that

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } \Omega, \\ -\nabla \cdot u &= g \quad \text{in } \Omega. \end{aligned} \tag{1.2}$$

In the *mixed formulation*, the problem reads as follows: find a pair $(u,p) \in H^1_0(\Omega)^2 \times L_{2,0}(\Omega)$ such that

$$\begin{array}{lcll} a(u,v) + b(v,p) & = & \langle f, v\rangle & \text{for all } v \in H^1_0(\Omega)^2, \\ b(u,q) & = & \langle g, q\rangle & \text{for all } q \in L_{2,0}(\Omega), \end{array} \tag{1.3}$$

where

$$\begin{aligned} a(u,v) &:= (\nabla u, \nabla v) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)dx, \\ b(v,q) &:= -(\nabla \cdot v, q) = -\sum_{i=1}^{2} \int_\Omega q(x) \frac{\partial}{\partial x_i} v_i(x)dx. \end{aligned}$$

For further information concerning the theory and the numerical treatment of the Stokes equations, the reader is referred, e.g., to Girault and Raviart [14] and to Teman [21].

The main result of this paper shows that the Besov regularity of u and p, respectively, is again much higher than the Sobolev regularity, so that the use of adaptive schemes is completely justified. More precisely, it turns out that under some further technical conditions u and p have the optimal regularity in the interesting Besov scale, i.e., for $f \in H^m(\Omega)^2,\ g \in H^{m+1}(\Omega)$, one has $u \in B^s_\tau(L_\tau(\Omega))^2,\ s < m+2,\ p \in B^s_\tau(L_\tau(\Omega)),\ s < m+1,\ 1/\tau = s/2 + 1/2$.

2 A New Regularity Theorem

Our aim is to investigate the dependence of the regularity of the pair (u,p) in the scale $B^s_\tau(L_\tau(\Omega)),\ 1/\tau = s/2+1/2$, of Besov spaces on the smoothness of f and g and on the shape of the domain Ω. Before we can state our main result, some preparations are necessary. Let the segments of $\partial\Omega$ be denoted by $\overline{\Gamma}_l,\ \Gamma_l$ open, $l = 1,\ldots,N$, numbered in positive orientation. Furthermore, let S_l denote the endpoint of Γ_l. Let us now suppose that $f \in H^m(\Omega)^2$ and $g \in H^{m+1}(\Omega)$ for some $m \in \mathbf{N}$. By using the regularity theory for smooth domains, see, e.g., [18] for details, we first observe that $u \in H^{m+2}(\tilde{\Omega})^2,\ p \in H^{m+1}(\tilde{\Omega})$ for any subdomain $\tilde{\Omega}$ of Ω with smooth boundary not containing a vertex of Ω. Then the well–known embeddings of Besov spaces $H^\alpha(\Omega) = B^\alpha_2(L_2(\Omega)) \hookrightarrow B^s_\tau(L_\tau(\Omega)),\ s < \alpha,\ \tau < 2$, give the estimates

$$\begin{aligned} u \in B^s_\tau(L_\tau(\tilde{\Omega}))^2, &\qquad 1/\tau = s/2+1/2,\ s < m+2, \\ p \in B^s_\tau(L_\tau(\tilde{\Omega})), &\qquad 1/\tau = s/2+1/2,\ s < m+1. \end{aligned} \tag{2.1}$$

Therefore it remains to study the regularity of u and p near the vertices. By the usual decomposition technique using suitable C^∞ truncation functions, it turns out that u and p can be written as

$$u = u_I + u_B, \qquad u_B = \sum_{l=1}^N u_l, \tag{2.2}$$

$$p = p_I + p_B, \qquad p_B = \sum_{l=1}^N p_l, \tag{2.3}$$

where the functions u_l and p_l are supported in the neighbourhood of the vertex S_l and are solutions to a modified Stokes problem, see Osborn [20] for details. Since (u,p) equals (u_l,p_l) in the vicinity of S_l, we see that the study of p and u near the vertex S_l is reduced to the study of the Stokes problem in a sector. Therefore the remaining results in this paper will all be stated for the Stokes equation in a sector.

We need some further notations. By a change of coordinates, we may assume that the vertex S_l is placed at 0 and that one of the sides of the corresponding sector V lies on the positive x_1–axis. Let ω denote the measure of the interior angle of V. Furthermore, let λ_j denote one of the roots of the transcendental equation

$$v(z) := \sinh^2(z^2\omega) - z^2\sin^2(\omega) = 0, \tag{2.4}$$

which lie in the upper half plane. Moreover, m_j is defined to be the order of λ_j as a zero of $v(z)$. Finally, we define the weighted Sobolev space $W^{m,\alpha}(V)$ to be the set of all functions for which the following norm is finite:

$$\|w\|_{W^{m,\alpha}(V)} := \sum_{\nu=0}^{m} \int_V r^{\alpha-2(m-\nu)} \Big(\sum_{|\mu|=\nu} |D^\mu w|^2 \Big) dx, \qquad r := (x_1^2 + x_2^2)^{1/2}. \tag{2.5}$$

Then the main result reads as follows.

Theorem 2.1 *Suppose that* $f \in W_0^{m,0}(V)^2$, $g \in W_0^{m+1,0}(V)$ *and that no* λ_j *lies on the line* $\Im z = m+1$ *in the complex plane. Let* (u,p) *denote the solution to the Stokes problem*

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } V, \\ -\nabla \cdot u &= g \quad \text{in } V. \end{aligned} \tag{2.6}$$

Then

$$\begin{aligned} u &\in B^s_\tau(L_\tau(V))^2, \quad \text{for all} \quad s < m+2, \ 1/\tau = s/2 + 1/2, \\ p &\in B^s_\tau(L_\tau(V)), \quad \text{for all} \quad s < m+1, \ 1/\tau = s/2 + 1/2. \end{aligned}$$

Proof: The proof is based on the following characterization of the solution space to (2.6) which was derived by Osborn [20]. Similar results have also been obtained by Grisvard [15] and Kondrat'ev [17].

Theorem 2.2 *Suppose that the conditions of Theorem* 2.1 *are satisfied. Let* (r,θ) *denote polar coordinates in* V. *Then* u *and* p *have expansions* $u = u_R + u_S$, $p = p_R + p_S$, *where* $u_R \in W_0^{m+2,0}(V)^2$, $p_R \in W^{m+1,0}(V)$ *and*

$$u_S = \sum_{0<\Im\lambda_j<m+1} \sum_{l=0}^{m_j-1} C^u_{j,l}(\theta) r^{-i\lambda_j} \log^l(r), \tag{2.7}$$

$$p_S = \sum_{0<\Im\lambda_j<m+1} \sum_{l=0}^{m_j-1} C^p_{j,l}(\theta) r^{-i\lambda_j-1} \log^l(r), \tag{2.8}$$

where $C^u_{j,l}(\theta)$ *and* $C^p_{j,l}(\theta)$ *are* C^∞ *functions of* θ.

We have to establish Besov regularity for u_R, u_S, p_R and p_S. The functions u_R and p_R can be treated as above by using suitable embeddings. It remains to study the singular parts u_S and p_S. It turns out that these parts, although not very smooth in the usual Sobolev scale, have arbitrary high regularity in the specific scale of Besov spaces we are interested in.

Theorem 2.3 *Suppose that the conditions of Theorem* 2.1 *are satisfied. Then for the functions* u_S *and* p_S *according to* (2.7) *and* (2.8), *respectively, the following holds:*

$$\begin{aligned} u_S &\in B^s_\tau(L_\tau(V))^2, \quad 1/\tau = s/2 + 1/2, \ \text{for all } s > 0, \\ p_S &\in B^s_\tau(L_\tau(V)), \quad 1/\tau = s/2 + 1/2, \ \text{for all } s > 0. \end{aligned}$$

By employing Theorem 2.3 which will be proved in Section 3, the result follows.
□

Remark 2.1 *The reader should observe that, in contrary to the usual Sobolev regularity, the Besov regularity of u and p is independent of the shape of the domain and depends only on the smoothness of the functions f and g.*

3 Proof of Theorem 2.3

The proof can be performed by employing the ideas developed in [4]. We shall briefly discuss the most important steps. We only present the arguments for the function p_S according to (2.8), the function u_S can be treated analogously. It is sufficient to establish Besov regularity for a function $h(r, \theta)$ of the form

$$h(r,\theta) = C(\theta) r^{-i\gamma - 1} \log^l(r), \tag{3.1}$$

where $C(\theta)$ is a C^∞ function and $\Im(\gamma) > 0$. We want to use the fact that function spaces such as Besov spaces can be characterized by wavelet expansions. Let Ψ be the set of $2^d - 1$ functions built in the usual way by tensor products from the univariate, compactly supported, orthonormal Daubechies wavelets, see [9, 19]. Then the functions

$$\eta_I := \eta_{j,k} := 2^{jd/2} \eta(2^j \cdot - k), \quad I = 2^{-j}k + 2^{-j}[0,1]^d, \quad k \in \mathbf{Z}^d, j \in \mathbf{Z}, \eta \in \Psi, \tag{3.2}$$

form an orthonormal basis for $L_2(\mathbf{R}^d)$. If the functions $\eta \in \Psi$ are sufficiently smooth (which can always be achieved, see [9] for details), then a function F is in the Besov space $B^\alpha_\tau(L_\tau(\mathbf{R}^d)), 1/\tau = \alpha/d + 1/2$, if and only if

$$\|P_0(F)\|_{L_\tau(\mathbf{R}^d)} + \left(\sum_{\eta \in \Psi} \sum_{I \in \mathcal{D}^+} |\langle F, \eta_I \rangle|^\tau \right)^{1/\tau} < \infty, \tag{3.3}$$

where $\mathcal{D}^+$ denotes the set of all dyadic cubes of measure < 1 and P_0 is a projector onto a suitable subspace of $L_2(\mathbf{R}^d)$, see, e.g., [19] for the case $\tau > 1$ and [13] for the general case. According to (3.3), we have to estimate the wavelet coefficients of a function h of the form (3.1). By employing a suitable extension technique, we may view $h(\theta, r)$ as a function on all of $\mathbf{R}^2$, see [4] for details. It can be shown that for this extended function the term $\|P_0(h)\|_{L_\tau(\mathbf{R}^d)}$ is always finite, see [8]. Therefore it remains to estimate the second term in (3.3), i.e., we have to show that

$$\sum_{(I,\eta) \in \Lambda} |\langle h, \eta_I \rangle|^\tau < \infty, \tag{3.4}$$

where Λ denotes the set of all pairs (I, η), $I \in \mathcal{D}^+$, $\eta \in \Psi$ for which $Q(I) \cap V \neq \emptyset$. Here $Q(I)$ denotes a suitable cube which contains the support of η_I. Let us start by estimating one wavelet coefficient. By using the vanishing moment property of wavelets, see again [9] for details, and employing a classical Whitney–type estimate for the error of approximation by polynomials on cubes, it turns out that there exists a polynomial P_I of total degree $< n$ such that

$$\begin{aligned} |\langle h, \eta_I \rangle| &\leq \|h - P_I\|_{L_2(Q(I))} \|\eta_I\|_{L_2(Q(I))} \\ &\lesssim |Q(I)|^{(n+1)/2} |h|_{W^n(L_\infty(Q(I)))} \\ &\lesssim 2^{-j(n+1)} |h|_{W^n(L_\infty(Q(I)))}. \end{aligned} \tag{3.5}$$

(By ' $\lesssim$ ' we clearly indicate inequality up to constants). Now we have to sum these expressions. First, we fix a refinement level j by considering the set

$$\Lambda_j := \{(I, \eta) \in \Lambda \mid |I| = 2^{-2j}\}. \tag{3.6}$$

For each level, we cover V by layers, i.e., we define

$$\Lambda_{j,k} \;:=\; \{(I, \eta) \in \Lambda_j \mid k2^{-j} \leq \delta_I < (k+1)2^{-j}\}, \tag{3.7}$$

where δ_I denotes the distance of the cube $Q(I)$ to zero,

$$\delta_I := \inf_{x \in Q(I)} r(x).$$

We first consider the sets

$$\Lambda_j^o := \Lambda_j \backslash \Lambda_{j,c}, \;\; \Lambda_{j,c} := \{(I, \eta) \in \Lambda_j \mid \delta_I < c2^{-j}\} \tag{3.8}$$

for some suitable constant c and estimate $|h|_{W^n(L_\infty(Q(I)))}$ for a typical cube $Q(I)$, $(I, \eta) \in \Lambda_j^o$. By using polar coordinates and Leibniz' rule, we obtain for $|\beta| = n$ on $Q(I)$

$$\begin{aligned} |D^\beta h| &\lesssim \sum_{\nu=0}^{n} r^{-(n-\nu)} \left| \left(\frac{d}{dr}\right)^{\nu} (r^{-i\gamma-1} \log^l(r)) \right| \\ &\lesssim \sum_{\nu=0}^{n} r^{-(n-\nu)} \sum_{\mu=0}^{\nu} r^{\Im\gamma-1-(\nu-\mu)} \left| \left(\frac{d}{dr}\right)^{\mu} \log^l(r) \right| \\ &\lesssim r^{-n+\Im\gamma-1} \sum_{\nu=0}^{\min(n,l-1)} |\log^{l-\nu}(r)| \\ &\lesssim r^{-n+\Im\gamma-1-\epsilon} \end{aligned}$$

for some suitable small $\epsilon > 0$. Hence

$$|h|_{W^n(L_\infty(Q(I)))} \lesssim \delta_I^{\Im\gamma-n-1-\epsilon} \qquad \text{for} \;\; (I, \eta) \in \Lambda_j^o. \tag{3.9}$$

Furthermore, one has

$$|\Lambda_{j,k}| \lesssim k \tag{3.10}$$

so that, by combining (3.5), (3.9) and (3.10), we obtain

$$\begin{aligned}
\sum_{(I,\eta)\in\Lambda_j^\circ} |\langle h,\eta_I\rangle|^\tau &\lesssim \sum_{k=k_1}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,k}} 2^{-j(n+1)\tau}\delta_I^{(\Im\gamma-n-1-\epsilon)\tau} \\
&\lesssim \sum_{k=k_1}^{\infty} k\cdot 2^{-j(n+1)\tau}(k\cdot 2^{-j})^{(\Im\gamma-n-1-\epsilon)\tau} \\
&\lesssim 2^{-j(-\epsilon+\Im\gamma)\tau}\sum_{k=k_1}^{\infty} k^{1+(\Im\gamma-n-1-\epsilon)\tau}, \qquad (3.11)
\end{aligned}$$

where k_1 depends on the constant c in (3.8). If we choose n large enough, the sum involving k is clearly finite. Summing over all refinement levels we are left with a geometric series which is convergent if we choose $\epsilon < \Im\gamma$ which is clearly possible.

It remains to study the sets $\Lambda_{j,c}$. It follows from (2.8) that $h \in H^s(V)$, for some sufficiently small s. Using this fact and following the lines of the proof of Theorem 3.2 in [8], we obtain the condition

$$-2s\tau/(2-\tau) < 0$$

which is clearly satisfied. The theorem is proved. □

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