Generalized Coorbit Theory, Banach Frames, and the Relation to α–Modulation Spaces *

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Abstract

This paper is concerned with generalizations and specific applications of the coorbit space theory based on group representations modulo quotients that has been developed quite recently. We show that the general theory applied to the affine Weyl–Heisenberg group gives rise to families of smoothness spaces that can be identified with α-modulation spaces.

Key Words: Square integrable group representations, time–frequency analysis, atomic decompositions, (Banach) frames, homogeneous spaces, weighted coorbit spaces, smoothness spaces.

AMS Subject classification: 57S25, 42C15, 42C40, 46E15, 46E35.

1 Introduction

In recent years, the construction and the application of frames has become a field of increasing importance [4, 11]. In general, given a Hilbert space $\mathcal{H}$, a collection of elements $\{e_i\}_{i\in\mathbb{Z}}$ is called a *frame* if there exist constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, e_i\rangle_{\mathcal{H}}|^2 \leq A_2\|f\|_{\mathcal{H}}^2. \tag{1.1}$$

In contrary to the classical Riesz basis setting, the representation of an element in the Hilbert space by means of the frame elements is not necessarily unique, i.e., the frame approach allows some redundancies. In some applications, these redundancies might be a disadvantage, however, usually this weak point is more than compensated by an enormous gain of flexibility.

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This higher flexibility which, e.g., allows very natural constructions of frames on manifolds, is one of the reasons why many people have been attracted by the frame approach for designing robust methods in innovative applications. The foundation of modern frame theory in Banach spaces was layed by Feichtinger and Gröchenig in a series of papers [16, 17, 18, 19]. Given a Hilbert space $\mathcal{H}$, the first step is to find a suitable group G that admits a (square) integrable representation in $\mathcal{H}$ and therefore gives rise to a generalized (continuous) wavelet transform. Then, so–called *coorbit spaces* can be defined by collecting all functions for which this wavelet transform is contained in some weighted L_p–space. Finally, a judicious discretization of the representation produces the desired frames for the coorbit spaces. This approach works fine on the whole Euclidean plane and covers, e.g., the classical wavelet and Weyl–Heisenberg frames. In two recent papers [9, 10], three of us have developed a generalization of the Feichtinger/Gröchenig theory to quotient spaces. Moreover, [35] contains a generalization into a different direction, which covers classical coorbit spaces restricted to elements that are invariant under the action of symmetry groups. A further development that starts with a general abstract frame was provided by two of us in [25]. In [37] the classical coorbit theory was generalized with the aim to treat also quasi-Banach spaces.

The main ingredient in [9, 10] was the concept of representations modulo quotients as, e.g., developed by Ali et al. [1, 2, 3] and Torresani [42]. This approach can be used to construct Gabor frames and modulation spaces on the spheres, see again [9, 10]. However, there is another possible application of the general coorbit space theory modulo quotients that we want to discuss in this paper, namely the construction of new smoothness spaces. It is well–known that the coorbit spaces associated with the affine group are the Besov spaces [26, 43, 44], whereas the smoothness spaces related with the Weyl–Heisenberg group are the modulation spaces [29]. Therefore one natural idea would be to construct some mixed forms of smoothness spaces, i.e., spaces lying 'somewhere in between' Besov and modulation spaces. Following the Feichtinger/Gröchenig approach, the first step would be to find a group that contains both, the affine and the Weyl–Heisenberg group, and square integrable representations generated by combinations of dilations, translations, and modulations. One possible candidate would be the so–called affine Weyl–Heisenberg group. Unfortunately, as shown by Torresani [40], no representation of this group is ever square integrable. However, one possible remedy has been suggested in [9, 10, 32, 40, 41]: why not factoring out a suitable closed subgroup and work with quotients? Then, by varying the subgroup and/or specific Borel sections we obtain indeed some kinds of mixed spaces.

The aim of this paper is twofold. First, we want to generalize the coorbit theory in [9, 10] in order to fill some gaps that have been left before:

- We show that the whole analysis can be carried out even if the quotient space does not possess an invariant measure.
- As proposed in [25], we generalize the theory from the strictly square integrable to the square integrable setting.
- We explain how some technical assumptions made in [9, 10, 25] can be satisfied in practice.

Secondly, we want to apply the whole machinery to construct and to analyze certain mixed smoothness spaces:

- We show that factoring out a natural subgroup of the affine Weyl–Heisenberg group, equipped with specific sections, indeed produces a square integrable representation. Hence, the associated coorbit spaces, i.e., the mixed smoothness spaces, are well–defined.

- We prove that it is possible to construct suitable Banach frames for the coorbit spaces.
- We discuss the relations of these smoothness spaces to classical function spaces such as modulation spaces [29] and Besov spaces [26, 43, 44]. Moreover we show how α–modulation spaces [13, 14, 20, 23, 27, 32, 34, 38], that include Besov and modulation spaces as particular cases, are fully characterized by our construction.

Our paper is organized as follows. In Section 2, we collect some basic facts on square integrable group representations modulo quotients as far as they are needed for our purpose. Then, in Section 3, we introduce and discuss the associated coorbit spaces. The presentation essentially follows the lines of [9, 10], however, the nontrivial operator A_σ occurring in our setting requires special care. In particular, it turns out that two different scales of coorbit spaces have to be considered. Section 4 contains the main results of this paper. We state and prove a decomposition and a reconstruction theorem and establish the frame bounds. To this end, we have to introduce and to analyze certain approximation operators. We also show that specific coverings of the embedded quotient that are needed always exist. In Section 5, the abstract theory developed so far is applied to the affine Weyl–Heisenberg group. We show that factoring out a certain subgroup yields a square integrable setting. Then, we prove that all the integrability conditions on the kernel and the oscillating part can be satisfied, so that we have established the coorbit spaces and the associated Banach frames. Finally, in Section 6 we show that such coorbit spaces coincide with certain α-modulation spaces and derive corresponding Banach frames and atomic decompositions by an application of the general discretization machinery illustrated in Section 4.

2 General Theory

Let G be a locally compact group with left Haar measure ν and let $\mathcal{H}$ be a separable Hilbert space. A *strongly continuous unitary representation* of G on $\mathcal{H}$ is a mapping $\mathcal{U}$ from G into the unitary operators on $\mathcal{H}$ for which $\mathcal{U}(g\tilde{g}) = \mathcal{U}(g)\mathcal{U}(\tilde{g})$ for all $g, \tilde{g} \in G$ and the mapping $g \mapsto \mathcal{U}(g)f$ is continuous for all $f \in \mathcal{H}$. Further, we say that $\mathcal{U}$ is *square integrable* if there exists $\psi \in \mathcal{H}\backslash\{0\}$ such that

$$\int_G |\langle \psi, \mathcal{U}(g)\psi\rangle_{\mathcal{H}}|^2 d\nu(g) < \infty.$$

For the classical integral transforms like the short time Fourier transform and wavelet transform related to the reduced Weyl-Heisenberg-group and the affine group, respectively, the representations in question are in fact square-integrable. However, for integral transforms related to group representations on L_2-spaces on manifolds, for example on the sphere, square integrability fails to hold. In other words, the corresponding group is too large.

A way to overcome this fact, is to make the group G smaller, i.e., to factor out a suitable closed subgroup H. In this way, we restrict the representation to a quotient $X := G/H$. The space X always carries a natural measure. There exists a G-invariant Radon-measure μ on X, which is unique up to multiplication with a constant, if and only if $\Delta_G|H = \Delta_H$, where Δ_G denotes the modular function of G. In the general case one still has a suitable substitute, called a quasi-invariant measure. Suppose that μ is a Radon measure on X. Then for $g \in G$

the translate μ_g is defined by $\mu_g(E) = \mu(gE)$ for a measurable set E. If all measures $\mu_g, g \in G$ are equivalent, i.e., they have the same null sets, then μ is called a *quasi-invariant measure*. If there exists a continuous function $\lambda : G \times X \to (0, \infty)$ such that $d\mu_g(x) = \lambda(g, x)d\mu(x)$ for all $g \in G, x \in X$ then μ is called *strongly quasi-invariant*. The function λ satisfies the cocycle property [21, formula (2.60)]

$$\lambda(g\tilde{g}, x) \;=\; \lambda(g, \tilde{g}x)\lambda(\tilde{g}, x).$$

Every pair G, H admits a strongly quasi-invariant measure. For its construction we refer to [21]. In the following, we drop the indication of the measure μ when writing $L_p(X)$.

Since the representation is not defined directly on X (unless H is a normal subgroup and the kernel of the representation) we need to introduce a section $\sigma : X \to G$ which assigns to each coset a point lying in it. In other words, if $\Pi : G \to X$ denotes the canonical projection then $\Pi \circ \sigma = id$. In general the section σ cannot be chosen to be continuous but it is always possible to choose it measurable or even continuous on some dense open subset of X. However, in many examples the section will be continuous. The action of an element $g \in G$ on $\sigma(x)$ for $x \in X$ can be written as

$$g\sigma(x) \;=\; \sigma(gx)h(g, x)$$

for some element $h(g, x) \in H$ which clearly is given by

$$h(g, x) \;=\; \sigma(gx)^{-1}g\sigma(x). \tag{2.1}$$

Let a quasi-invariant measure μ on X and a section σ be given. Then a unitary representation $\mathcal{U}$ of G on $\mathcal{H}$ is called *square-integrable modulo* (H, σ) if there exists a function $\psi \in \mathcal{H}$ such that the self-adjoint operator $A_\sigma : \mathcal{H} \to \mathcal{H}$ (dependent on σ and ψ) weakly defined by

$$A_\sigma f \;:=\; \int_X \langle f, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}}\, \mathcal{U}(\sigma(x))\psi \, d\mu(x),$$

i.e.,

$$\langle A_\sigma f, g\rangle_{\mathcal{H}} \;=\; \int_X \langle f, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}}\, \langle \mathcal{U}(\sigma(x))\psi, g\rangle_{\mathcal{H}}\, d\mu(x) \quad \text{for all } f, g \in \mathcal{H} \tag{2.2}$$

is bounded and has a bounded inverse A_σ^{-1}. Then we have that

$$\langle A_\sigma f, f\rangle_{\mathcal{H}} = \int_X |\langle f, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}}|^2 d\mu(x) < \infty \quad \text{for all } f \in \mathcal{H}. \tag{2.3}$$

The function ψ is called *admissible*. If A_σ is a multiple of the identity then ψ is called *strictly admissible*.

In [9, 10] the investigations have been carried through under the assumption that A_σ is a multiple of the identity. However, for many applications, e.g., the wavelet transform on the sphere, it turns out that this assumption is not necessarily true. Therefore one aim of this paper is to generalize the approach in [9, 10] to more general operators A_σ, see also [25].

The *wavelet transform* or voice transform is defined by

$$V_\psi f(x) \;:=\; \langle f, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}}, \quad x \in X.$$

By (2.3), we have that $V_\psi f \in L_2(X)$. The main ingredient in the coorbit space theory is a reproducing formula. In the case that A_σ is a multiple of the identity one uses V_ψ to define the reproducing kernel. In the general case we define a second transform

$$W_\psi f(x) \;:=\; V_\psi(A_\sigma^{-1} f)(x) \;=\; \langle A_\sigma^{-1} f, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}} \;=\; \langle f, A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}}, \quad x \in X.$$

Using $A_\sigma^{-1} f$ and $A_\sigma^{-1} g$ instead of f and g in (2.2), respectively, we obtain that

$$\langle f, g\rangle_{\mathcal{H}} \;=\; \langle W_\psi f, V_\psi g\rangle \;=\; \langle V_\psi f, W_\psi g\rangle, \quad f, g \in \mathcal{H}, \tag{2.4}$$

where $\langle \cdot, \cdot\rangle = \int_X F(x)\overline{G(x)} d\mu(x)$ whenever the integral exists. By (2.4), we see that

$$\begin{aligned} V_\psi f(x) &= \langle f, \mathcal{U}(\sigma(x))\psi\rangle \;=\; \langle V_\psi f, W_\psi(\mathcal{U}(\sigma(x))\psi)\rangle \;=\; \langle V_\psi f, R(x,\cdot)\rangle, \\ W_\psi f(x) &= \langle f, A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi\rangle \;=\; \langle W_\psi f, W_\psi(\mathcal{U}(\sigma(x))\psi)\rangle \;=\; \langle W_\psi f, R(x,\cdot)\rangle, \end{aligned}$$

where

$$R(x,y) \;=\; R_\psi(x,y) \;:=\; W_\psi(\mathcal{U}(\sigma(x))\psi)(y) \;=\; \langle A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\psi\rangle. \tag{2.5}$$

Clearly, $R(y,x) = \overline{R(x,y)}$ for all $x, y \in X$. Moreover, we have for all $x, y \in X$ that

$$|R(x,y)| \le |||A_\sigma^{-1}|||\, \|\psi\|_{\mathcal{H}}^2. \tag{2.6}$$

The following facts on square-integrable representations $\mathcal{U}$ modulo (H, σ) and admissible functions ψ are well-known, see [2, Theorem 7.3.1].

- The set
$$S_\sigma := \{\mathcal{U}(\sigma(x))\psi : x \in X\} \tag{2.7}$$
is total in $\mathcal{H}$, i.e., $S_\sigma^\perp = \{0\}$. Since A_σ is continuously invertible, we see that also the set $A_\sigma^{-1}(S_\sigma) = \{A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi : x \in X\}$ is total in $\mathcal{H}$.
- The mappings V_ψ and W_ψ are bijective mappings of $\mathcal{H}$ onto the reproducing kernel Hilbert space
$$\mathcal{M}_2 := \{F \in L_2(X) : \langle F, R(x,\cdot)\rangle \;=\; F(x) \text{ a.e.}\}.$$
- By (2.4), the mappings $\tilde{V}_\psi : L_2(X) \to \mathcal{H}$ and $\tilde{W}_\psi : L_2(X) \to \mathcal{H}$ defined by
$$\begin{aligned} \tilde{V}_\psi F &:= \int_X F(x) A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi\, d\mu(x), \\ \tilde{W}_\psi F &:= \int_X F(x)\mathcal{U}(\sigma(x))\psi\, d\mu(x), \end{aligned}$$
i.e.,
$$\langle \tilde{V}_\psi F, g\rangle_{\mathcal{H}} = \langle F, W_\psi g\rangle, \quad \langle \tilde{W}_\psi F, g\rangle_{\mathcal{H}} = \langle F, V_\psi g\rangle \tag{2.8}$$
fulfill
$$V_\psi \tilde{V}_\psi F(x) = \langle F, R(x,\cdot)\rangle, \quad W_\psi \tilde{W}_\psi F(x) = \langle F, R(x,\cdot)\rangle \quad \text{for all } F \in \mathcal{M}_2$$
and
$$f = \tilde{V}_\psi V_\psi f, \quad f = \tilde{W}_\psi W_\psi f \quad \text{for all } f \in \mathcal{H}.$$
Clearly, they are adjoint mappings, $\tilde{V}_\psi = W_\psi^*$ and $\tilde{W}_\psi = V_\psi^*$.

3 Coorbit Spaces

The coorbit spaces that we will define are smoothness spaces on some manifold. In order to measure smoothness we will also need to plug in weight functions on X. For technical reasons we assume in the following that G and, hence, also $X = G/H$ is σ-compact.

For some positive measurable weight function v on X and $1 \le p \le \infty$, let

$$L_{p,v}(X) := \{f \text{ measurable } : fv \in L_p(X)\}$$

with the natural norm

$$\begin{aligned} \|f\|_{L_{p,v}} &:= \left(\int_X |f(x)|^p v(x)^p d\mu(x)\right)^{1/p}, \quad 1 \le p < \infty, \\ \|f\|_{L_{\infty,v}} &:= \operatorname*{ess\,sup}_{x\in X} |f(x)|v(x). \end{aligned}$$

The generalized Young inequality (Schur test) for $L_{p,v}$ will be a major tool in the sequel:

Theorem 3.1. *Let K be some kernel on $X \times X$. We associate to K the integral operator*

$$K(F)(x) := \int_X K(x,y)F(y)d\mu(y).$$

If K satisfies

$$\operatorname*{ess\,sup}_{x\in X} \int_X |K(x,y)|\frac{v(x)}{v(y)}d\mu(y) \le C_K < \infty, \tag{3.1}$$

then K is a continuous operator on $L_{\infty,v}(X)$. If K satisfies

$$\operatorname*{ess\,sup}_{y\in X} \int_X |K(x,y)|\frac{v(x)}{v(y)}d\mu(x) \le C_K < \infty, \tag{3.2}$$

then K is a continuous operator on $L_{1,v}(X)$. If K satisfies both (3.1) *and* (3.2) *then K is a continuous operator on $L_{p,v}(X)$, $1 \le p \le \infty$, and satisfies*

$$\|K(F)\|_{L_{p,v}(X)} \le C_K\|F\|_{L_{p,v}(X)}. \tag{3.3}$$

For the proof see e.g. [10].

First we need to introduce the "reservoir" for our coorbit spaces. To this end, let w be some weight function on X satisfying $w(x) \ge 1$ for all $x \in X$. Throughout this paper, we impose the fundamental condition

$$\operatorname*{ess\,sup}_{y\in X} \int_X |R(x,y)|\frac{w(x)}{w(y)}\, d\mu(x) < \infty. \tag{3.4}$$

For $\psi \in \mathcal{H}$ with property (3.4) (for $R = R_\psi$) we define the spaces

$$\begin{aligned} \mathcal{H}_{1,w} &:= \{f \in \mathcal{H} : W_\psi(f) \in L_{1,w}(X)\}, \\ \mathcal{K}_{1,w} &:= \{f \in \mathcal{H} : V_\psi(f) \in L_{1,w}(X)\} \end{aligned}$$

with norms

$$\|f\|_{\mathcal{H}_{1,w}} := \|W_\psi f\|_{L_{1,w}}, \quad \|f\|_{\mathcal{K}_{1,w}} := \|V_\psi f\|_{L_{1,w}}, \tag{3.5}$$

respectively. By (3.4), the sets S_σ and $A_\sigma^{-1} S_\sigma$ are contained in $\mathcal{H}_{1,w}$ and $\mathcal{K}_{1,w}$, respectively. Since these sets are total in $\mathcal{H}$, we conclude that $\mathcal{H}_{1,w}$ and $\mathcal{K}_{1,w}$ are dense in $\mathcal{H}$. Moreover, we have by (2.4) that

$$\|f\|_{\mathcal{H}}^2 = \int_X W_\psi f \, \overline{V_\psi f} \, d\mu(x) \le \int_X |W_\psi f| \, |V_\psi f| \, d\mu(x)$$

and since $w(x) \ge 1$ and $|V_\psi f| \le \|f\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}}$, $|W_\psi f| \le \|f\|_{\mathcal{H}} \, |||A_\sigma^{-1}||| \, \|\psi\|_{\mathcal{H}}$, we obtain that

$$\|f\|_{\mathcal{H}} \le \|\psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}_{1,w}} \,, \quad \|f\|_{\mathcal{H}} \le \|\psi\|_{\mathcal{H}} \, |||A_\sigma^{-1}||| \, \|f\|_{\mathcal{K}_{1,w}} .$$

Thus, both $\mathcal{H}_{1,w}$ and $\mathcal{K}_{1,w}$ are continuousely embedded in $\mathcal{H}$.

Remark 3.2. *In particular, $\mathcal{H}_{1,w}$ and $\mathcal{K}_{1,w}$ are Banach spaces. In fact, if $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_{1,w}$, then $(F_n) = (W_\psi f_n)$ is a Cauchy sequence in $L_{1,w}(X)$. Therefore, there exists $F \in L_{1,w}(X)$ such that $F_n \to F$. Since $f_n \in \mathcal{H}$ we have $\langle F_n, R(x, \cdot) \rangle = F_n(x)$. By (3.4) and the generalized Young inequality, the application of the kernel R is a continuous operator on $L_{1,w}(X)$ yielding*

$$F(x) = \langle F, R(x, \cdot) \rangle. \tag{3.6}$$

¿From (3.6), (2.6), and the fact that $w(y) \ge 1$ we obtain immediately that $F \in L_\infty(X) \cap L_{1,w}(X) \subset L_2(X)$. Therefore there exists $f \in \mathcal{H}$ such that $F = W_\psi f$ and since $F \in L_{1,w}(X)$ we have $f \in \mathcal{H}_{1,w}$. Similarly it can be shown that $\mathcal{K}_{1,w}$ is a Banach space.

Introducing the anti-dual spaces $\mathcal{H}_{1,w}'$ and $\mathcal{K}_{1,w}'$ (the spaces of all bounded conjugate-linear functionals on $\mathcal{H}_{1,w}$ and $\mathcal{K}_{1,w}$, respectively) we thus have the continuous embeddings

$$\begin{array}{ccccc} \mathcal{H}_{1,w} & \subset & \mathcal{H} & \subset & \mathcal{H}_{1,w}', \\ \mathcal{K}_{1,w} & \subset & \mathcal{H} & \subset & \mathcal{K}_{1,w}'. \end{array} \tag{3.7}$$

(Hereby we identify an element $f \in \mathcal{H}$ with a functional in $\mathcal{H}_{1,w}'$ by $\langle f, g \rangle_{\mathcal{H}_{1,w}' \times \mathcal{H}_{1,w}} = \langle f, g \rangle_{\mathcal{H}}$, $g \in \mathcal{H}_{1,w}$.) Moreover, $\mathcal{H}_{1,w}$ is norm dense in $\mathcal{H}$ and $\mathcal{H}$ is weak-$*$ dense in $\mathcal{H}_{1,w}'$, and similarly for $\mathcal{K}_{1,w}$ and $\mathcal{K}_{1,w}'$. In other words, $(\mathcal{H}_{1,w}, \mathcal{H}, \mathcal{H}_{1,w}')$ and $(\mathcal{K}_{1,w}, \mathcal{H}, \mathcal{K}_{1,w}')$ form Gelfand triples. By definition, we see that $\mathcal{H}_{1,w} = A_\sigma \mathcal{K}_{1,w}$ and therefore $\mathcal{K}_{1,w}' = A_\sigma^* \mathcal{H}_{1,w}'$.

We can extend the operators V_ψ and W_ψ to $\mathcal{H}_{1,w}'$ and $\mathcal{K}_{1,w}'$, respectively, by setting

$$\begin{aligned} V_\psi f(x) &:= \langle f, \mathcal{U}(\sigma(x))\psi \rangle_{\mathcal{H}_{1,w}' \times \mathcal{H}_{1,w}}, \\ W_\psi f(x) &:= \langle f, A_\sigma^{-1} \mathcal{U}(\sigma(x))\psi \rangle_{\mathcal{K}_{1,w}' \times \mathcal{K}_{1,w}} \;=\; \langle (A_\sigma^{-1})^* f, \mathcal{U}(\sigma(x))\psi \rangle_{\mathcal{H}_{1,w}' \times \mathcal{H}_{1,w}}. \end{aligned}$$

By (3.4), these expressions are well-defined. Further, we conclude by the following inequality

$$\begin{aligned} \|V_\psi f\|_{L_{\infty,1/w}} &= \operatorname*{ess\,sup}_{x \in X} |V_\psi f(x)| \, w(x)^{-1} \;=\; \operatorname*{ess\,sup}_{x \in X} |\langle f, \mathcal{U}(\sigma(x))\psi \rangle_{\mathcal{H}_{1,w}' \times \mathcal{H}_{1,w}}| \, w(x)^{-1} \\ &\le \|f\|_{\mathcal{H}_{1,w}'} \operatorname*{ess\,sup}_{x \in X} \|\mathcal{U}(\sigma(x))\psi\|_{\mathcal{H}_{1,w}} \, w(x)^{-1} \\ &= \|f\|_{\mathcal{H}_{1,w}'} \operatorname*{ess\,sup}_{x \in X} \int_X |W_\psi(\mathcal{U}(\sigma(x))\psi)(y)| \, \frac{w(y)}{w(x)} d\mu(y) \;\le\; C \, \|f\|_{\mathcal{H}_{1,w}'} \end{aligned}$$

that $V_\psi : \mathcal{H}_{1,w}' \to L_{\infty,1/w}(X)$ is a continuous operator, see also [25, Lemma 1]. In a completely similar way, we can show that $W_\psi : \mathcal{K}_{1,w}' \to L_{\infty,1/w}(X)$ is a continuous operator. Now the operators $\tilde{V}_\psi$ and $\tilde{W}_\psi$ in (2.8) can be weakly extended to $L_{\infty,1/w}(X)$ by

$$\langle \tilde{V}_\psi F, g \rangle_{\mathcal{H}_{1,w}' \times \mathcal{H}_{1,w}} := \langle F, W_\psi g \rangle \,, \quad \langle \tilde{W}_\psi F, g \rangle_{\mathcal{K}_{1,w}' \times \mathcal{K}_{1,w}} := \langle F, V_\psi g \rangle, \quad F \in L_{\infty,1/w}. \tag{3.8}$$

By following the lines of the proof of Lemma 2 in [10], it can be shown that both, $\tilde{W}_\psi$ and $\tilde{V}_\psi$, are bounded operators on $L_{\infty,1/w}(X)$. Then we obtain for $F \in L_{\infty,1/w}(X)$ that

$$
\begin{aligned}
V_\psi \tilde{V}_\psi F(x) &= \langle \tilde{V}_\psi F, \mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{H}'_{1,w}\times\mathcal{H}_{1,w}} = \langle F, W_\psi(\mathcal{U}(\sigma(x))\psi)\rangle = \langle F, R(x,\cdot)\rangle, \qquad (3.9)\\
W_\psi \tilde{W}_\psi F(x) &= \langle \tilde{W}_\psi F, A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}} = \langle F, V_\psi(A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi)\rangle = \langle F, R(x,\cdot)\rangle.
\end{aligned}
$$

By (3.8) we have for $f \in \mathcal{H}'_{1,w}$ and $g \in \mathcal{H}_{1,w}$ that

$$
\langle \tilde{V}_\psi V_\psi f, g\rangle_{\mathcal{H}'_{1,w}\times\mathcal{H}_{1,w}} = \langle V_\psi f, W_\psi g\rangle. \tag{3.10}
$$

Now we introduce the coorbit spaces. Let v be a positive measurable weight function on X (not necessarily $v(x) \geq 1$) and let

$$
m(x,y) := \max\left\{\frac{v(x)}{v(y)}, \frac{v(y)}{v(x)}\right\}.
$$

We impose the fundamental condition

$$
\sup_{y\in X}\int_X |R(x,y)|\, m(x,y)\, d\mu(x) \;\leq\; C_\psi < \infty. \tag{3.11}
$$

Remark 3.3. *Since $|R(x,y)| = |R(y,x)|$, the condition that the left-hand side of* (3.11) *is uniformly bounded is equivalent to*

$$
\sup_{y\in X}\int_X |R(x,y)|\frac{v(x)}{v(y)} d\mu(x) < \infty \quad \text{and} \quad \sup_{x\in X}\int_X |R(x,y)|\frac{v(x)}{v(y)} d\mu(y) < \infty. \tag{3.12}
$$

Indeed, it is obvious that (3.11) *implies* (3.12). *For the converse observe that* (3.12) *implies*

$$
\begin{aligned}
&\sup_{y\in X}\int_X |R(x,y)|\, m(x,y)\, d\mu(x) \;\leq\; \sup_{y\in X}\int_X |R(x,y)|\left(\frac{v(x)}{v(y)} + \frac{v(y)}{v(x)}\right) d\mu(x)\\
&\leq \sup_{y\in X}\int_X |R(x,y)|\frac{v(x)}{v(y)} d\mu(x) + \sup_{y\in X}\int_X |R(y,x)|\frac{v(y)}{v(x)} d\mu(x) \;<\; \infty.
\end{aligned}
$$

The conditions (3.12) imply by Theorem 3.1 that $F \mapsto \langle F, R(x,\cdot)\rangle$ is continuous on $L_{p,v}$, $1 \leq p \leq \infty$. Additionally, we require that there exists a weight function w associated to v satisfying condition (3.4), $w(x) \geq 1$ for all $x \in X$, and

$$
\{\langle F, R(x,\cdot)\rangle \,:\, F \in L_{p,v}\} \subset L_{\infty,1/w} \tag{3.13}
$$

for all $1 \leq p \leq \infty$.

Remark 3.4. i) *If $v(x) \geq 1$ then the choice $w(x) = 1$ is valid. Indeed, if $F \in L_{p,v}$ then Hölder's inequality with $1/p + 1/q = 1$ and* (2.6) *yield*

$$
\begin{aligned}
&|\int_X F(y)\overline{R(x,y)}\, d\mu(y)| \;\leq\; \int_X |F(y)|\, v(y)\, |R(x,y)|\, d\mu(y)\\
&\leq \left(\int_X |F(y)|^p\, v^p(y)\, |R(x,y)| d\mu(y)\right)^{1/p} \left(\int_X |R(x,y)|\, d\mu(y)\right)^{1/q}\\
&\leq |||A_\sigma^{-1}|||^{1/p}\, \|\psi\|_{\mathcal{H}}^{2/p}\, \|F\|_{L_v^p} \left(\int_X |R(x,y)|\, m(x,y)\, d\mu(y)\right)^{1/q} \;\leq\; C\|F\|_{L_v^p}.
\end{aligned}
$$

Moreover, (3.11) *clearly implies* (3.4) *if* $w = 1$.
ii) *In the general case (no lower bound on* v*) we will show later in Lemma 4.5 that under some conditions on* v *and* R *needed also for the discretization method there exists a natural choice of a weight function* w *associated to* v *satisfying* (3.13).

For $1 \le p \le \infty$ we define the coorbit spaces

$$\begin{aligned} \mathcal{H}_{p,v} &:= \{f \in \mathcal{K}'_{1,w} : W_\psi f \in L_{p,v}(X)\}, \\ \mathcal{K}_{p,v} &:= \{f \in \mathcal{H}'_{1,w} : V_\psi f \in L_{p,v}(X)\} \end{aligned}$$

with norms

$$\|f\|_{\mathcal{H}_{p,v}} := \|W_\psi f\|_{L_{p,v}}\,, \quad \|f\|_{\mathcal{K}_{p,v}} := \|V_\psi f\|_{L_{p,v}}.$$

These are really norms since $\|f\|_{\mathcal{K}_{p,v}} = 0$ implies $V_\psi f = 0$ which in turn implies $f = 0$ since $\{\mathcal{U}(\sigma(x))\psi : x \in X\}$ is total in $\mathcal{H}_{1,w}$ (see Theorem 1 in [16, 25]) and similarly for the second norm.

The basic ingredient in the coorbit theory is a correspondence principle between the spaces $\mathcal{H}_{p,v}$, $\mathcal{K}_{p,v}$ and certain subspaces of functions on the coset space X, which are defined by means of the reproducing kernel R. For $1 \le p \le \infty$ and ψ with property (3.11) let

$$\mathcal{M}_{p,v} := \{F \in L_{p,v}(X) : \langle F, R(x,\cdot)\rangle = F(x) \text{ a.e. }\}.$$

By the generalized Young inequality, the expression $\langle F, R(x,\cdot)\rangle$ defines a function in $L_{p,v}(X)$. The correspondence principle can now be formulated analogously to Theorem 3.1 in [9] as follows.

Theorem 3.5. *Let* $\psi \in \mathcal{H}$ *be given such that the corresponding kernel* R *satisfies* (3.11).

i) *The following relations hold true*

$$\begin{aligned} \langle V_\psi f, R(x,\cdot)\rangle &= V_\psi f(x), \quad f \in \mathcal{K}_{p,v}, \\ \langle W_\psi f, R(x,\cdot)\rangle &= W_\psi f(x), \quad f \in \mathcal{H}_{p,v}, \end{aligned}$$

i.e., $V_\psi f$, $W_\psi f \in \mathcal{M}_{p,v}$.

ii) *For every* $F \in \mathcal{M}_{p,v}$, $1 \le p \le \infty$, *there exists a uniquely determined element* $f \in \mathcal{K}_{p,v}$ *such that* $F = V_\psi f$ *and a uniquely determined element* $f \in \mathcal{H}_{p,v}$ *such that* $F = W_\psi f$.

iii) *Both* $(\mathcal{H}_{p,v}, \|\cdot\|_{\mathcal{H}_{p,v}})$ *and* $(\mathcal{K}_{p,v}, \|\cdot\|_{\mathcal{K}_{p,v}})$ *are Banach spaces.*

Proof: i) Since $\mathcal{K}_{p,v}$ is a subspace of $\mathcal{H}'_{1,w}$ and $\mathcal{H}_{p,v}$ is a subspace of $\mathcal{K}'_{1,w}$ it is enough to prove the assertion for $\mathcal{H}'_{1,w}$ and $\mathcal{K}'_{1,w}$. For these spaces, however, the result is shown in Lemma 3 in [25] (using the σ-compactness of X).

ii) By (3.13) we have that $\mathcal{M}_{p,v} \subset L_{\infty,1/w}(X)$. By (3.9) we obtain that $F = V_\psi(\tilde{V}_\psi F)$, where $\tilde{V}_\psi F \in \mathcal{H}'_{1,w}$ and since $F \in L_{p,v}(X)$ also $\tilde{V}_\psi F \in \mathcal{K}_{p,v}$. The uniqueness condition follows by definition of $\mathcal{K}_{p,v}$. To show the assertion for $F = W_\psi f$, $f \in \mathcal{H}_{p,v}$, we can follow the same lines.

iii) It follows from i) and ii), for example, as in the proof of [16, Theorem (Properties of coorbit spaces)] or [25, Proposition 2]. ■

By (3.9) we see that $V_\psi \tilde{V}_\psi$ and $W_\psi \tilde{W}_\psi$ are identities on $\mathcal{M}_{p,v}$. Since we have for $f \in \mathcal{K}_{p,v}$ that $V_\psi f \in \mathcal{M}_{p,v}$ it follows $V_\psi \tilde{V}_\psi V_\psi f = V_\psi f$. Now V_ψ is injective on $\mathcal{H}'_{1,w}$ so that $\tilde{V}_\psi V_\psi$ is the identity on $\mathcal{K}_{p,v}$. Similarly we obtain that $\tilde{W}_\psi W_\psi$ is the identity on $\mathcal{H}_{p,v}$.

Remark 3.6. i) (Equivalence of $\mathcal{H}_{p,v}$ and $\mathcal{K}_{p,v}$) *By* [25, Proposition 3]*, if*

$$\operatorname*{ess\,sup}_{x\in X}\int_X |\langle \mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\psi\rangle| m(x,y)d\mu(x) \;<\; \infty \tag{3.14}$$

and

$$\operatorname*{ess\,sup}_{x\in X}\int_X |\langle A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi, A_\sigma^{-1}\mathcal{U}(\sigma(y))\psi\rangle| m(x,y)d\mu(x) \;<\; \infty, \tag{3.15}$$

then $\mathcal{H}_{p,v} = \mathcal{K}_{p,v}$ for all $p \in [1,\infty]$ with equivalent norms. In this case the continuous frame S_σ introduced in (2.7) *is called* intrinsically localized *(with intrinsically localized canonical dual). We refer to* [25] *for details on the theory of localized continuous frames and their properties for the characterization of generalized coorbit spaces.*

ii) (Independence of ψ) *Let $\phi,\psi \in \mathcal{H}\setminus\{0\}$. Suppose that* (3.14) *and* (3.15) *hold both for ϕ and ψ, and additionally*

$$\max\left\{\operatorname*{ess\,sup}_{x\in X}\textstyle\int_X |\langle \mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\phi\rangle|\, m(x,y)\, d\mu(y),\right.$$
$$\left.\operatorname*{ess\,sup}_{y\in X}\textstyle\int_X |\langle \mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\phi\rangle|\, m(x,y)\, d\mu(x)\right\} \;<\; \infty$$

then $\mathcal{H}_{p,v}$ $(= \mathcal{K}_{p,v})$ does not depend on whether we take ϕ or ψ for its definition (with equivalent norms for ϕ and ψ), see [10, Lemma 3] *and* [25, Proposition 4].

4 Main Results

In this section, we state and prove the main results of this paper, i.e., we show that judicious discretizations of the continuous wavelet transform give rise to atomic decompositions and Banach frames for the coorbit spaces introduced in Section 3. The corresponding Theorems 4.6 and 4.7 are presented in Subsection 4.1. The proofs are based on certain discretizations and approximations which are also introduced in Subsection 4.1.

4.1 Discretizations and Approximations

The main aim of this section is to discretize the continuous transform V_ψ, i.e., to arrive at an atomic decomposition of the coorbit spaces or even to construct coherent Banach frames of the form $\{\mathcal{U}(\sigma(x_i))\psi : x_i \in X\}$.

A major tool is that of a bounded uniform partition of unity. This tool is well-established on groups, see [12]. However, we need an adaption to homogeneous spaces. In the following we fix a section σ. A sequence $(x_i)_{i\in I} \subset X$ is called *U-dense* if

$$\bigcup_{i\in I} \sigma(x_i)U \supset \sigma(X) \tag{4.1}$$

for some relatively compact neighborhood U of $e \in G$ with non–void interior and it is called *relatively separated*, if

$$\sup_{g\in G} \#\{i \in I : g \in \sigma(x_i)L\} \;\leq\; C_L < \infty \tag{4.2}$$

for all compact subsets $L \subset G$.

Remark 4.1. *The finite overlap condition* (4.2) *is equivalent to*

$$\sup_{j\in I} \#\{i \in I : \sigma(x_i)L \cap \sigma(x_j)L \neq \emptyset\} \leq C_L, \tag{4.3}$$

possibly with a different constant C_L from that of (4.2)*. Indeed, assume* (4.2) *and fix j. One has $\sigma(x_i)L \cap \sigma(x_j)L \neq \emptyset$ if and only if $\sigma(x_j) \in \sigma(x_i)LL^{-1}$. Then by* (4.2) *the number of such i cannot exceed $C_{LL^{-1}}$. Conversely, assume* (4.3) *and let $g \in \sigma(x_j)L$. Then clearly $g \in \sigma(x_i)L$ for at most C_L different indices i by* (4.3)*. Thus,* (4.2) *is satisfied.*

We have the following theorem.

Theorem 4.2. *There exist relatively separated and U-dense sequences $(x_i)_{i\in I} \subset X$ for all (σ–compact) locally compact groups G, all closed subgroups H and all relatively compact neighborhoods $U \subset G$ of $e \in G$ with non–void interior.*

Proof: We adapt the proof in [33]. We assume here that G is σ–compact. For the general case one has to use Zorn's lemma.

Consider $G_0 := \bigcup_{n\in\mathbb{N}} U^n$. This is an open and hence closed subgroup of G. Then G is the disjoint union $G = \bigcup_{s\in S} sG_0$, where $S \subset G$ is a countable set by the σ–compactness of G.

First we construct a covering

$$s\overline{U} \cap \sigma(X) \subset \bigcup_{i=1}^{N_1} \sigma(x_i)U \cap \sigma(X)$$

by the following procedure: if $L_1 := s\overline{U} \cap \sigma(X) \neq \emptyset$, then we choose a point $x_1 \in X$ such that $\sigma(x_1) \in L_1$. Next we form $L_2 := (s\overline{U} \setminus \sigma(x_1)U) \cap \sigma(X)$. If $L_2 \neq \emptyset$, then we choose $x_2 \in X$ such that $\sigma(x_2) \in L_2$ and so on. Let $W^2 \subset U$ with $W = W^{-1}$. Then our process terminates after

$$N_1 \leq \nu((sU \cap \sigma(X))W)/\nu(W) \tag{4.4}$$

steps with $(s\overline{U} \setminus \bigcup_{i=1}^{N_1} \sigma(x_i)U) \cap \sigma(X) = \emptyset$ by the following argument: we see that

$$\sigma(x_j)W \cap \sigma(x_i)W = \emptyset, \quad i \neq j. \tag{4.5}$$

The contrary would imply $\sigma(x_j)w_1 = \sigma(x_i)w_2$ for some $w_1, w_2 \in W$, i.e., $\sigma(x_j) \in \sigma(x_i)W^2 \subset \sigma(x_i)U$ which is not possible by construction. Further, we have by construction for all $i = 1, \ldots, N_1$ that $\sigma(x_i)W \subset (s\overline{U} \cap \sigma(X))W$ and consequently

$$\bigcup_{i=1}^{N_1} \sigma(x_i)W \subset (s\overline{U} \cap \sigma(X))W.$$

Together with (4.5) this implies (4.4).

Next we consider $L_{N_1+1} := (s\overline{U^2} \setminus \bigcup_{i=1}^{N_1} \sigma(x_i)U) \cap \sigma(X)$. If $L_{N_1+1} \neq \emptyset$, then we choose $x_{N_1+1} \in X$ such that $\sigma(x_{N_1+1}) \in L_{N_1+1}$. Continuing like this we obtain a covering of $sG_0 \cap \sigma(X)$ of the form

$$sG_0 \cap \sigma(X) \subset \bigcup_{i\in I_s} \sigma(x_i)U.$$

and performing this for all $s \in S$ we get a covering of $\sigma(X)$ of the form

$$\sigma(X) \subset \bigcup_{i \in I} \sigma(x_i)U$$

with some countable index set I.

Finally we prove that $(x_i)_{i \in I}$ is relatively separated. For an arbitrary compact set L let $g \in \sigma(x_i)L$. Then $\sigma(x_i) \in gL^{-1}$ and $\sigma(x_i)W \subset gL^{-1}W$. If $g \in \sigma(x_j)L$ for some $j \neq i$ then clearly $\sigma(x_j)W \subset gL^{-1}W$. Hence, regarding (4.5), the number of x_i's such that $\sigma(x_i)W$ fits into $gL^{-1}W$ is bounded by $C_L := \nu(L^{-1}W)/\nu(W)$. This completes the proof. ■

Remark 4.3. *Condition* (4.3) *means that the family* $\{\sigma(x_i)U\}_{i \in I}$ *is an admissible covering in the sense of* [20], *Def.* 2.1. *Therefore, we may conclude from Lemma* 2.9 *in* [20] *that there exists a splitting* $I = \bigcup_{r=1}^{r_0} I_r$ *such that*

$$\sigma(x_i)U \cap \sigma(x_j)U = \emptyset \qquad \text{for } i, j \in I_r \text{ and } i \neq j. \tag{4.6}$$

It is standard to construct a bounded partition of unity corresponding to some U-dense and relatively separated sequence $(x_i)_{i \in I}$, i.e., a sequence of (continuous) functions $\phi_i, i \in I$, on G such that

(a) $0 \leq \phi_i(g) \leq 1$ for all $g \in G$,

(b) $\operatorname{supp} \phi_i \subset \sigma(x_i)U$,

(c) $\sum_{i \in I} \phi_i(\sigma(x)) = 1$ for all $x \in X$.

For shorter notation we define $\tau_i := \phi_i \circ \sigma$. Also we introduce the following subsets of X:

$$X_i \;:=\; \{x \in X : \sigma(x) \in \sigma(x_i)U\}.$$

Clearly, these sets form a covering of X with uniformly finite overlap. Moreover, we observe that $x_i \in X_i$ and $\operatorname{supp} \tau_i \subset X_i$.

Lemma 4.4. i) *It holds* $X_i \subset \sigma(x_i)\Pi(U)$.
ii) *If* $\sigma(\sigma(x_i)\Pi(U)) \subset \sigma(x_i)U$ *then* $\sigma(x_i)\Pi(U) = X_i$.

Proof: i) Let $x \in X_i$. Then there exists $u \in U$ such that $\sigma(x) = \sigma(x_i)u$. Thus, with h as defined in (2.1) we obtain

$$u \;=\; \sigma(x_i)^{-1}\sigma(x) \;=\; \sigma(\sigma(x_i)^{-1}x)h(\sigma(x_i)^{-1}, x).$$

Since $\Pi \circ \sigma = id$ and $h(\sigma(x_i)^{-1}, x) \in H$, application of Π yields $\Pi(u) = \sigma(x_i)^{-1}x$, i.e., $x = \sigma(x_i)\Pi(u)$.
ii) We only need to prove $\sigma(x_i)\Pi(U) \subset X_i$. Let $x = \sigma(x_i)\Pi(u)$ for some $u \in U$. Then our condition implies $\sigma(x) = \sigma(\sigma(x_i)\Pi(u)) = \sigma(x_i)u'$ for some $u' \in U$. Thus, $x \in X_i$. ■

In order to carry through the discretization machinery we have to impose the following condition on the weight function v. We require that

$$\frac{v(x)}{v(y)} \leq D \quad \text{ for all } x, y \in X_i,\; i \in I \tag{4.7}$$

for some constant $D < \infty$ that is independent of $i \in I$. In particular, v is bounded on X_i. In the terminology of Feichtinger and Gröbner [20] this means that v is moderate with respect to the covering $\{X_i\}_{i \in I}$.

For some relatively compact set U we further introduce the kernel dependent on ψ

$$\mathrm{osc}_U(x,y) \;:=\; \sup_{u \in U} |\langle A_\sigma^{-1} \mathcal{U}(\sigma(x))(Id - \mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y))\psi\rangle|, \tag{4.8}$$

which can be viewed as an adapted version of a modulus of continuity of the kernel R.

Under a condition on osc_U needed later in the discretization Theorems 4.6 and 4.7 we can determine a weight function w such that (3.13) is satisfied.

Lemma 4.5. *Let v satisfy* (4.7) *and let*

$$w(x) \;:=\; \max\left\{1,\; \sum_{i \in I} v(x_i)^{-1} \max\{1, \mu(X_i)^{-1}\} \chi_{X_i}(x)\right\} \tag{4.9}$$

with χ_{X_i} denoting the characteristic function of X_i fulfill (3.4) *and*

$$\operatorname*{ess\,sup}_{x \in X} \int_X \mathrm{osc}_{U^{-1}U}(y,x) \frac{w(y)}{w(x)} d\mu(y) \;<\; \infty. \tag{4.10}$$

Then (3.13) *is satisfied. In particular, assuming* (3.11)*, the space $\mathcal{M}_{p,v}$ is continuously embedded into $L_{\infty,1/w}(X)$.*

Moreover, if $\mu(X_i) \geq C > 0$ for all $i \in I$, then the weight w is equivalent to $x \mapsto \max\{1, v^{-1}(x)\}$ and (3.11) *already implies* (3.4).

Proof: For $R_i(x,y) := R(x,y)\tau_i(y)$ we have that

$$|\langle F, R_i(x,\cdot)\rangle| \;=\; |\int_{X_i} \overline{R(x,y)} F(y)\tau_i(y) d\mu(y)| \leq \|\chi_{X_i} F\|_{L^1} \sup_{z \in X_i} |R(x,z)|. \tag{4.11}$$

Let us first treat the first factor on the right hand side. We fix some index $k \in I$. ¿From (4.7) it follows $1 \leq D(\operatorname{ess\,sup}_{x \in X_k} v(x)^{-1}) v(y)$ for all $y \in X_k$. Since $\mu(X_k) < \infty$ we conclude by Hölder's inequality that

$$\|\chi_{X_k} F\|_{L_1} \;\leq\; D \sup_{x \in X_k} v(x)^{-1} \|\chi_{X_k} F v\|_{L_1} \;\leq\; C_k \|Fv\|_{L_p} \;=\; C_k \|F\|_{L_{p,v}} \tag{4.12}$$

for some constant C_k depending only on X_k. We define the kernel

$$K_i(x,y) \;:=\; \mu(X_k)^{-1} \chi_{X_k}(x) \chi_{X_i}(y), \quad x,y \in X, i \in I.$$

Setting $K_i^*(x,y) := K_i(y,x)$ we obtain

$$\int_X K_i^*(x,y) \chi_{X_k}(y) d\mu(y) \;=\; \mu(X_k)^{-1} \int_X \chi_{X_k}(y) \chi_{X_k}(y) \chi_{X_i}(x) d\mu(y) \;=\; \chi_{X_i}(x)$$

and further using $K_i(F)$ as integral operator with kernel K_i applied to F and (4.12)

$$\|\chi_{X_i} F\|_{L_1} \;=\; \langle K_i^*(\chi_{X_k}), |F|\rangle \;=\; \langle \chi_{X_k}, K_i(|F|)\rangle \;=\; \|\chi_{X_k} K_i(|F|)\|_{L_1} \;\leq\; C_k \|K_i(|F|)\|_{L_{p,v}}.$$

Let us estimate the operator norm of K_i on $L_{p,v}$ by the generalized Young inequality. By (4.7), we obtain for the integral with respect to x

$$\int_X K_i(x,y)\frac{v(x)}{v(y)}d\mu(x) \;=\; \mu(X_k)^{-1}\int_X \chi_{X_k}(x)\chi_{X_i}(y)\frac{v(x)}{v(y)}d\mu(x) \le D^2\frac{v(x_i)}{v(x_k)},$$

and for the integral with respect to y

$$\int_X K_i(x,y)\frac{v(x)}{v(y)}d\mu(y) = \mu(X_k)^{-1}\int_X \chi_{X_k}(x)\chi_{X_i}(y)\frac{v(x)}{v(y)}d\mu(y) \;\le\; D^2\mu(X_k)^{-1}\mu(X_i)\frac{v(x_k)}{v(x_i)}.$$

Now (3.3) implies

$$\|K_i(|F|)\|_{L_{p,v}} \le \tilde{C}_k\, v(x_i)^{-1}\,\max\{1,\mu(X_i)\}\,\|F\|_{L_{p,v}}.$$

As k was arbitrary we have altogether shown that

$$\|\chi_{X_i}F\|_{L_1} \le C\|F\|_{L_{p,v}}v(x_i)^{-1}\max\{1,\mu(X_i)\} \quad \text{for all } F\in L_{p,v}$$

with some constant $C>0$. Now we consider the second term on the right hand side of (4.11). For all $y,z\in X_i$ the triangle inequality yields

$$|R(x,z)| \;\le\; |R(x,z)-R(x,y)|+|R(x,y)| \;=\; |\langle A_\sigma^{-1}(\mathcal{U}(\sigma(z))-\mathcal{U}(\sigma(y))\psi,\mathcal{U}(\sigma(x))\psi\rangle|+|R(x,y)|.$$

Moreover, $y,z\in X_i$ imply $\sigma(z)\in\sigma(y)U^{-1}U$. Thus,

$$\begin{aligned}\sup_{z\in X_i}|R(x,z)| \;&\le\; \sup_{u\in U^{-1}U}|\langle A_\sigma^{-1}\mathcal{U}(\sigma(y))(\mathcal{U}(u^{-1})-Id)\psi,\mathcal{U}(\sigma(x))\psi\rangle| + |R(x,y)|\\ &=\; \mathrm{osc}_{U^{-1}U}(y,x)+|R(x,y)|\end{aligned}$$

for all $y\in X_i$. This shows

$$\sup_{z\in X_i}|R(x,z)| \;\le\; \mu(X_i)^{-1}\int_X \chi_{X_i}(y)\,(\mathrm{osc}_{U^{-1}U}(y,x)+|R(x,y)|)\,d\mu(y).$$

Pasting the pieces together and using $\sum_{i\in I}\tau_i(y)=1$ we obtain

$$\begin{aligned}&|\langle F,R(x,\cdot)\rangle| \;=\; |\sum_{i\in I}\langle F,R_i(x,\cdot)\rangle|\\ \le\;& \sum_{i\in I}\|\chi_{X_i}F\|_{L_1}\,\mu(X_i)^{-1}\int_X \chi_{X_i}(y)\,(\mathrm{osc}_{U^{-1}U}(y,x)+|R(x,y)|)\;d\mu(y)\\ \le\;& C\|F\|_{L_{p,v}}\int_X(\mathrm{osc}_{U^{-1}U}(y,x)+|R(x,y)|)\left(\sum_{i\in I}v(x_i)^{-1}\max\{1,\mu(X_i)\}\mu(X_i)^{-1}\chi_{X_i}(y)\right)d\mu(y).\end{aligned}$$

Clearly, the function $x\mapsto\sum_{i\in I}v(x_i)^{-1}\max\{1,\mu(X_i)\}\mu(X_i)^{-1}\chi_{X_i}(x)$ is contained in $L_{\infty,1/w}$ by definition (4.9) of w. Thus, by our assumption on R and $\mathrm{osc}_{U^{-1}U}$, the generalized Young inequality shows that $\langle F,R(x,\cdot)\rangle$ is contained in $L_{\infty,1/w}$.

The assertion that w is equivalent to $\max\{1,v^{-1}\}$ provided $\mu(X_i)\ge C>0$ follows immediately from (4.7). Moreover, by checking different cases we see that

$$\frac{\max\{1,v^{-1}(x)\}}{\max\{1,v^{-1}(y)\}} \;\le\; \max\left\{\frac{v(x)}{v(y)},\frac{v(y)}{v(x)}\right\},$$

which shows that (3.11) implies (3.4) for our choice of the weight w. ■

For simpler notation we introduce the numbers

$$a_i \;:=\; \mu(X_i).$$

Let $\ell_{p,va^{1/p}}$ denote the space of sequences over I for which

$$\|(\eta_i)_{i\in I}\|_{\ell_{p,va^{1/p}}} \;:=\; \|(\eta_i v(x_i) a_i^{1/p})_{i\in I}\|_{\ell_p(I)} < \infty.$$

The space $\ell_{p,va^{1/p-1}}$ is defined analogously. Clearly, if $(a_i)_{i\in I}$ is bounded from above and below then $\ell_{p,va^{1/p}} = \ell_{p,va^{1/p-1}} = \ell_{p,v}$ with equivalent norms. In particular, this is the case if μ is an invariant measure and the condition in Lemma 4.4 ii) is satisfied (implying $X_i = \sigma(x_i)\Pi(U)$).

Now we are ready to state our main results. The first one is a decomposition theorem which says that discretizing the representation by means of an U–dense set indeed produces an atomic decomposition of $\mathcal{H}_{p,v}$.

Theorem 4.6. *Let G be a locally compact, topological Hausdorff group with closed subgroup H and let v be a weight function on $X = G/H$. Further, let $\mathcal{U}$ be a square integrable representation of G mod (H,σ) with admissible function ψ. Assume that the kernel R fulfills (3.11) and (3.4) with an associated weight $w(x) \geq 1$ satisfying (3.13). Let a relatively compact neighborhood U of the identity in G be chosen such that*

$$\int_X \mathrm{osc}_U(y,x)\frac{v(x)}{v(y)}\,d\mu(x) \leq \gamma \quad \textit{and} \quad \int_X \mathrm{osc}_U(y,x)\frac{v(x)}{v(y)}\,d\mu(y) \leq \gamma, \tag{4.13}$$

where $\gamma < 1$. Let $(x_i)_{i\in I}$ be a U–dense, relatively separated family and assume that v satisfies (4.7).

Then $\mathcal{H}_{p,v}$, $1 \leq p \leq \infty$, has the following atomic decomposition: if $f \in \mathcal{H}_{p,v}$, $1 \leq p \leq \infty$, then f can be represented as

$$f = \sum_{i\in I} c_i \mathcal{U}(\sigma(x_i))\psi,$$

where the sequence of coefficients $(c_i)_{i\in I} = (c_i(f))_{i\in I} \in \ell_{p,va^{1/p-1}}$ depends linearly on f and satisfies

$$||(c_i)_{i\in I}||_{\ell_{p,va^{1/p-1}}} \leq A||f||_{\mathcal{H}_{p,v}}. \tag{4.14}$$

If $(c_i)_{i\in I} \in \ell_{p,va^{1/p-1}}$, then $f = \sum_{i\in I} c_i\mathcal{U}(\sigma(x_i))\psi$ is contained in $\mathcal{H}_{p,v}$ and

$$||f||_{\mathcal{H}_{p,v}} \leq B||(c_i)_{i\in I}||_{\ell_{p,va^{1/p-1}}}. \tag{4.15}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by its moments and how f can be reconstructed from these moments. This question is answered by the following theorem which shows that $\{\psi_i := A_\sigma^{-1}\mathcal{U}(\sigma(x_i))\psi : i \in I\}$ indeed give rise to Banach frames.

Theorem 4.7. *Impose the same assumptions as in Theorem 4.6 with*

$$\int_X \mathrm{osc}_U(x,y)\frac{v(x)}{v(y)}\,d\mu(x) \leq \frac{\tilde{\gamma}}{C_\psi} \quad \textit{and} \quad \int_X \mathrm{osc}_U(x,y)\frac{v(x)}{v(y)}\,d\mu(y) \leq \frac{\tilde{\gamma}}{C_\psi}, \tag{4.16}$$

where $\tilde{\gamma} < 1$, *instead of* (4.13)

Then the set

$$\{\psi_i := A_\sigma^{-1}\mathcal{U}(\sigma(x_i))\psi : i \in I\}$$

is a Banach frame for $\mathcal{H}_{p,v}$. *This means that*

i) $f \in \mathcal{H}_{p,v}$ *if and only if* $(\langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}})_{i\in I} \in \ell_{p,va^{1/p}}$;

ii) *there exist two constants* $0 < A' \leq B' < \infty$ *such that*

$$A' \|f\|_{\mathcal{H}_{p,v}} \leq \|(\langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}})_{i\in I}\|_{\ell_{p,va^{1/p}}} \leq B' \|f\|_{\mathcal{H}_{p,v}}; \tag{4.17}$$

iii) *there exists a bounded, linear reconstruction operator* $\mathcal{S}$ *from* $\ell_{p,va^{1/p}}$ *to* $\mathcal{H}_{p,v}$ *such that* $\mathcal{S}\left((\langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}})_{i\in I}\right) = f$.

Remark 4.8. i) *By interchanging the roles of* $\mathcal{H}_{p,v}$ *and* $\mathcal{K}_{p,v}$ *and of* $U(\sigma(x))\psi$ *and* $A_\sigma^{-1}U(\sigma(x))\psi$, *similar decomposition and reconstruction theorems can also be developed for the spaces* $\mathcal{K}_{p,v}$.
ii)*Further information concerning Banach frames can be found in* [28] *and in* [4].

The proofs of the Theorems 4.6 and 4.7 consist of several parts that will be presented in the sequel. The main ingredient of the following is that the operator which maps $F \in \mathcal{M}_{p,v}$ onto the function $\langle F, R(x,\cdot)\rangle$ is the identity on $\mathcal{M}_{p,v}$. The idea now is to approximate this operator (which is given by an integral) by a sum. As in [10] we use the following two approximation operators

$$\begin{aligned}
T_\phi F(x) &:= \sum_{i\in I}\langle F, \tau_i\rangle R(x_i, x) \\
&= \sum_{i\in I}\int_X F(y)\tau_i(y)\, d\mu(y) R(x_i, x), \\
S_\phi F(x) &:= \sum_{i\in I} F(x_i)\langle \tau_i, R(x,\cdot)\rangle \\
&= \sum_{i\in I}\int_X F(x_i)\tau_i(y) R(y, x)\, d\mu(y).
\end{aligned}$$

We have to prove that these operators are invertible under certain conditions and afterwards we use the correspondence principle (Theorem 3.5) to obtain an atomic decomposition of the coorbit space or even a Banach frame.

Lemma 4.9. i) *Suppose that there exists* $\gamma < 1$ *such that* (4.13) *holds. Then* $\|Id - T_\phi\|_{\mathcal{M}_{p,v}\to\mathcal{M}_{p,v}} \leq \gamma < 1$. *In particular,* T_ϕ *is bounded with bounded inverse.*
ii) *Suppose that* R *fulfills* (3.11) *and that there exists* $\tilde{\gamma} < 1$ *such that* (4.16) *holds where* C_ψ *is the constant in* (3.11). *Then* $\|Id - S_\phi\|_{\mathcal{M}_{p,v}\to\mathcal{M}_{p,v}} \leq \tilde{\gamma} < 1$. *In particular,* S_ϕ *is bounded with bounded inverse.*

<u>Proof:</u> We proceed analogously as in the proof of Theorem 4 in [10]. Using the reproducing formula on $\mathcal{M}_{p,v}$ and the fact that $(\tau_i)_{i\in I}$ is a partition of unity on X we obtain for $F \in \mathcal{M}_{p,v}$

$$F(x) = \int_X F(y)\overline{R(x,y)} d\mu(y) = \sum_{i\in I}\int_X F(y)\tau_i(y) R(y,x)\, d\mu(y).$$

It follows immediately that

$$\begin{aligned} F(x) - T_\phi F(x) &= \sum_{i\in I} \int_X F(y)\tau_i(y)[R(y,x) - R(x_i,x)]\,d\mu(y), \\ F(x) - S_\phi F(x) &= \sum_{i\in I} \int_X [F(y) - F(x_i)]\tau_i(y)R(y,x)\,d\mu(y). \end{aligned} \tag{4.18}$$

Let us first consider $F - T_\phi F$. By definition of R we obtain

$$\begin{aligned} |F(x) - T_\phi F(x)| &\le \sum_{i\in I} \int_X |F(y)|\,\tau_i(y)\,|R(y,x) - R(x_i,x)|\,d\mu(y) \\ &= \sum_{i\in I} \int_X |F(y)|\,\tau_i(y)\,|\langle A_\sigma^{-1}\big(\mathcal{U}(\sigma(y)) - \mathcal{U}(\sigma(x_i))\big)\psi, \mathcal{U}(\sigma(x))\psi\rangle|\,d\mu(y). \end{aligned}$$

Since $\operatorname{supp}\phi_i \subset \sigma(x_i)U$ we are only interested in those $y \in X$ such that $\sigma(y) \in \sigma(x_i)U$ which implies $\sigma(y) = \sigma(x_i)u$ for some $u \in U$ or equivalently $\sigma(x_i) = \sigma(y)u^{-1}$. Hence, we have

$$\begin{aligned} &|F(x) - T_\phi F(x)| \\ &\le \sum_{i\in I} \int_X |F(y)|\tau_i(y) \sup_{u\in U} |\langle A_\sigma^{-1}\big(\mathcal{U}(\sigma(y)) - \mathcal{U}(\sigma(y)u^{-1})\big)\psi, \mathcal{U}(\sigma(x))\psi\rangle|\,d\mu(y) \\ &= \sum_{i\in I} \int_X |F(y)|\tau_i(y)\operatorname{osc}_U(y,x)\,d\mu(y) \;=\; \int_X |F(y)|\operatorname{osc}_U(y,x)\,d\mu(y). \end{aligned}$$

By (4.13) and the generalized Young inequality, we obtain

$$\|F - T_\phi F\|_{\mathcal{M}_{p,v}} = \|(Id - T_\phi)F\|_{\mathcal{M}_{p,v}} \le \gamma\|F\|_{\mathcal{M}_{p,v}}.$$

Hence, $\|Id - T_\phi\|_{\mathcal{M}_{p,v}\to\mathcal{M}_{p,v}} \le \gamma < 1$ and thus T_ϕ is boundedly invertible on $\mathcal{M}_{p,v}$.

Let us now consider $F - S_\phi F$. Since $F \in \mathcal{M}_{p,v}$, we obtain using the reproducing formula and the definition of R

$$|F(y) - F(x_i)| \le \int_X |F(r)||\langle A_\sigma^{-1}\left(\mathcal{U}(\sigma(y)) - \mathcal{U}(\sigma(x_i))\right)\psi, \mathcal{U}(\sigma(r))\psi\rangle|\,d\mu(r).$$

By (4.18) we only need to consider those y with $\sigma(y) \in \sigma(x_i)U$, i.e., $\sigma(x_i) = \sigma(y)u^{-1}$ for some $u \in U$. Hence, we have

$$\begin{aligned} |F(y) - F(x_i)| &\le \int_X |F(r)| \sup_{u\in U} |\langle A_\sigma^{-1}\left(\mathcal{U}(\sigma(y)) - \mathcal{U}(\sigma(y)u^{-1})\right)\psi, \mathcal{U}(\sigma(r))\psi\rangle|\,d\mu(r) \\ &= \int_X |F(r)|\operatorname{osc}_U(y,r)\,d\mu(r). \end{aligned}$$

Since (ϕ_i) is a partition of unity, by (3.11), (4.16) and the generalized Young inequality we deduce

$$\|F - S_\phi F\|_{\mathcal{M}_{p,v}} \le C_\psi \|\sum_{i\in I} |F(\cdot) - F(x_i)|\tau_i(\cdot)\|_{L_{p,v}} \le \tilde{\gamma}\|F\|_{\mathcal{M}_{p,v}}$$

which completes the proof. ■

We further need some auxiliary statements.

Lemma 4.10. *Let $1 \le p \le \infty$. There exist constants $0 < C_1 \le C_2 < \infty$ such that*

$$C_1 \|(\eta_i)_{i\in I}\|_{\ell_{p,va^{1/p}}} \;\le\; \|\sum_{i\in I} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}} \;\le\; C_2 \|(\eta_i)_{i\in I}\|_{\ell_{p,va^{1/p}}} \tag{4.19}$$

for all sequences $(\eta_i)_{i\in I}$.

Proof: Since $(x_i)_{i\in I}$ is a relatively separated family, there exists a splitting $I = \bigcup_{r=1}^{r_0} I_r$ such that $X_i \cap X_j = \emptyset$ for $i, j \in I_r$ and $i \ne j$. Using (4.7) we obtain for $1 \le p < \infty$

$$\begin{aligned}
\|\sum_{i\in I} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}} &= \|\sum_{r=1}^{r_0}\sum_{i\in I_r} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}} \le \sum_{r=1}^{r_0} \|\sum_{i\in I_r} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}} \\
&= \sum_{r=1}^{r_0}\left(\int_X \sum_{i\in I_r} |\eta_i|^p\,\chi_{X_i}(x) v(x)^p d\mu(x)\right)^{1/p} = \sum_{r=1}^{r_0}\left(\sum_{i\in I_r} |\eta_i|^p \int_{X_i} v(x)^p d\mu(x)\right)^{1/p} \\
&\le D\sum_{r=1}^{r_0}\left(\sum_{i\in I_r} |\eta_i|^p\,\mu(X_i) v(x_i)^p\right)^{1/p} \le CD\left(\sum_{i\in I} |\eta_i|^p\, v(x_i)^p a_i\right)^{1/p}.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
\left(\sum_{i\in I} |\eta_i|^p\, v(x_i)^p a_i\right)^{1/p} &\le D\sum_{r=1}^{r_0}\left(\sum_{i\in I_r} |\eta_i|^p \int_{X_i} v(x)^p d\mu(x)\right)^{1/p} = D\sum_{r=1}^{r_0} \|\sum_{i\in I_r} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}} \\
&\le CD \|\sum_{i\in I} |\eta_i|\,\chi_{X_i}\|_{L_{p,v}}.
\end{aligned}$$

The proof for $p = \infty$ is similar. ■

The following lemma is taken from [25, Lemma 3(b)], compare also [17] or [36, Lemma 4.5.8(b)].

Lemma 4.11. *A bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{K}'_{1,w}$ is weak-$*$ convergent to $f \in \mathcal{K}'_{1,w}$ if and only if $W_\psi f_n$ converges pointwise to $W_\psi f$.*

4.2 Proof of Theorem 4.6

First we take care of the bounds (4.14) and (4.15).

Lemma 4.12. *Suppose that the assumptions of Theorem 4.6 are fulfilled. Let $F \in L_{p,v}$. Then there exists a constant $A < \infty$ such that the following inequality holds true*

$$\|(\langle F, \tau_i\rangle)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} \le A\|F\|_{L_{p,v}}.$$

Proof: Lemma 4.10 yields

$$\begin{aligned}
\|(\langle F, \tau_i\rangle)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} &\le \|(\langle |F|, \tau_i\rangle)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} \\
&\le \frac{1}{C_1}\|\sum_{i\in I} \langle |F|, \tau_i\rangle \chi_{X_i} a_i^{-1}\|_{L_{p,v}}.
\end{aligned}$$

Further, we see for an arbitrary fixed $x \in X$ that

$$\sum_{i\in I} \langle |F|, \tau_i\rangle \chi_{X_i}(x) a_i^{-1} = \sum_{i\in I_x} \langle |F|, \tau_i\rangle \mu(X_i)^{-1} ,$$

where $I_x := \{i \in I : \ x \in X_i\}$. Since $(x_i)_{i\in I}$ is a relatively separated family, we see, by using the notation of the proof of Lemma 4.10, that $\# I_x \leq r_0$ and consequently

$$\sum_{i\in I_x} \langle |F|, \tau_i\rangle \mu(X_i)^{-1} \leq \langle |F|, K(x,\cdot)\rangle$$

with

$$K(x,y) := \sum_{i\in I_x} \mu(X_i)^{-1} \chi_{X_i}(y) = \sum_{i\in I} \mu(X_i)^{-1} \chi_{X_i}(x) \chi_{X_i}(y).$$

Let us check the Schur type conditions for K. Using (4.7) the integral with respect to x yields

$$\begin{aligned} \int_X K(x,y) \frac{v(x)}{v(y)} d\mu(x) &= \sum_{i\in I} \chi_{X_i}(y) \frac{v(x_i)}{v(y)} \mu(X_i)^{-1} \int_X \chi_{X_i}(x) \frac{v(x)}{v(x_i)} d\mu(x) \\ &\leq D \sum_{i\in I} \chi_{X_i}(y) \frac{v(x_i)}{v(y)} \leq C_U D^2, \end{aligned}$$

where C_U is the constant from (4.2). The estimation for the integral with respect to y is almost the same.

Therefore the weighted Young inequality implies that

$$\|(\langle F, \tau_i\rangle)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} \leq C \, \|\langle |F|, K(x,\cdot)\rangle\|_{L_{p,v}} \leq C \, \|F\|_{L_{p,v}} .$$

■

Lemma 4.13. *Under the assumptions of Theorem 4.6 there exists a constant $B < \infty$ such that for any sequence $(c_i)_{i\in I} \in \ell_{p,va^{1/p-1}}$, $1 \leq p \leq \infty$, the following inequality holds true*

$$\| \sum_{i\in I} c_i R(x_i,\cdot)\|_{L_{p,v}} \leq B \, \|(c_i)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} .$$

If $1 \leq p < \infty$ then the sum on the left hand side converges in norm, and if $p = \infty$ then it converges pointwise.

Proof: First observe that for all $y \in X_i$ we have $\sigma(x_i) = \sigma(y) u^{-1}$ for some $u \in U$. Thus we get

$$\begin{aligned} |R(x_i,x)| &\leq |R(x_i,x) - R(y,x)| + |R(y,x)| \\ &= |\langle A_\sigma^{-1}(\mathcal{U}(\sigma(y)u^{-1})\psi - \mathcal{U}(\sigma(y))\psi), \mathcal{U}(\sigma(x))\psi\rangle| + |R(y,x)| \\ &\leq \mathrm{osc}_U(y,x) + |R(y,x)|. \end{aligned} \tag{4.20}$$

Further we obtain

$$\begin{aligned} |\sum_{i\in I} c_i R(x_i,x)| &= |\sum_{i\in I} c_i R(x_i,x) \mu(X_i)^{-1} \int_{X_i} d\mu(y)| \\ &\leq \sum_{i\in I} |c_i| \mu(X_i)^{-1} \int_X \chi_{X_i}(y)(\mathrm{osc}_U(y,x) + |R(y,x)|) d\mu(y) \\ &= \int_X (\mathrm{osc}_U(y,x) + |R(y,x)|) \left(\sum_{i\in I} |c_i| \mu(X_i)^{-1} \chi_{X_i}(y) \right) d\mu(y). \end{aligned} \tag{4.21}$$

Using the conditions (3.11) and (4.13) on R and osc_U we conclude by the generalized Young inequality

$$\|\sum_{i\in I} c_i R(x_i,\cdot)\|_{L_{p,v}} \leq (\gamma + C_\psi)\|\sum_{i\in I}|c_i|\mu(X_i)^{-1}\chi_{X_i}\|_{L_{p,v}}$$

and by Lemma 4.10 that

$$\|\sum_{i\in I} c_i R(x_i,\cdot)\|_{L_{p,v}} \leq B\,\|(c_i\mu(X_i)^{-1})_{i\in I}\|_{\ell_{p,va^{1/p}}} = B\,\|(c_i)_{i\in I}\|_{\ell_{p,va^{1/p-1}}}.$$

If $1 \leq p < \infty$ then the finite sequences are dense in $\ell_{p,va^{1/p-1}}$ and it is easy to see from the last inequality that the sum $\sum c_i R(x,x_i)$ converges in the norm of $L_{p,v}$. If $p = \infty$ then it follows once more from (4.21) and conditions (3.11) and (4.13) that $\sup_{x\in X}|\sum_{i\in I} c_i R(x_i,x)|v(x) < \infty$, in particular the series converges pointwise. ■

Let us now prove Theorem 4.6.

Proof of Theorem 4.6: Assume $f \in \mathcal{H}_{p,v}$ so that $W_\psi f \in L_{p,v}$. Further, by Lemma 4.9, the operator T_ϕ is bounded with bounded inverse. By definition of T_ϕ we obtain

$$W_\psi f(x) \;=\; T_\phi T_\phi^{-1} W_\psi f(x) \;=\; \sum_{i\in I}\langle T_\phi^{-1} W_\psi f, \tau_i\rangle R(x_i,x). \tag{4.22}$$

Applying Lemma 4.12 with $F := T_\phi^{-1} W_\psi f \in \mathcal{M}_{p,v}$ yields

$$\begin{aligned} \|(\langle T_\phi^{-1} W_\psi f,\tau_i\rangle)_{i\in I}\|_{\ell_{p,va^{1/p-1}}} &\leq A\,\|T_\phi^{-1} W_\psi f\|_{L_{p,v}} \leq A\,|||T_\phi^{-1}|||\;\|W_\psi f\|_{L_{p,v}} \\ &\leq A\,|||T_\phi^{-1}|||\;\|f\|_{\mathcal{H}_{p,v}}\;. \end{aligned}$$

Now assume $1 \leq p < \infty$. Then it follows from Lemma 4.13 that the series on the right hand side of (4.22) converges in the norm of $L_{p,v}$. Since $R(x_i,x) = W_\psi(U(\sigma(x_i)\psi))(x)$ and $\tilde{W}_\psi W_\psi$ is the identity on $\mathcal{H}_{p,v}$, equation (4.22) yields

$$f \;=\; \tilde{W}_\psi\left(\sum_{i\in I} c_i(f) W_\psi U(\sigma(x_i))\psi\right)$$

with $c_i(f) := \langle T_\phi^{-1} W_\psi f,\tau_i\rangle$. As $\tilde{W}_\psi$ is continuous on $L_{p,v}$, we obtain

$$f \;=\; \sum_{i\in I} c_i(f) U(\sigma(x_i))\psi$$

and the series on the right hand side is norm convergent.

If $p = \infty$ then the series on the right hand side of (4.22) is pointwise convergent by Lemma 4.13. As $\mathcal{H}_{\infty,v}$ is a subspace of $\mathcal{K}'_{1,w}$ it follows from Lemma 4.11 that the partial sums of $\sum_{i\in I} c_i(f)U(\sigma(x_i))\psi$ converge to f in the weak-$*$ topology of $\mathcal{K}'_{1,w}$.

Finally, we conclude from Lemma 4.13 that for $(c_i)_{i\in I} \in \ell_{p,va^{1/p-1}}$

$$\|\sum_{i\in I} c_i U(\sigma(x_i))\psi\|_{\mathcal{H}_{p,v}} \;=\; \|\sum_{i\in I} c_i R(x_i,x)\|_{L_{p,v}} \;\leq\; B\|(c_i)_{i\in I}\|_{\ell_{p,va^{1/p-1}}}.$$

This shows (4.15) and we are done. ■

4.3 Proof of Theorem 4.7

We first prove the frame bounds.

Lemma 4.14. *Suppose that the same assumptions of Theorem 4.7 are fulfilled. For $i \in I$, let $\psi_i := A_\sigma^{-1}\mathcal{U}(\sigma(x_i))\psi$. Then, for $f \in \mathcal{H}_{p,v}$, there exists a constant $B' < \infty$ such that*

$$|| \left(\langle f, \psi_i \rangle_{\mathcal{K}'_{1,w} \times \mathcal{K}_{1,w}}\right)_{i \in I} ||_{\ell_{p,va^{1/p}}} \leq B' ||f||_{\mathcal{H}_{p,v}}.$$

<u>Proof:</u> Let $F := W_\psi f$. Then the assertion is equivalent to

$$|| \left(F(x_i)\right)_{i \in I} ||_{\ell_{p,va^{1/p}}} \leq B' ||F||_{L_{p,v}}. \tag{4.23}$$

We want to use Lemma 4.10 for the proof. By the reproducing property of R we obtain

$$\begin{aligned}\sum_{i \in I} |F(x_i)| \chi_{X_i}(x) &= \sum_{i \in I} |\int_X F(y) R(x_i, y) d\mu(y)| \chi_{X_i}(x) \\ &\leq \int_X |F(y)| \sum_{i \in I} |R(x_i, y)| \chi_{X_i}(x) d\mu(y). \end{aligned} \tag{4.24}$$

This suggests to investigate the kernel

$$K(x, y) := \sum_{i \in I} |R(x_i, y)| \chi_{X_i}(x).$$

Similarly as in (4.20) we obtain for all $x \in X_i$ that

$$|R(x_i, y)| = |R(x_i, y) - R(x, y)| + |R(x, y)| \leq \operatorname{osc}_U(x, y) + |R(x, y)|.$$

Thus,

$$K(x, y) \leq \sum_{i \in I} (\operatorname{osc}_U(x, y) + |R(x, y)|) \chi_{X_i}(x) \leq C_U (\operatorname{osc}_U(x, y) + |R(x, y)|).$$

Using (3.11) and (4.16) we see by the generalized Young inequality that the integral operator associated to the kernel K acts as a bounded operator on $L_{p,v}$. Hence we conclude together with Lemma 4.10 that

$$\|(F(x_i))_{i \in I}\|_{\ell_{p,va^{1/p}}} \leq C_2 \| \sum_{i \in I} |F(x_i)| \chi_{X_i} \|_{L_{p,v}} \leq B' \|F\|_{L_{p,v}}.$$

∎

Lemma 4.15. *Let the assumptions of Theorem 4.7 be true. Suppose that* (4.16) *and* (3.11) *are satisfied and that v and w are related by* (3.13). *Let $f \in \mathcal{K}'_{1,w}$ and $\psi_i := A_\sigma^{-1}\mathcal{U}(\sigma(x_i))\psi$ for $i \in I$. If $\left(\langle f, \psi_i \rangle_{\mathcal{K}'_{1,w} \times \mathcal{K}_{1,w}}\right)_{i \in I} \in \ell_{p,va^{1/p}}$, then $f \in \mathcal{H}_{p,v}$ and there exists a constant $A' > 0$ such that*

$$||f||_{\mathcal{H}_{p,v}} \leq \frac{1}{A'} || \left(\langle f, \psi_i \rangle_{\mathcal{K}'_{1,w} \times \mathcal{K}_{1,w}}\right)_{i \in I} ||_{\ell_{p,va^{1/p}}}.$$

Proof: By Lemma 4.9 and (4.16) the operator S_ϕ is boundedly invertible and we obtain

$$\|f\|_{\mathcal{H}_{p,v}} \;=\; \|W_\psi f\|_{L_{p,v}} \;=\; \|S_\phi^{-1} S_\phi W_\psi f\|_{L_{p,v}} \;\leq\; |||S_\phi^{-1}|||\, \|S_\phi W_\psi f\|_{L_{p,v}}.$$

Setting $c_i := \langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}} \;=\; W_\psi f(x_i)$ we obtain by definition of S_ϕ that

$$S_\phi W_\psi f(x) \;=\; \sum_{i\in I} c_i \int_X \tau_i(y) R(x,y)\, d\mu(y) \;=\; \int_X R(x,y) \sum_{i\in I} c_i \tau_i(y)\, d\mu(y).$$

Applying the generalized Young inequality with (3.11) and Lemma 4.10 we obtain

$$\|S_\phi W_\psi f\|_{L_{p,v}} \;\leq\; C_\psi \|\sum_{i\in I} |c_i| \tau_i\|_{L_{p,v}} \;\leq\; C_\psi \|\sum_{i\in I} |c_i| \chi_{X_i}\|_{L_{p,v}} \;\leq\; C_\psi C_2 \|(c_i)_{i\in I}\|_{\ell_{p,va^{1/p}}}. \tag{4.25}$$

Altogether, this proves the assertion. ∎

Proof of Theorem 4.7: It remains to prove the existence of a linear bounded reconstruction operator (part (iii) of Theorem 4.7). By Lemma 4.9 the operator S_ϕ is boundedly invertible on $\mathcal{M}_{p,v}$. Thus for $f \in \mathcal{H}_{p,v}$ we obtain

$$W_\psi f \;=\; S_\phi^{-1} S_\phi W_\psi f \;=\; S_\phi^{-1}\left(\sum_{i\in I} W_\psi f(x_i) \langle \tau_i, R(x,\cdot)\rangle\right).$$

As $\tilde{W}_\psi W_\psi$ is the identity on $\mathcal{H}_{p,v}$ and $W_\psi f(x_i) = \langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}}$ we obtain

$$f \;=\; \tilde{W}_\psi S_\phi^{-1}\left(\sum_{i\in I} \langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}} \langle \tau_i, R(x,\cdot)\rangle\right).$$

Clearly, this is a reconstruction of f from the coefficients $(\langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}})_{i\in I} \in \ell_{p,va^{1/p}}$. Moreover, it is not difficult to see that $\tilde{W}_\psi$ is a continuous operator from $\mathcal{M}_{p,v}$ onto $\mathcal{H}_{p,v}$. Thus, it follows from the continuity of S_ϕ^{-1} and Lemma 4.15 (resp. its proof) that

$$\mathcal{S}(c_i)_{i\in I} \;:=\; \tilde{W}_\psi S_\phi^{-1}\left(\sum_{i\in I} c_i \langle \tau_i, R(x_i,\cdot)\rangle\right)$$

is bounded from $\ell_{p,va^{1/p}}$ into $\mathcal{H}_{p,v}$.

We finally remark that we may actually reconstruct f by a series if $1 \leq p < \infty$. Indeed, it follows from the density of the finite sequences in $\ell_{p,va^{1/p}}$ and (4.25) that the series $\sum_{i\in I} c_i \langle \tau_i, R(x_i, x)\rangle$ is norm convergent in $\mathcal{M}_{p,v}$. This allows to interchange the series with the application of the operator $\tilde{W}_\psi S_\phi^{-1}$ to obtain

$$f \;=\; \sum_{i\in I} \langle f, \psi_i\rangle_{\mathcal{K}'_{1,w}\times\mathcal{K}_{1,w}} e_i$$

where $e_i = \tilde{W}_\psi(E_i)$ and $E_i(x) := S_\phi^{-1}(\langle \tau_i, R(x,\cdot)\rangle)$. This completes the proof of Theorem 4.7.
∎

4.4 Reformulation of the Integrability Conditions

In order to apply the results of the previous sections we have to prove that the kernels $K = R, \mathrm{osc}_U$ fulfil

$$\sup_{y\in X}\int_X |K(x,y)|m(x,y)d\mu(x) \leq C, \quad \sup_{y\in X}\int_X |K(y,x)|m(y,x)d\mu(x) \leq C. \tag{4.26}$$

For symmetry reasons both integrals are the same in case $K = R$. By Remark 3.6 also the kernels

$$\begin{aligned} Z_0(x,y) &= \langle \mathcal{U}(\sigma(x))\psi, U(\sigma(y))\psi\rangle, \\ Z_2(x,y) &= \langle A_\sigma^{-1}\mathcal{U}(\sigma(x))\psi, A_\sigma^{-1}\mathcal{U}(\sigma(y))\psi\rangle \;=\; \langle A_\sigma^{-2}\mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\psi\rangle, \\ Z^{\psi,\phi}(x,y) &= \langle \mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\phi\rangle \end{aligned}$$

are of interest. Observe that these kernels and R can be written as

$$Z_\kappa^{\psi,\phi}(x,y) \;=\; \langle A_\sigma^{-\kappa}\mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\phi\rangle \tag{4.27}$$

with $\kappa = 0,1,2$ and possibly $\phi = \psi$. We remark that the definition of A_σ is dependent on ψ, so here it is meant that A_σ is formed with respect to ψ and not with respect to ϕ. For $x \in X$, let

$$B_\sigma^\kappa(x) \;=\; \mathcal{U}(\sigma(x)^{-1})A_\sigma^{-\kappa}\mathcal{U}(\sigma(x)), \quad \kappa = 0,1,2. \tag{4.28}$$

As A_σ^{-1} is a self-adjoint operator also $B_\sigma^\kappa(x)$ is self-adjoint for all $x \in X$. Clearly, $B_\sigma^0(x) = Id$. For simplicity we set $B_\sigma(x) := B_\sigma^1(x)$.

Using the fact that we are working on a homogeneous space and that our kernels are related to some group representation we may reformulate these conditions under the assumption that μ is an invariant measure.

Lemma 4.16. *Assume that μ is an invariant measure on X.*

i) *It holds*

$$\int_X |Z_\kappa^{\psi,\phi}(x,y)|m(x,y)d\mu(x) \;=\; \int_X |\langle \mathcal{U}(\sigma(x)k(x,y))\psi, B_\sigma^\kappa(y)\phi\rangle| m(\sigma(y)x,y)d\mu(x), \tag{4.29}$$

where

$$k(x,y) \;:=\; (\sigma(y)\sigma(x))^{-1}\sigma(\sigma(y)x) \;=\; h(\sigma(y),x)^{-1} \;\in H, \tag{4.30}$$

with $h \in H$ defined by (2.1).

ii) *It holds*

$$\begin{aligned} &\int_X \mathrm{osc}_U(x,y)m(x,y)d\mu(x) \\ &= \int_X \sup_{u\in U} |\langle \mathcal{U}(\sigma(x)k(x,y))(Id - \mathcal{U}(u^{-1}))\psi, B_\sigma(y)\psi\rangle| m(\sigma(y)x,y)d\mu(x) \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} &\int_X \mathrm{osc}_U(y,x)m(y,x)d\mu(x) \\ &= \int_X \sup_{u\in U} |\langle B_\sigma(y)(Id - \mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(x)k(x,y))\psi\rangle| m(\sigma(y)x,y)d\mu(x). \end{aligned} \tag{4.32}$$

Proof: i) We first rewrite the kernel $Z_\kappa^{\psi,\phi}$ in the following way

$$\begin{aligned} Z_\kappa^{\psi,\phi}(x,y) &= \langle A_\sigma^{-\kappa}\mathcal{U}(\sigma(x))\psi, \mathcal{U}(\sigma(y))\phi\rangle = \langle \mathcal{U}(\sigma(y)^{-1})A_\sigma^{-\kappa}\mathcal{U}(\sigma(x))\psi, \phi\rangle \\ &= \langle B_\sigma^\kappa(y)\mathcal{U}(\sigma(y)^{-1}\sigma(x))\psi, \phi\rangle = \langle \mathcal{U}(\sigma(y)^{-1}\sigma(x))\psi, B_\sigma^\kappa(y)\phi\rangle. \end{aligned}$$

Integrating and using the invariance of the measure μ yields

$$\begin{aligned} \int_X |Z_\kappa^{\psi,\phi}(x,y)|m(x,y)d\mu(x) &= \int_X |\langle \mathcal{U}(\sigma(y)^{-1}\sigma(x))\psi, B_\sigma^\kappa(y)\phi\rangle| m(x,y)d\mu(x) \\ &= \int_X |\langle \mathcal{U}(\sigma(y)^{-1}\sigma(\sigma(y)x)\psi, B_\sigma^\kappa(y)\phi\rangle| m(\sigma(y)x,y)d\mu(x) \\ &= \int_X |\langle \mathcal{U}(\sigma(x)k(x,y))\psi, B_\sigma^\kappa(y)\phi\rangle| m(\sigma(y)x,y)d\mu(x). \end{aligned}$$

The latter integral equals the right hand side of (4.29).

ii) We have

$$\begin{aligned} \mathrm{osc}_U(x,y) &= \sup_{u\in U} |\langle A_\sigma^{-1}\mathcal{U}(\sigma(x))(Id-\mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y))\psi\rangle| \\ &= \sup_{u\in U} |\langle \mathcal{U}(\sigma(x))(Id-\mathcal{U}(u^{-1}))\psi, A_\sigma^{-1}\mathcal{U}(\sigma(y))\psi\rangle| \\ &= \sup_{u\in U} |\langle \mathcal{U}(\sigma(x))(Id-\mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y))B_\sigma(y)\psi\rangle| \\ &= \sup_{u\in U} |\langle \mathcal{U}(\sigma(y)^{-1}\sigma(x))(Id-\mathcal{U}(u^{-1}))\psi, B_\sigma(y)\psi\rangle|. \end{aligned}$$

Using the invariance of the measure μ we obtain

$$\begin{aligned} &\int_X \mathrm{osc}_U(x,y)m(x,y)d\mu(x) \\ = &\int_X \sup_{u\in U} |\langle \mathcal{U}(\sigma(y)^{-1}\sigma(x))(Id-\mathcal{U}(u^{-1}))\psi, B_\sigma(y)\psi\rangle| m(x,y)d\mu(x) \\ = &\int_X \sup_{u\in U} |\langle \mathcal{U}(\sigma(y)^{-1}\sigma(\sigma(y)x))(Id-\mathcal{U}(u^{-1}))\psi, B_\sigma(y)\psi\rangle| m(\sigma(y)x,y)d\mu(x) \\ = &\int_X \sup_{u\in U} |\langle \mathcal{U}(\sigma(x)k(x,y))(Id-\mathcal{U}(u^{-1}))\psi, B_\sigma(y)\psi\rangle| m(\sigma(y)x,y)d\mu(x). \end{aligned}$$

This yields (4.31). On the other hand, we have

$$\begin{aligned} \mathrm{osc}_U(y,x) &= \sup_{u\in U} |\langle A_\sigma^{-1}\mathcal{U}(\sigma(y))(Id-\mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(x))\psi\rangle| \\ &= \sup_{u\in U} |\langle \mathcal{U}(\sigma(y))B_\sigma(y)(Id-\mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(x))\psi\rangle| \\ &= \sup_{u\in U} |\langle B_\sigma(y)(Id-\mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y)^{-1}\sigma(x))\psi\rangle|. \end{aligned}$$

Using once more the invariance of μ and the symmetry of m we obtain

$$
\begin{aligned}
& \int_X \operatorname{osc}_U(y,x) m(y,x) d\mu(x) \\
= & \int_X \sup_{u\in U} |\langle B_\sigma(y)(Id - \mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y)^{-1}\sigma(x))\psi\rangle| m(y,x) d\mu(x) \\
= & \int_X \sup_{u\in U} |\langle B_\sigma(y)(Id - \mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(y)^{-1}\sigma(\sigma(y)x))\psi\rangle| m(y,\sigma(y)x) d\mu(x) \\
= & \int_X \sup_{u\in U} |\langle B_\sigma(y)(Id - \mathcal{U}(u^{-1}))\psi, \mathcal{U}(\sigma(x)k(x,y))\psi\rangle| m(\sigma(y)x,y) d\mu(x).
\end{aligned}
$$

■

5 A Coorbit Theory on Homogeneous Spaces Associated to the Affine Weyl-Heisenberg Group

In this section, we will need the following unitary operators of modulation, translation and dilation on $L_2(\mathbb{R})$ and their Fourier transformed versions:

$$
\begin{aligned}
M_\omega f(t) &= e^{2\pi i\omega\cdot t} f(t), & (M_\omega f)\hat{}(\xi) &= T_\omega \hat{f}(\xi), \\
T_x f(t) &= f(t-x), & (T_x f)\hat{}(\xi) &= M_{-x}\hat{f}(\xi), \\
D_a f(t) &= |a|^{-1/2} f(t/a), & (D_a f)\hat{}(\xi) &= D_{1/a}\hat{f}(\xi).
\end{aligned}
$$

We limit the analysis to the one–dimensional case, i.e., we consider the group with the generic element $g = (x,\omega,a,\varphi)$, where $x,\omega,\varphi \in \mathbb{R}$ and $a \in \mathbb{R}_+$, and the group law

$$(x,\omega,a,\varphi)\circ(x',\omega',a',\varphi') = (x + ax', \omega + a^{-1}\omega', aa', \varphi + \varphi' + \omega a x').$$

This group is called the *affine Weyl-Heisenberg group* and is denoted by G_{aWH}. The inverse element of $g \in G_{aWH}$ is given by

$$g^{-1} = (-a^{-1}x, -a\omega, a^{-1}, -\varphi + x\omega).$$

The affine Weyl-Heisenberg group is topologically isomorphic to

$$G_{aWH} \simeq \mathbb{R}^{2+1} \times \mathbb{R}_+$$

and is a unimodular group with Haar measure $d\nu(x,\omega,a,\varphi) = dx d\omega \frac{da}{a} d\varphi$. The representation of G_{aWH} on $L_2(\mathbb{R})$ given by

$$
\begin{aligned}
\mathcal{U}(x,\omega,a,\varphi)f(t) &= e^{2\pi i\varphi} T_x M_\omega D_a f(t) \\
&= a^{-1/2} e^{2\pi i(\omega(t-x)+\varphi)} f\left(\frac{t-x}{a}\right)
\end{aligned}
\tag{5.1}
$$

is called the *Stone-von-Neumann representation.* Unfortunately, this representation $\mathcal{U}$ is not square integrable. Therefore, several homogeneous spaces of G_{aWH} were considered, see, e.g., [31, 39, 40]. Here we restrict our attention to the homogeneous space G_{aWH}/H with

$$H := \{(0,0,a,\varphi) \in G_{aWH}\}. \tag{5.2}$$

Since G is unimodular and H as an Abelian group is also unimodular the natural measure $d\mu = dxd\omega$ on G_{aWH}/H is G_{aWH}–invariant. Let $\beta : G_{aWH}/H \to \mathbb{R}_+$ be a Borel function and $\sigma(x, \omega) = (x, \omega, \beta(x, \omega), 0)$ the corresponding Borel section. By [30], it is more convenient to consider sections independent of x, i.e.,

$$\sigma(x, \omega) = (x, \omega, \beta(\omega), 0). \tag{5.3}$$

In this case, for $\psi \in L_2(\mathbb{R})$, the operator A_σ in (2.2) can be written as a Fourier multiplier operator, i.e.,

$$(A_\sigma f)\hat{} = m_\beta \hat{f}$$

in the weak sense with the symbol

$$m_\beta(\xi) := \int_{\mathbb{R}} |\hat{\psi}\big(\beta(\omega)(\xi - \omega)\big)|^2 \beta(\omega) d\omega. \tag{5.4}$$

Moreover, one can check that A_σ is bounded with bounded inverse if and only if $m_\beta(\xi)$ is bounded from above and below, i.e.,

$$C_1 \le m_\beta(\xi) \le C_2 \quad \text{a.e.} \tag{5.5}$$

for constants $0 < C_1, C_2 < \infty$. In other words, ψ is admissible if and only if (5.5) is fulfilled. Here we refer also to [30, 32, 39, 40, 41].

In the following, we are interested in the specific section σ given by the function

$$\beta(\omega) = \beta_\alpha(\omega) = (1 + |\omega|)^{-\alpha}, \quad \alpha \in [0, 1). \tag{5.6}$$

We want to verify the admissibility of some special functions ψ. To this end, we prove the following auxiliary lemma for the argument of $\hat{\psi}$ in (5.4).

Lemma 5.1. *Let*

$$r_\xi(\omega) := \beta(\omega)(\xi - \omega) = (1 + |\omega|)^{-\alpha}(\xi - \omega).$$

Then, for any fixed $A > 0$, there exists $\xi_A > 0$ such that for all $\xi \ge \xi_A$ the function r_ξ is invertible on $\mathcal{A} := \{\omega : r_\xi(\omega) \in [-A, A]\}$. The inverse function $r_\xi^{(-1)}$ of r_ξ on $[-A, A]$ has the form

$$r_\xi^{(-1)}(x) = -xg(\xi, x) + \xi$$

with some function $g(x, \xi)$ satisfying

$$xg(\xi, x) + g(\xi, x)^{1/\alpha} = 1 + \xi\,. \tag{5.7}$$

Furthermore, g fulfills

$$\lim_{\xi \to \infty} \xi^{-\alpha} g(\xi, x) = 1 \tag{5.8}$$

uniformly in $x \in [-A, A]$.

Proof: Since $\alpha \in [0, 1)$, there exists $\xi_A > 0$ such that for $\xi \ge \xi_A$ the set $\mathcal{A}$ contains only positive values of ω. Now we have for $\omega > 0$ that

$$r'_\xi(\omega) = -(1 + \omega)^{-\alpha} \left(1 + \alpha \frac{\xi - \omega}{1 + \omega}\right)$$

and it is easy to check that the right-hand side is negative for $\omega > -(1+\alpha\xi)/(1-\alpha)$ and hence for $\omega > 0$. Thus, r_ξ is monotonically decreasing on $\mathcal{A}$ and has therefore an inverse.

We see that indeed

$$r_\xi(-xg(\xi,x)+\xi) \;=\; (1-xg(\xi,x)+\xi)^{-\alpha}(\xi+xg(\xi,x)-\xi) \;=\; (g(\xi,x)^{1/\alpha})^{-\alpha}xg(\xi,x) \;=\; x$$

such that $r_\xi^{(-1)}$ has the claimed form. Since $r_\xi^{(-1)}$ has only function values in $\mathcal{A}$ we have that

$$0 < r_\xi^{(-1)}(x) \;=\; -xg(\xi,x)+\xi \;=\; g(\xi,x)^{1/\alpha}-1,$$

so that $g(\xi,x) > 1$ for all $\xi \geq \xi_A$ and all $x \in [-A,A]$. Together with (5.7) this implies for $x \in [-A,A]$ and $\alpha \in (0,1)$ that

$$1 \;\leq\; (1+A)^{1-\alpha}\,\frac{g(\xi,x)^{1/\alpha-1}}{(1+\xi)^{1-\alpha}}. \tag{5.9}$$

Furthermore, we obtain

$$\frac{g(\xi,x)^{1/\alpha}}{1+\xi} = \left|1-x\,\frac{g(\xi,x)}{1+\xi}\right| \leq 1+|x|\,\frac{g(\xi,x)}{1+\xi}(1+A)^{1-\alpha}\,\frac{g(\xi,x)^{1/\alpha-1}}{(1+\xi)^{1-\alpha}} \leq 1+\frac{|x|}{2A}\,\frac{g(\xi,x)^{1/\alpha}}{1+\xi}$$

provided that $\xi \geq (2A)^{\frac{1}{1-\alpha}}(1+A)-1$. This shows that

$$1 \geq \left(1-\frac{|x|}{2A}\right)\frac{g(\xi,x)^{1/\alpha}}{1+\xi} \geq \frac{1}{2}\frac{g(\xi,x)^{1/\alpha}}{1+\xi}, \qquad x \in [-A,A],$$

and consequently $g(\xi,x)/(1+\xi)^\alpha \leq 2^\alpha$. Hence we obtain

$$\frac{g(\xi,x)}{1+\xi} = \frac{1}{(1+\xi)^{1-\alpha}}\frac{g(\xi,x)}{(1+\xi)^\alpha} \leq 2^\alpha\frac{1}{(1+\xi)^{1-\alpha}} \to 0, \qquad \xi\to\infty$$

uniformly in $x \in [-A,A]$. By (5.7), we have for all $x \in [-A,A]$ that

$$x\lim_{\xi\to\infty}\frac{g(\xi,x)}{1+\xi} + \lim_{\xi\to\infty}\frac{g(\xi,x)^{1/\alpha}}{1+\xi} \;=\; 1 \tag{5.10}$$

which implies $\lim\limits_{\xi\to\infty} g(\xi,x)^{1/\alpha}/(1+\xi) = 1$ uniformly in $x \in [-A,A]$. This is equivalent to (5.8).

■

Now we can prove our admissibility condition.

Theorem 5.2. *Let the Borel section σ be given by* (5.3) *with β defined by* (5.6)*. Further, let ψ be a non-zero L_2 function whose Fourier transform is compactly supported. Then ψ is admissible, i.e., it satisfies* (5.5)*.*

Proof: We will only perform the analysis for ξ tending to $+\infty$. A simple integral transform shows that

$$m_\beta(-\xi) \;=\; \int_{\mathbb{R}} |\hat\psi(\beta(\omega)(\omega-\xi))|^2\beta(\omega)d\omega.$$

So the analysis for ξ tending to $-\infty$ will require only slight changes. Note that if $|\hat{\psi}|$ is an even function then $m_\beta(-\xi) = m_\beta(\xi)$ anyway.

Assume that $\operatorname{supp}\hat{\psi} \subset [-A, A]$. Then, by Lemma 5.1, we may substitute $x = r_\xi(\omega)$ in (5.4) for $\xi \geq \xi_A > 0$. This yields

$$m_\beta(\xi) \;=\; \int_{\mathbb{R}} |\hat{\psi}(r_\xi(\omega))|^2 \beta(\omega) d\omega \;=\; -\int_{\mathbb{R}} |\hat{\psi}(x)|^2 \beta(r_\xi^{(-1)}(x))(r_\xi^{(-1)})'(x) dx. \tag{5.11}$$

As in the previous lemma, for $\xi \geq \xi_A$ only positive values of ω will contribute to the first integral. Further, for $\omega > 0$, we have

$$r_\xi'(\omega) \;=\; \beta'(\omega)(\xi - \omega) - \beta(\omega) \;=\; -\beta(\omega)(\alpha(1+\omega)^{-1}(\xi - \omega) + 1) \tag{5.12}$$

which gives

$$\left(r_\xi^{(-1)}\right)'(x) \;=\; \frac{1}{r_\xi'(r_\xi^{(-1)}(x))} \;=\; -\beta(r_\xi^{(-1)}(x))^{-1}\left(\alpha\frac{\xi - r_\xi^{(-1)}(x)}{1 + r_\xi^{(-1)}(x)} + 1\right)^{-1}. \tag{5.13}$$

Thus, for $\xi \geq \xi_A > 0$, we have

$$m_\beta(\xi) \;=\; \int_{-A}^{A} |\hat{\psi}(x)|^2 \, G(\xi, x) dx \tag{5.14}$$

with

$$G(\xi, x) \;:=\; (1 + \alpha L(\xi, x))^{-1}, \quad L(\xi, x) \;:=\; \frac{\xi - r_\xi^{(-1)}(x)}{1 + r_\xi^{(-1)}(x)} \;=\; x g(\xi, x)^{1-1/\alpha}, \tag{5.15}$$

where the last equality follows by Lemma 5.1. Now (5.8) shows that

$$\lim_{\xi\to\infty} G(\xi, x) \;=\; \lim_{\xi\to\infty} \frac{1}{1 + \alpha x g(\xi, x)^{1-1/\alpha}} \;=\; \lim_{\xi\to\infty} \frac{1}{1 + \alpha x \xi^{\alpha(1-1/\alpha)}} \;=\; 1 \tag{5.16}$$

uniformly in $x \in [-A, A]$ and consequently

$$\lim_{\xi\to\infty} m_\beta(\xi) \;=\; \int_{-A}^{A} |\hat{\psi}(x)|^2 dx$$

for any L_2 function ψ with compact support in the Fourier domain. As m_β is always positive and continuous this shows that m_β is bounded from below and above for any ψ with $\operatorname{supp}\hat{\psi}$ compact. ■

In the following, we will consider coorbit spaces with respect to the following weight functions on $X = G_{aWH}/H \simeq \mathbb{R}^2$,

$$v(x, \omega) \;=\; v_s(\omega) \;=\; (1 + |\omega|)^s, \quad s \in \mathbb{R}. \tag{5.17}$$

We will see later that v_s satisfies (4.7). Associated to the weights v_s we define the functions m_s by

$$m_s((x,\omega),(\tilde{x},\tilde{\omega})) \;=\; m_s(\omega,\tilde{\omega}) \;=\; \max\left\{\frac{v_s(\omega)}{v_s(\tilde{\omega})}, \frac{v_s(\tilde{\omega})}{v_s(\omega)}\right\} \;=\; \max\left\{\left(\frac{1+|\omega|}{1+|\tilde{\omega}|}\right), \left(\frac{1+|\tilde{\omega}|}{1+|\omega|}\right)\right\}^{|s|}.$$

5.1 Integrability of the Kernels

In order to establish coorbit spaces, we need to verify the integrability of R and osc. The integrability conditions needed for the definition of the coorbit spaces and for the descretization are settled in the following theorems.

Theorem 5.3. *Let the Borel section σ be given by* (5.3) *with β defined by* (5.6)*. Further, let ψ, ϕ be non-zero L_2 functions whose Fourier transforms are compactly supported C^2-functions. Then the kernels $Z_\kappa^{\psi,\phi}$, $\kappa = 0, 1, 2$, defined by* (4.27)*, satisfy*

$$\sup_{(\tilde{x},\tilde{\omega})\in\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |Z_\kappa^{\psi,\phi}((x,\omega),(\tilde{x},\tilde{\omega}))| m_s(\omega,\tilde{\omega}) dx d\omega \; < \; \infty, \tag{5.18}$$

In particular, $R = R_\psi = Z_1^{\psi,\psi}$ satisfies the integrability condition (3.11)*.*

Theorem 5.4. *Let the Borel section σ be given by* (5.3) *with β defined by* (5.6)*. Further, let $\psi \in L_2$ with $\operatorname{supp}\hat{\psi}$ compact and $\hat{\psi} \in C^2$. Denote by osc_U the associated kernel defined in* (4.8)*. For any $\delta > 0$ there exists some neighborhood U of $e \in G$ such that*

$$\sup_{(\tilde{x},\tilde{\omega})\in\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{osc}_U((x,\omega),(\tilde{x},\tilde{\omega}))\, m(\omega,\tilde{\omega}) dx d\omega \quad < \quad \delta, \tag{5.19}$$

$$\sup_{(\tilde{x},\tilde{\omega})\in\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{osc}_U((\tilde{x},\tilde{\omega}),(x,\omega))\, m(\tilde{\omega},\omega) dx d\omega \quad < \quad \delta. \tag{5.20}$$

We will develop the proofs of these theorems in several steps. A basic ingredient is Lemma 4.16. So let us investigate B_σ^κ and $k(x,y)$ with respect to the setting of this section.

Lemma 5.5. *Let the Borel section σ be given by* (5.3)*. Then the operator $B_\sigma^\kappa(x,\omega)$ in* (4.28) *is a Fourier multiplier, i.e.,*

$$(B_\sigma^\kappa(x,\omega)f)\hat{}(\xi) \; = \; h_{x,\omega}^\kappa(\xi)\hat{f}(\xi) \tag{5.21}$$

with

$$h_{x,\omega}(\xi) \; := \; m_\beta^{-1}\big(\beta(\omega)^{-1}(\xi + \omega\beta(\omega))\big) \tag{5.22}$$

Consequently, if ψ is admissible, then we have $0 < 1/C_2 \le h_{x,\omega}(\xi) \le 1/C_1 < \infty$ a.e. with the constants C_1, C_2 from (5.5)*.*

Proof: Using the notation $\widetilde{D}_a f(t) = f(t/a)$ we have

$$\begin{aligned}
(B_\sigma^\kappa(x,\omega)f)\hat{} \; &= \; (T_{-\beta(\omega)^{-1}x} M_{-\beta(\omega)\omega} D_{\beta(\omega)^{-1}} A_\sigma^{-\kappa} T_x M_\omega D_{\beta(\omega)} f)\hat{}\, e^{2\pi i x\omega} \\
&= \; M_{\beta(\omega)^{-1}x} T_{-\beta(\omega)\omega} D_{\beta(\omega)} \left(m_\beta^{-\kappa} M_{-x} T_\omega D_{\beta(\omega)^{-1}} \hat{f} \right) e^{2\pi i x\omega} \\
&= \; \left(T_{-\beta(\omega)\omega} \widetilde{D}_{\beta(\omega)} m_\beta^{-\kappa} \right) M_{\beta(\omega)^{-1}x} T_{-\beta(\omega)\omega} M_{-\beta(\omega)^{-1}x} T_{\beta(\omega)\omega} \hat{f} e^{2\pi i x\omega} \\
&= \; \left(T_{-\beta(\omega)\omega} \widetilde{D}_{\beta(\omega)} m_\beta^{-\kappa} \right) \hat{f}
\end{aligned}$$

The last term is exactly the right hand side of (5.21). ∎

Lemma 5.6. *Let the Borel section σ be given by* (5.3)*. Then the kernel k in* (4.30) *fulfills*

$$k((x,\omega),(\tilde{x},\tilde{\omega})) \;=\; \Big(0,0,\beta(\tilde{\omega})^{-1}\beta(\omega)^{-1}\beta\big(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega\big),*\Big),$$

where $*$ denotes some function in $x,\omega,\tilde{x},\tilde{\omega}$.

Proof: A simple computation shows

$$\begin{aligned}
&[\sigma(\tilde{x},\tilde{\omega})\sigma(x,\omega)]^{-1} \;=\; [(\tilde{x},\tilde{\omega},\beta(\tilde{\omega}),0)\cdot(x,\omega,\beta(\omega),0)]^{-1}\\
=\;& [(\tilde{x}+\beta(\tilde{\omega})x,\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega,\beta(\tilde{\omega})\beta(\omega),*)]^{-1}\\
=\;& (-\beta(\tilde{\omega})^{-1}\beta(\omega)^{-1}(\tilde{x}+\beta(\tilde{\omega})x),-\beta(\tilde{\omega})\beta(\omega)(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega),\beta(\tilde{\omega})^{-1}\beta(\omega)^{-1},*).
\end{aligned}$$

Moreover, we have

$$\sigma(\tilde{x},\tilde{\omega})(x,\omega)=(\tilde{x}+\beta(\tilde{\omega})x,\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega) \tag{5.23}$$

and hence,

$$\sigma(\sigma(\tilde{x},\tilde{\omega})(x,\omega))=(\tilde{x}+\beta(\tilde{\omega})x,\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega,\beta(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega),0).$$

Thus, we obtain

$$\begin{aligned}
k((x,\omega),(\tilde{x},\tilde{\omega})) \;&=\; [\sigma(\tilde{x},\tilde{\omega})\sigma(x,\omega)]^{-1}\sigma(\sigma(\tilde{x},\tilde{\omega})(x,\omega))\\
&=\; \Big(0,0,\beta(\tilde{\omega})^{-1}\beta(\omega)^{-1}\beta\big(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega\big),*\Big).
\end{aligned}$$

This concludes the proof. ■

We remark that for the expression appearing in the integrals (4.29), (4.31) and (4.32) this shows

$$\sigma(x,\omega)k((x,\omega),(\tilde{x},\tilde{\omega})) \;=\; \Big(x,\omega,\beta(\tilde{\omega})^{-1}\beta\big(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega\big),*\Big). \tag{5.24}$$

Thus, it seems useful to define

$$\theta(\omega,\tilde{\omega}) \;:=\; \beta(\tilde{\omega})^{-1}\beta(\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega). \tag{5.25}$$

Choosing β as in (5.6) we have the following auxiliary result.

Lemma 5.7. *Let β be defined by* (5.6)*. Then for any $A>0$ there exists $\omega_0>0$ such that for all $|\omega|>\omega_0$ and all $\tilde{\omega}\in\mathbb{R}$*

$$A(1+\theta(\omega,\tilde{\omega})^{-1})\le|\omega|. \tag{5.26}$$

Moreover, there exists $\omega_1>0$ such that $\theta(\omega,\tilde{\omega})\le C$ for all $|\omega|\le\omega_1<\infty$ with a constant C independent of $\tilde{\omega}$.

Proof: For our special choice of β we obtain

$$\begin{aligned}
\theta(\omega,\tilde{\omega})^{-1} \;&=\; \left(\frac{1+|\tilde{\omega}+(1+|\tilde{\omega}|)^{\alpha}\omega|}{1+|\tilde{\omega}|}\right)^{\alpha}\\
&\le\; \left(1+\frac{|\omega|}{(1+|\tilde{\omega}|)^{1-\alpha}}\right)^{\alpha} \;\le\; (1+|\omega|)^{\alpha}.
\end{aligned}$$

Since $\alpha<1$, the right-hand side grows less than linearly in $|\omega|$ independently of $\tilde{\omega}$. This yields the first assertion.

The last assertion of the lemma is easy to see by the explicit form of θ. ■

Let us now consider the function m_s.

Lemma 5.8. *Let β be given by* (5.6) *and v_s by* (5.17)*. Then m_s can be estimated by*

$$m_s(\sigma(\tilde{x},\tilde{\omega})(x,\omega),(\tilde{x},\tilde{\omega})) \;\le\; (1+|\omega|)^{\frac{|s|}{1-\alpha}}$$

for all $\tilde{x},\tilde{\omega},x,\omega \in \mathbb{R}$ and $s \in \mathbb{R}$.

Proof: By (5.23) and definition of m_s we obtain

$$m_s(\sigma(\tilde{x},\tilde{\omega})(x,\omega),(\tilde{x},\tilde{\omega})) \;=\; \max\left\{\left(\frac{1+|\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega|}{1+|\tilde{\omega}|}\right)^{|s|},\left(\frac{1+|\tilde{\omega}|}{1+|\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega|}\right)^{|s|}\right\}.$$

As in the previous proof we see that

$$\left(\frac{1+|\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega|}{1+|\tilde{\omega}|}\right)^{|s|} \;\le\; \left(1+\frac{|\omega|}{(1+|\tilde{\omega}|)^{1-\alpha}}\right)^{|s|} \;\le\; (1+|\omega|)^{|s|} \;\le\; (1+|\omega|)^{\frac{|s|}{1-\alpha}}.$$

We claim that

$$\sup_{\tilde{\omega}\in\mathbb{R}} \frac{1+|\tilde{\omega}|}{1+|\tilde{\omega}+\beta(\tilde{\omega})^{-1}\omega|} \;\le\; (1+|\omega|)^{\frac{1}{1-\alpha}}.$$

Of course, this would prove the assertion. The substitution $\tilde{\omega}\to-\tilde{\omega}$ shows that it suffices to prove the claim for $\omega<0$. It can be easily seen that in this case the supremum is attained for some $\tilde{\omega}>0$. Consequently, it remains to consider

$$f_\omega(\tilde{\omega}) := \frac{1+\tilde{\omega}}{1+|\tilde{\omega}-\beta(\tilde{\omega})^{-1}\omega|}, \quad \omega,\tilde{\omega}>0.$$

For $\tilde{\omega}-\beta(\tilde{\omega})^{-1}\omega \ge 0$, i.e., $\tilde{\omega}/(1+\tilde{\omega})^\alpha \ge \omega$, we obtain that

$$f'_\omega(\tilde{\omega}) \;=\; \frac{-(1-\alpha)\omega(1+\tilde{\omega})^{-\alpha}}{((1+\tilde{\omega})^{1-\alpha}-\omega)^2} \;<\; 0$$

and for $\tilde{\omega}-\beta(\tilde{\omega})^{-1}\omega<0$ that

$$f'_\omega(\tilde{\omega}) \;=\; \frac{2+(1-\alpha)\omega(1+\tilde{\omega})^\alpha}{(1-\tilde{\omega}+\omega(1+\tilde{\omega})^\alpha)^2} \;>\; 0.$$

In other words, for fixed $\omega>0$, the function f_ω is strictly monotonically decreasing in the first case and strictly monotonically increasing in the second case. Consequently, since $\tilde{\omega}/(1+\tilde{\omega})^\alpha$ is monotonically increasing, the function f_ω attains its maximum for the positive solution $\tilde{\omega}$ of $\tilde{\omega}/(1+\tilde{\omega})^\alpha=\omega$ and the maximum is given by $1+\tilde{\omega}$. Finally, we conclude for $\omega,\tilde{\omega}>0$ that

$$\omega = \frac{\tilde{\omega}}{(1+\tilde{\omega})^\alpha} \;=\; (1+\tilde{\omega})^{1-\alpha}-\frac{1}{(1+\tilde{\omega})^\alpha},$$

so that

$$1+\omega = (1+\tilde{\omega})^{1-\alpha}+1-\frac{1}{(1+\tilde{\omega})^\alpha} \;\ge\; (1+\tilde{\omega})^{1-\alpha},$$

and hence

$$1+\tilde{\omega} \le (1+\omega)^{\frac{1}{1-\alpha}}.$$

This finishes the proof. ■

The proof of the following auxiliary lemma is quite technical and, hence, postponed to the appendix.

Lemma 5.9. *Let $\psi \in L_2$ such that* $\operatorname{supp}\hat{\psi} \subset [-A, A]$ *and $\hat{\psi}$ is twice continuously differentiable. Then it holds*

$$|(m_\beta^{-1})'(\xi)| \le C \min\{1, |\xi|^{-2+\alpha}\}, \tag{5.27}$$
$$|(m_\beta^{-1})''(\xi)| \le C \min\{1, |\xi|^{-3+\alpha}\}. \tag{5.28}$$

Lemma 5.10. *Let $\psi \in L_2$ with* $\operatorname{supp}\hat{\psi}$ *compact and $\hat{\psi} \in C^2$. Form the corresponding multiplier symbol $m_\beta = m_{\beta,\psi}$ by (5.4) and the function $h_{x,\omega} = h^{\psi}_{x,\omega}$ by (5.22). Then it holds*

$$\sup_{\xi \in [-A,A]} |h^{(k)}_{x,\omega}(\xi)| \le C, \quad k = 0, 1, 2$$

for any $A > 0$ and constants C independent of x, ω. (Here $h^{(k)}_{x,\omega}$ denotes the k-th derivative of $h_{x,\omega}$.)

Proof: The condition for $k = 0$ is satisfied since m_β is bounded. Using (5.27) and (5.28) we obtain

$$\begin{aligned}
\sup_{t\in[-A,A]} |h^{(k)}_{x,\omega}(t)| &= \sup_{t\in[-A,A]} \beta^{-k}(\omega)|(m_\beta^{-1})^{(k)}(\beta(\omega)^{-1}t + \omega)| \\
&\le C \sup_{t\in[-A,A]} \min\{\beta^{-k}(\omega), \beta^{-k}(\omega)|\beta(\omega)^{-1}t + \omega|^{-(k+1-\alpha)}\} \\
&\le C \sup_{t\in[-A,A]} \min\{(1+|\omega|)^{k\alpha}, (1+|\omega|)^{-\alpha(1-\alpha)}\, |t + (1+|\omega|)^{-\alpha}\omega|^{-(k+1-\alpha)}\} \\
&\le \tilde{C},
\end{aligned}$$

the constant $\tilde{C}$ being independent of ω. ■

Now, we are prepared to prove the Theorems 5.3 and 5.4.

Proof of Theorem 5.3: We assume that $\operatorname{supp}\hat{\psi}$ and $\operatorname{supp}\hat{\phi}$ are contained in $[-A, A]$. By (5.24) we have

$$\mathcal{U}(\sigma(x,\omega)k((x,\omega),(\tilde{x},\tilde{\omega}))) = e^{2\pi i *} T_x M_\omega D_{\theta(\omega,\tilde{\omega})}$$

with $\theta(\omega,\tilde{\omega})$ defined in (5.25). Using Lemma 4.16 i), Lemma 5.8, the Plancherel theorem and Lemma 5.5 we can estimate the integral in (5.18) by

$$\begin{aligned}
&\sup_{(\tilde{x},\tilde{\omega})\in\mathbb{R}^2} \int_{\mathbb{R}}\int_{\mathbb{R}} |Z^{\psi,\phi}_\kappa((x,\omega),(\tilde{x},\tilde{\omega}))| m_s(\omega,\tilde{\omega})\,dx\,d\omega \\
=& \int_{\mathbb{R}}\int_{\mathbb{R}} |\langle T_x M_\omega D_{\theta(\omega,\tilde{\omega})}\psi, B^\kappa_\sigma(\tilde{x},\tilde{\omega})\phi\rangle|\, m_s(\sigma(\tilde{x},\tilde{\omega})(x,\omega),(\tilde{x},\tilde{\omega}))\,dx\,d\omega \\
\le& \int_{\mathbb{R}}\int_{\mathbb{R}} |\langle M_{-x} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}}\hat{\psi}, h^\kappa_{\tilde{x},\tilde{\omega}}\hat{\phi}\rangle| (1+|\omega|)^{\frac{|s|}{1-\alpha}}\,dx\,d\omega.
\end{aligned}$$

The scalar product in the last integral equals

$$K(x,\omega,\tilde{x},\tilde{\omega}) := \mathcal{F}((T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}}\hat{\psi})\,\overline{h^\kappa_{\tilde{x},\tilde{\omega}}\,\hat{\phi}})(x). \tag{5.29}$$

It holds $\operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}}\hat{\psi} \subset [\omega - \theta(\omega,\tilde{\omega})^{-1}A, \omega + \theta(\omega,\tilde{\omega})^{-1}A]$. Choose ω_0 as in Lemma 5.7. Then for all $|\omega| > \omega_0$ it holds $\operatorname{supp}\hat{\phi} \cap \operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}}\hat{\psi} = \emptyset$ by (5.26) and the expression in (5.29) vanishes. Consequently, as a function of ω the kernel K has support contained in a compact set which is independent of $x, \tilde{x}, \tilde{\omega}$ and the integration with respect to ω is only over this compact set. Furthermore, the inequality

$$|\hat{f}(x)| \leq \min \left\{ \frac{\|f^{(2)}\|_{L_1}}{|x|^2}, \|f\|_{L_1} \right\} \tag{5.30}$$

suggests to consider the L_1-norm of

$$G_{\tilde{x},\tilde{\omega},\omega}(t) := (T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})(t) \overline{h^\kappa_{\tilde{x},\tilde{\omega}}(t)} \overline{\hat{\phi}(t)}$$

and of its derivatives. In particular, the L_1-norm of this function and its second derivative have to possess a bound which is uniform in $\tilde{x}, \tilde{\omega}, \omega$. Since $\hat{\psi}$ and $\hat{\phi}$ are twice differentiable, $G_{\tilde{x},\tilde{\omega},\omega}$ is a product of three C^2 functions. By the rule for the derivative of a product of functions it suffices to prove that the derivative of order $k = 0, 1, 2$ of one of the factors has a uniform L_1-bound and the others have uniform L_∞-bound with respect to $\tilde{x}, \tilde{\omega}, \omega$. For the last factor $\hat{\phi}$ the uniform L_1-bound of all of its derivatives is clear by assumption on ϕ and since it does not depend on $\tilde{x}, \tilde{\omega}, \omega$. For the first factor it holds

$$(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})^{(k)}(t) = \theta(\omega, \tilde{\omega})^{1/2+k}\, \hat{\psi}^{(k)}(\theta(\omega, \tilde{\omega})(t - \omega)).$$

Since we only need to consider $|\omega| \leq \omega_0$ and since $\theta(\omega, \tilde{\omega}) \leq C$ for all $\tilde{\omega} \in \mathbb{R}$ and $|\omega| < \omega_0$ by the second part of Lemma 5.7, we clearly have

$$\|(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})^{(k)}\|_{L_\infty} \leq C \|\hat{\psi}^{(k)}\|_{L_\infty} \leq C_k \tag{5.31}$$

for a constant C_k independent of ω and $\tilde{\omega}$. Since $\operatorname{supp} \hat{\phi} \subset [-A, A]$ we only need L_∞ bounds for $(h^\kappa_{x,\omega})^{(k)}$ on $[-A, A]$, $k = 0, 1, 2$. These bounds follows from Lemma 5.10. (If $\kappa = 2$ then $(h^2_{x,\omega})' = 2h'_{x,\omega} h_{x,\omega}$ and $(h^2_{x,\omega})'' = 2(h''_{x,\omega} h_{x,\omega} + (h'_{x,\omega})^2)$ so the bounds for $h^2_{x,\omega}$ follow from the ones for $h_{x,\omega}$.) Hence, it holds

$$|K(x, \omega, \tilde{x}, \tilde{\omega})| \leq \chi_{[-\omega_0,\omega_0]}(\omega) \min \left\{ C_1, \frac{C_2}{|x|^2} \right\}.$$

Thus, we finally obtain

$$\begin{aligned}
&\sup_{(\tilde{x},\tilde{\omega}) \in \mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |Z^{\psi,\phi}_\kappa((x, \omega), (\tilde{x}, \tilde{\omega}))| m_s(\omega, \tilde{\omega}) dx d\omega \\
&\leq \sup_{(\tilde{x},\tilde{\omega}) \in \mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |K(x, \omega, \tilde{x}, \tilde{\omega})| (1 + |\omega|)^{\frac{|s|}{1-\alpha}} dx d\omega \\
&\leq \int_{-\omega_0}^{\omega_0} (1 + |\omega|)^{\frac{|s|}{1-\alpha}} d\omega \int_{\mathbb{R}} \min \left\{ C_1, \frac{C_2}{|x|^2} \right\} dx < \infty.
\end{aligned}$$

■

Proof of Theorem 5.4: Using Lemma 4.16 ii) and proceeding as in the proof of Theorem 5.3 we have to integrate with respect to x, ω the two kernels

$$\begin{aligned}
K_1(x, \omega, \tilde{x}, \tilde{\omega}) &:= \sup_{u \in U} |\mathcal{F}\big((T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)) \overline{h_{\tilde{x},\tilde{\omega}} \hat{\psi}}\big)(x)|, \\
K_2(x, \omega, \tilde{x}, \tilde{\omega}) &:= \sup_{u \in U} |\mathcal{F}\big(\overline{h_{\tilde{x},\tilde{\omega}} \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)} (T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})\big)(x)|.
\end{aligned}$$

Now we have for $u^{-1} := (\hat{x}, \hat{\omega}, \hat{a}, \hat{\varphi}) \in U^{-1}$ that

$$\begin{aligned} \mathcal{F}\left(\mathcal{U}(u^{-1})\psi\right)(t) &= e^{2\pi i \hat{\varphi}} \mathcal{F}\left(T_{\hat{x}} M_{\hat{\omega}} D_{\hat{a}} \psi\right)(t) \\ &= e^{2\pi i \hat{\varphi}} M_{-\hat{x}} T_{\hat{\omega}} D_{\hat{a}^{-1}} \hat{\psi}(t) \;=\; \hat{a}^{1/2}\, e^{2\pi i \hat{\varphi}}\, e^{-2\pi i \hat{x} t} \hat{\psi}(\hat{a}(t - \hat{\omega})). \end{aligned} \tag{5.32}$$

Let $\operatorname{supp} \hat{\psi} \subset [-A, A]$. Then $\operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi}$ behaves as in the previous proof and

$$\begin{aligned} \operatorname{supp} \mathcal{F}\left(\mathcal{U}(u^{-1})\psi\right) &\subset [\hat{\omega} - \hat{a}^{-1}A, \hat{\omega} + \hat{a}^{-1}A], \\ \operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} [\mathcal{F}\left(\mathcal{U}(u^{-1})\psi\right)] &\subset \left[\omega + \theta^{-1}\hat{\omega} - \hat{a}^{-1}\theta^{-1}A, \omega + \theta^{-1}\hat{\omega} + \hat{a}^{-1}\theta^{-1}A\right]. \end{aligned}$$

Both kernels vanish for $|\omega| > \omega_U$ if for those ω

i) $\operatorname{supp} \hat{\psi} \cap \operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi} = \emptyset$,

ii) $\operatorname{supp} \hat{\psi} \cap \operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} [\mathcal{F}\left(\mathcal{U}(u^{-1})\psi\right)] = \emptyset$

iii) $\operatorname{supp} \mathcal{F}\left(\mathcal{U}(u^{-1})\psi\right) \cap \operatorname{supp} T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi} = \emptyset$.

Property i) is fulfilled for $|\omega| > \omega_0$ with ω_0 related to A as in Lemma 5.7. Let $U^{-1} \subset [-\varepsilon_x, \varepsilon_x] \times [-\varepsilon_\omega, \varepsilon_\omega] \times [1 - \varepsilon_a, 1 + \varepsilon_a] \times [-\varepsilon_\varphi, \varepsilon_\varphi]$. Then it can be checked that we can modify Lemma 5.7 as follows: for any $A > 0$ there exists $\omega_1 > 0$ such that for all $|\omega| > \omega_1$ it holds $A + \theta^{-1}(\frac{A}{1-\varepsilon_a} + \varepsilon_\omega) < |\omega|$. Then ii) is fulfilled for $|\omega| > \omega_1$. Finally, iii) holds for $|\omega| \geq \varepsilon_\omega + A(\theta^{-1} + \frac{1}{1-\varepsilon_a})$. Once again, by using a modification of Lemma 5.7, it can be shown that this is satisfied for $|\omega| > \omega_2$. Now we can choose ω_U as the maximum of the right-hand sides.

Following further the arguments in the previous proof, in particular (5.30), it remains to consider

$$\begin{aligned} G^1_{\tilde{x},\tilde{\omega},\omega}(t) &:= (T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi))(t) \overline{h_{\tilde{x},\tilde{\omega}}(t) \hat{\psi}(t)}, \\ G^2_{\tilde{x},\tilde{\omega},\omega}(t) &:= \overline{h_{\tilde{x},\tilde{\omega}}(t) \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)(t)} (T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})(t). \end{aligned}$$

Again we use that G^i, $i = 1, 2$, is a product of three C^2 function so that it suffices to prove L_∞, resp. L_1 estimates for the derivatives of the three factors. By Lemma 5.10, we have $\|h^{(k)}_{\tilde{x},\tilde{\omega}}\|_\infty \leq C$, $k = 0, 1, 2$. Since $\hat{\psi}$ is a C^2 function with compact support, $\|\hat{\psi}^{(k)}\|_p$, $k = 0, 1, 2$; $p = 1, \infty$ are uniformly bounded. Moreover, for $|\omega| \leq \omega_U$, we see by (5.31) and since $\operatorname{supp}(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})^{(k)}$ is finite that $\|(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \hat{\psi})^{(k)}\|_p$, $k = 0, 1, 2$, $p = 1, \infty$, are uniformly bounded. Let us finally consider $\|\mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)^{(k)}\|_{L_\infty}$ and $\|(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi))^{(k)}\|_{L_\infty}$. By (5.31) we have for $|\omega| \leq \omega_U$ that

$$\|(T_\omega D_{\theta(\omega,\tilde{\omega})^{-1}} \mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi))^{(k)}\|_{L_\infty} \leq C \|\mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)^{(k)}\|_{L_\infty}$$

so that it remains to consider $\|\mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)^{(k)}\|_{L_\infty}$. We claim that

$$\lim_{u^{-1} \to e} \|(\mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi))^{(k)}\|_{L_\infty} = 0, \quad k = 0, 1, 2. \tag{5.33}$$

We obtain

$$\begin{aligned} \|\mathcal{F}(\psi - \mathcal{U}(u^{-1})\psi)^{(k)}\|_{L_\infty} &= \|\hat{\psi}^{(k)} - e^{2\pi i \hat{\varphi}} [M_{-\hat{x}} T_{\hat{\omega}} D_{\hat{a}^{-1}} \hat{\psi}]^{(k)}\|_{L_\infty} \\ &\leq |1 - e^{2\pi i \hat{\varphi}}|\, \|[M_{-\hat{x}} T_{\hat{\omega}} D_{\hat{a}^{-1}} \hat{\psi}]^{(k)}\|_{L_\infty} + \|\hat{\psi}^{(k)} - [M_{-\hat{x}} T_{\hat{\omega}} D_{\hat{a}^{-1}} \hat{\psi}]^{(k)}\|_{L_\infty} \end{aligned}$$

and replace the derivatives by

$$[M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}]^{(1)} = \hat{a}M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}^{(1)} - 2\pi i\hat{x}M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi},$$

$$[M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}]^{(2)} = \hat{a}^2M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}^{(2)} - 4\pi i\hat{x}\hat{a}M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}^{(1)} + (2\pi i\hat{x})^2M_{-\hat{x}}T_{\hat{\omega}}D_{\hat{a}^{-1}}\hat{\psi}.$$

Using the triangle inequality and the fact that for a continuous function f with compact support

$$\lim_{\varphi\to 0}|1-e^{2\pi i\varphi}| = \lim_{x\to 0}\|T_xf-f\|_{L_\infty} = \lim_{\omega\to 0}\|M_\omega f-f\|_{L_\infty} = \lim_{a\to 1}\|D_af-f\|_{L_\infty} = 0$$

we deduce (5.33). This finishes the proof. ■

Remark 5.11. *The proof shows also that for any choice of U the integrals* (5.19) *and* (5.20) *containing* osc_U *are finite (but not necessarily less than δ).*

5.2 U-Dense Sets Associated to α-Coverings

We want to discuss the geometry of suitable U-dense sets associated to the section defined with $\beta(\omega) = (1+|\omega|)^{-\alpha}$ as in (5.3). In particular, we want to provide a set

$$\mathcal{X} := \{(x_{j,k},\omega_j)\}_{(j,k)\in\mathbb{Z}^2} \subset \mathbb{R}\times\mathbb{R} \simeq X$$

such that the covering property (4.1) and the finite overlap property (4.2) are satisfied. As stated in Remark 4.1 condition (4.2) is equivalent to (4.3).

Theorem 5.12. *Assume $\alpha \in [0,1)$. For all $\varepsilon > 0$ small enough and for suitable constants $c,\ell > 0$ independent of $\varepsilon > 0$ (to be determined in the proof) denote $U(\varepsilon) := (-\varepsilon,\varepsilon)\times(-2\varepsilon c, 2\varepsilon c)\times((1+\ell\varepsilon)^{-1}, 1+\ell\varepsilon)\times(-\varepsilon,\varepsilon) \subset G_{aWH}$ a relatively compact neighborhood of $e := (0,0,1,0) \in G_{aWH}$ with non-void interior. Let us denote $\omega_j := p_\alpha(\varepsilon j)$ and $x_{j,k} := \varepsilon\beta(\omega_j)k$, where*

$$p_\alpha(\omega) := \mathrm{sgn}(\omega)\left((1+(1-\alpha)|\omega|)^{1/(1-\alpha)} - 1\right).$$

Then the set $\mathcal{X}$ has the properties

- C1) $\sigma(X) \subset \bigcup_{(j,k)\in\mathbb{Z}^2}\sigma(x_{j,k},\omega_j)U(\varepsilon)$,
- C2) $\sup_{(j',k')\in\mathbb{Z}^2}\#\{(j,k)\in\mathbb{Z}^2 : \sigma(x_{j,k},\omega_j)L\cap\sigma(x_{j',k'},\omega_{j'})L \neq \emptyset\} \le C_L < \infty$, *for all relatively compact $L \subset G_{aWH}$ with non-void interior.*

In other words, $\mathcal{X}$ is $U(\varepsilon)$-dense and relatively separated.

Proof: We split the proof into different steps.
Step 1. Frequency decomposition. The function $p_\alpha(\omega)$ is a continuous and monotone bijection of $\mathbb{R}$. Therefore it maps admissible coverings of $\mathbb{R}$ into admissible coverings of $\mathbb{R}$. This implies that $\tilde{\Omega}_j^\alpha(\varepsilon) := \{p_\alpha(\omega) : \omega \in (\varepsilon(j-1), \varepsilon(j+1))\}$ defines an admissible covering for $\mathbb{R}$. Now, observe that $s_\alpha(\omega) := \frac{dp_\alpha}{d\omega}(\omega) = (1+(1-\alpha)|\omega|)^{\alpha/(1-\alpha)}$, and it is immediate to show that

$$s_\alpha(\omega) = (\beta(p_\alpha(\omega)))^{-1}. \tag{5.34}$$

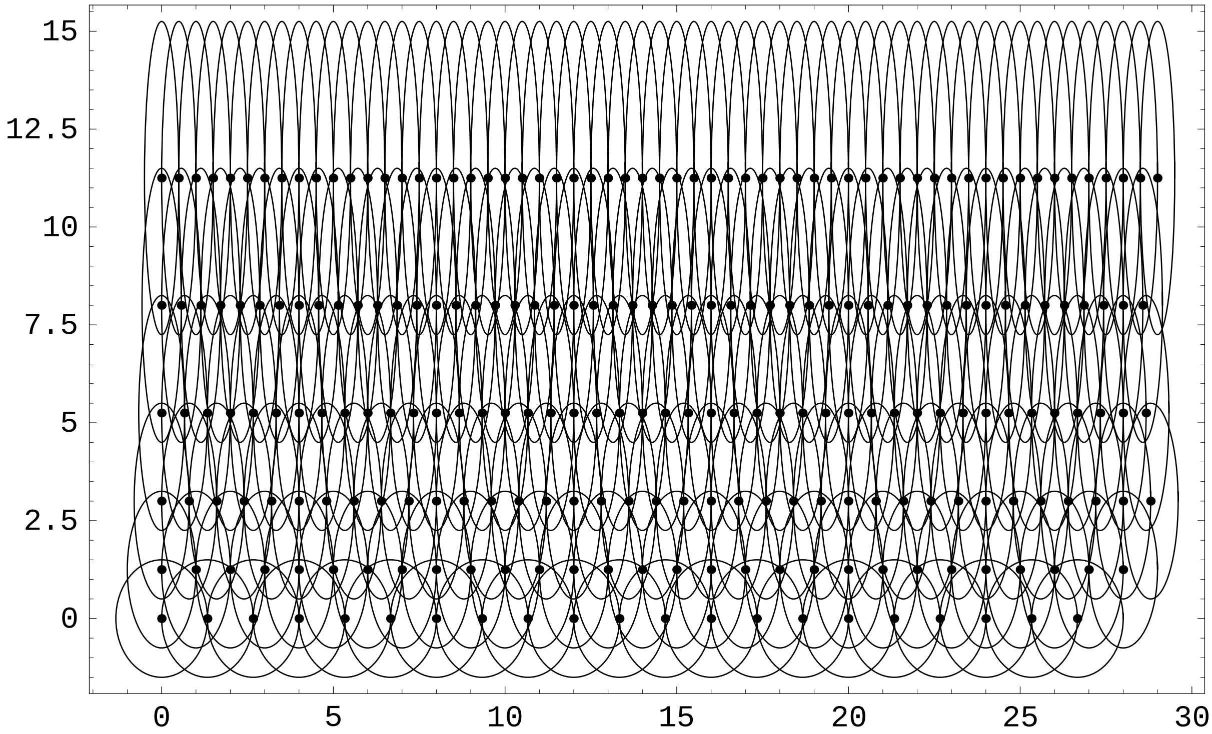

Figure 1: Example of admissible covering and U-dense set in the time-frequency plane for $\alpha = 1/2$.

By the mean value theorem one therefore has $\operatorname{diam}(\tilde{\Omega}_j^\alpha(\varepsilon)) = |p_\alpha(\varepsilon(j+1)) - p_\alpha(\varepsilon(j-1))| = 2\varepsilon s_\alpha(\xi)$, for some $\xi \in (\varepsilon(j-1), \varepsilon(j+1))$. Clearly, for all $\xi, \omega \in \varepsilon(j-1, j+1)$ it holds

$$\left(\frac{1+(1-\alpha)\varepsilon|j-1|}{1+(1-\alpha)\varepsilon|j+1|}\right)^{\alpha/(1-\alpha)} \leq \left(\frac{1+(1-\alpha)|\omega|}{1+(1-\alpha)|\xi|}\right)^{\alpha/(1-\alpha)} \leq \left(\frac{1+(1-\alpha)\varepsilon|j+1|}{1+(1-\alpha)\varepsilon|j-1|}\right)^{\alpha/(1-\alpha)}$$

and therefore, for all $\xi \in (j-1, j+1)$ we have $s_\alpha(\varepsilon\xi) \sim s_\alpha(\varepsilon j)$, uniformly with respect to $j \in \mathbb{Z}$ and $\varepsilon > 0$. Thus, $\operatorname{diam}(\tilde{\Omega}_j^\alpha(\varepsilon)) \sim 2\varepsilon s_\alpha(\varepsilon j)$ and $\Omega_j^\alpha(\varepsilon) := (p_\alpha(\varepsilon j) - 2\varepsilon c s_\alpha(\varepsilon j), p_\alpha(\varepsilon j) + 2\varepsilon c s_\alpha(\varepsilon j))$ defines an equivalent admissible covering [15] for $\mathbb{R}$ for a suitable constant $c > 0$ (independent of j and ε), and thus

$$\sup_{i \in \mathbb{Z}} \#\{j \in \mathbb{Z} : \Omega_i^\alpha(\varepsilon) \cap \Omega_j^\alpha(\varepsilon) \neq 0\} \leq N, \tag{5.35}$$

for some $N \in \mathbb{N}$. Denoting $\omega_j := p_\alpha(\varepsilon j)$, one can rewrite

$$\Omega_j^\alpha(\varepsilon) := (\omega_j - 2\varepsilon c(\beta(\omega_j))^{-1}, \omega_j + 2\varepsilon c(\beta(\omega_j))^{-1}). \tag{5.36}$$

Step 2. Time decomposition. For any fixed $j \in \mathbb{Z}$ let us consider $x_{j,k} := \varepsilon\beta(\omega_j)k, \quad k \in \mathbb{Z}$. It is immediate to show that for $N \in \mathbb{N}$

$$\sup_{j,k \in \mathbb{Z}} \#\{h \in \mathbb{Z} : x_{j,k} \leq x_{j-N,h} \leq x_{j,k+1}\} \leq C_N. \tag{5.37}$$

Define $T_{j,k}^\alpha(\varepsilon) := (x_{j,k} - \varepsilon\beta(\omega_j), x_{j,k} + \varepsilon\beta(\omega_j))$.

Step 3. Time-Frequency decomposition. Combining (5.35) and (5.37) one can show that

$$\Omega^\alpha_{j,k}(\varepsilon) := T^\alpha_{j,k}(\varepsilon) \times \Omega^\alpha_j(\varepsilon) \tag{5.38}$$

defines an admissible covering for $\mathbb{R} \times \mathbb{R} \simeq X$ (see Figure 1), and

$$\sup_{i,h \in \mathbb{Z}\times\mathbb{Z}} \#\{j,k \in \mathbb{Z} : \Omega^\alpha_{i,h}(\varepsilon) \cap \Omega^\alpha_{j,k}(\varepsilon) \neq \emptyset\} \leq M, \tag{5.39}$$

for some $M \in \mathbb{N}$.

Observe now that $\Omega^\alpha_{i,h}(\varepsilon) \cap \Omega^\alpha_{j,k}(\varepsilon) = \emptyset$ implies $\left(\Omega^\alpha_{i,h}(\varepsilon) \times V_1\right) \cap \left(\Omega^\alpha_{j,k}(\varepsilon) \times V_2\right) = \emptyset$ for all $V_1, V_2 \subset \mathbb{R}_+ \times \mathbb{R}$. Moreover, by a straightforward application of the group law in G_{aWH} we obtain

$$\sigma(x_{j,k}, \omega_j)U(\varepsilon) = \Omega^\alpha_{j,k}(\varepsilon) \times \beta(\omega_j)((1+\ell\varepsilon)^{-1}, 1+\ell\varepsilon) \times (-\varepsilon(1+\omega_j\beta(\omega_j)), \varepsilon(1+\omega_j\beta(\omega_j))). \tag{5.40}$$

Assume that $(x,\omega) \in X \simeq \mathbb{R} \times \mathbb{R}$, then there exists $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ such that $(x,\omega) \in \Omega^\alpha_{j,k}(\varepsilon)$. We want to show that $\beta(\omega) \in \beta(\omega_j)((1+\ell\varepsilon)^{-1}, 1+\ell\varepsilon)$ for some $\ell > 0$ independent of $|j| \geq 1$ and $\varepsilon > 0$. Since $\omega \in \Omega^\alpha_j(\varepsilon)$ and by (5.36), it holds

$$\left(\frac{1+|\omega_j|}{1+|\omega_j + \mathrm{sgn}(\omega_j)2\varepsilon c(\beta(\omega_j))^{-1}|}\right) \leq \frac{1+|\omega_j|}{1+|\omega|} \leq \left(\frac{1+|\omega_j|}{1+|\omega_j - \mathrm{sgn}(\omega_j)2\varepsilon c(\beta(\omega_j))^{-1}|}\right).$$

Since both the left and right estimates are finite (at least for all $\varepsilon > 0$ small enough), strictly positive, and tending to 1 for j tending to ∞, certainly there exists a constant $\ell > 0$ independent of j and ε such that

$$(1+\varepsilon\ell)^{-1} \leq \frac{\beta(\omega)}{\beta(\omega_j)} \leq (1+\varepsilon\ell).$$

This implies that $\beta(\omega) \in \beta(\omega_j)((1+\ell\varepsilon)^{-1}, 1+\ell\varepsilon)$ and by (5.40) that $\sigma(x,\omega) = (x,\omega,\beta(\omega),0) \in \sigma(x_{j,k},\omega_j)U(\varepsilon)$. This together with (5.39) finally implies that $\{\sigma(x_{j,k},\omega_j)U(\varepsilon)\}_{(j,k)\in\mathbb{Z}^2}$ is an admissible covering for $\sigma(X)$ and that C1) and C2) hold with $L = U(\varepsilon)$.

It remains to prove that C2) holds for *all* relatively compact L with nonvoid interior. By Remark 4.3 there exists a splitting $\mathbb{Z}^2 = \bigcup_{r=1}^{r_0} I_r$ such that $\sigma(x_{j,k},\omega_j)U(\varepsilon) \cap \sigma(x_{j',k'},\omega_{j'})U(\varepsilon) = \emptyset$ for all $(j,k),(j',k') \in I_r, (j,k) \neq (j',k')$. Therefore, as in the proof of Theorem 4.2, if $g \in \sigma(x_{j,k},\omega_j)L$ then $\sigma(x_{j,k},\omega_j)U(\varepsilon) \subset gL^{-1}U(\varepsilon)$ and

$$\#\{(j,k) \in I_r : g \in \sigma(x_{j,k},\omega_j)L\} \leq \frac{\nu(L^{-1}U(\varepsilon))}{\nu(U(\varepsilon))},$$

where ν denotes the Haar measure of G_{aWH}. Since $\{(j,k) \in \mathbb{Z}^2 : g \in \sigma(x_{j,k},\omega_j)L\} = \bigcup_{r=1}^{r_0}\{(j,k) \in I_r : g \in \sigma(x_{j,k},\omega_j)L\}$, we conclude $\#\{(j,k) \in \mathbb{Z}^2 : g \in \sigma(x_{j,k},\omega_j)L\} \leq C_L := r_0\frac{\nu(L^{-1}U(\varepsilon))}{U(\varepsilon)}$. ■

Remark 5.13. *The previous proof showed that actually*

$$X_{j,k} = \{(x,\omega) : \sigma(x,\omega) \in \sigma(x_{j,k},\omega_j)U(\varepsilon)\} = \sigma(x_{j,k},\omega_j)\Pi(U(\varepsilon)).$$

Since the measure μ on X is invariant this means that

$$a_{j,k} = \mu(X_{j,k}) = \mu(\Pi(U(\varepsilon))) = const.$$

Lemma 5.14. *The weight functions v_s in* (5.17) *satisfy the condition* (4.7)*.*

Proof: Since v_s depends only on ω and since $v_s = v^s$ it suffices to show that

$$\max_{\omega,\omega' \in \Omega_j^\alpha} \frac{1+|\omega|}{1+|\omega'|} =: m_j$$

has a uniform bound with respect to $j \in \mathbb{Z}$. For symmetry reasons we may restrict to $j > 0$. By (5.36) we obtain

$$m_j = \frac{1+|w_j + 2\varepsilon c(1+|\omega_j|)^\alpha|}{1+|w_j - 2\varepsilon c(1+|\omega_j|)^\alpha|}.$$

Since $0 \le \alpha < 1$ and $\omega_j \to \infty$ as $j \to \infty$ we have $\lim_{j\to\infty} m_j = 1$. In particular, m_j is uniformly bounded with respect to j. ■

5.3 Coorbit Spaces

Now we are prepared to introduce the coorbit spaces with respect to the sections associated to $\beta_\alpha(\omega) = (1+|\omega|)^{-\alpha}$, $0 \le \alpha < 1$ and the weight functions $v_s(\omega) = (1+|\omega|)^s$, $s \in \mathbb{R}$.

Indeed, Lemma 5.14 states that the moderateness condition (4.7) is satisfied for the weight function v_s. Choose a Schwartz function ψ with compactly supported Fourier transform. Then by Theorem 5.3 the kernel $R = R_\psi$ satisfies the integrability condition (3.11).

If $s \ge 0$ then by Remark 3.4 the weight function $w(x,\omega) \equiv 1$ satisfies the embedding condition (3.13), and also the integrability condition (3.4) with respect to $w \equiv 1$ is fulfilled. If $s < 0$ then by Lemma 4.5 a valid choice for the weight w is $w(x,\omega) = v_s(\omega)^{-1} = v_{|s|} = (1+|\omega|)^{|s|}$. The integrability condition (4.10) on $\mathrm{osc}_{U^{-1}U}$ is satisfied by Theorem 5.4 and Remark 5.11. The numbers $a_{j,k} = \mu(X_{j,k})$ are constant, in particular bounded from below, see Remark 5.13. Moreover, also in the case $s < 0$ the integrability condition (3.4) on R with respect to $w = v_{|s|}$ is satisfied, once again by Theorem 5.3. For convenience we define

$$w_s(\omega) := \begin{cases} 1 & \text{if } s \ge 0, \\ (1+|\omega|)^{|s|} & \text{if } s < 0. \end{cases}$$

The arguments above imply that the spaces $\mathcal{H}_{1,w_s}$ and $\mathcal{K}_{1,w_s}$, and hence also their duals $\mathcal{H}'_{1,w_s}$ and $\mathcal{K}'_{1,w_s}$, are well-defined for any $s \in \mathbb{R}$. When emphasizing the dependence on α we add an index, e.g. $\mathcal{H}'_{1,w_s,\alpha}$. Also, it is clear now from the above reasoning that the general coorbit spaces are well defined, i.e.,

$$\begin{aligned} \mathcal{H}_{p,v_s} &= \mathcal{H}_{p,v_s,\alpha} = \{f \in \mathcal{K}'_{1,w_s,\alpha} : V_\psi^\alpha \in L_{p,v_s}\}, \\ \mathcal{K}_{p,v_s} &= \mathcal{K}_{p,v_s,\alpha} = \{f \in \mathcal{H}'_{1,w_s,\alpha} : W_\psi^\alpha \in L_{p,v_s}\}. \end{aligned}$$

Moreover, it follows from Remark 3.6 and Theorem 5.3 that

$$\mathcal{H}_{p,v_s,\alpha} = \mathcal{K}_{p,v_s,\alpha},$$

and different choices of the Schwartz function ψ with compact support in the Fourier domain define the same spaces $\mathcal{H}_{p,v_s,\alpha}$ with equivalent norms. In the next section we will identify the coorbit spaces with α-modulation spaces.

6 The Frame Theory of α-Modulation Spaces $M^{s,\alpha}_{p,q}$

The α–modulation spaces are usally defined by means of the flexible Gabor–wavelet transform

$$V_\psi^\alpha(f)(x,\omega) = \langle f, \mathcal{U}(\sigma(x,\omega))\psi\rangle = \langle f, T_x M_\omega D_{\beta(\omega)}\psi\rangle. \tag{6.1}$$

It is easily verified that for $\alpha = 0$, i.e., $\beta(\omega) = 1$, the family

$$\{\mathcal{U}(\sigma(x,\omega))\psi = T_x M_\omega \psi : (x,\omega) \in X\}$$

is in fact a Gabor system and $V_\psi^0 f := \langle f, \mathcal{U}(\sigma(x,\omega))\psi\rangle$ coincides with the classical short time Fourier transform (STFT), while for $\alpha \to 1$ the family tends to the situation encountered in the wavelet context, where V_ψ^1 is just a slight modification of the continuous wavelet transform (CWT). The intermediate case $\alpha = 1/2$ appears in the literature as the Fourier-Bros-Iagolnitzer (FBI) transform [6, 32].

The introduction of a new class of function spaces defined as retract of weighted $L_{p,q}$ spaces by means of V_ψ^α has been suggested already in [32, 30, 31, 14, 23]. An application of [13, Theorem 4.3] shows that this class coincides with the family of so called α-modulation spaces $M^{s,\alpha}_{p,q}$ introduced independently by Gröbner [27, 15] and Päivärinta/Somersalo [38] as an "intermediate" family between modulation [29] and *inhomogeneous* Besov spaces [26, 43, 44]. In particular, one characterizes *α-modulation spaces* as follows. For $s \in \mathbb{R}$, for all $1 \le p, q \le \infty$, and for $\alpha \in [0,1]$

$$M^{s+\alpha(1/q-1/2),\alpha}_{p,q}(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) : V_\psi^\alpha(f) \in L^s_{p,q}(\mathbb{R}^2)\}, \tag{6.2}$$

$$\|f\|_{M^{s+\alpha(1/q-1/2),\alpha}_{p,q}} \asymp \|V_\psi^\alpha(f)\|_{L^s_{p,q}},$$

where ψ is a suitable Schwartz function and $L^s_{p,q}(\mathbb{R}^2)$ is the space of functions F on $\mathbb{R}^2$ such that

$$\|F\|_{L^s_{p,q}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,\omega)|^p dx\right)^{q/p} (1+|\omega|)^{sq} d\omega\right)^{1/q} < \infty.$$

For $\alpha = 0$, the space $M^{s,0}_{p,q}(\mathbb{R})$ coincides with the modulation space $M^s_{p,q}(\mathbb{R})$. For $\alpha \to 1$ the space $M^{s,1}_{p,q}(\mathbb{R})$ coincides with the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R})$.

Interesting analysis has been recently developed on the scale of such function spaces. The mapping properties on α-modulation spaces of pseudodifferential operators in certain Hörmander classes have been studied by Holschneider and Nazaret [32] and Borup [5] as generalizations of classical results of Cordoba and Fefferman [7]. Characterizations of α-modulation spaces by *brushlet unconditional bases* have been given by Nielsen and Borup [34], while a corresponding characterizing family of intermediate Banach frames and atomic decompositions between Gabor and wavelet frames was defined in [14, 23].

We want to show that the frames appearing in [14, 23] can also be derived by an application of our coorbit space theory. Therefore, let us finally apply our abstract discretization theorems to α-modulation spaces.

Theorem 6.1. *Let $1 \le p \le \infty$, $0 \le \alpha < 1$ and $s \in \mathbb{R}$. Let $\psi \in L_2$ with* $\operatorname{supp}\hat{\psi}$ *compact and $\hat{\psi} \in C^2$. Then the following holds true.*

1) *The coorbit spaces $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)},\alpha}$ can be identified with the α–modulation spaces $M^{s,\alpha}_{p,p}$.*

2) *There exists $\varepsilon_0 > 0$ with the following property: Let $\{(x_{j,k}, \omega_j)\}_{j,k\in\mathbb{Z}}$ denote the point set associated to any $0 < \varepsilon \le \varepsilon_0$ as constructed in Theorem* 5.12.

   i) (Atomic decomposition) *Any $f \in M^{s,\alpha}_{p,p}$ can be written as*

$$f \;=\; \sum_{(j,k)\in\mathbb{Z}^2} c_{j,k}(f) T_{x_{j,k}} M_{\omega_j} D_{\beta_\alpha(\omega_j)} \psi$$

   and there exist constants $0 < C_1, C_2 < \infty$ (independent of p) such that

$$C_1 \|f\|_{M^{s,\alpha}_{p,p}} \;\le\; \left(\sum_{(j,k)\in\mathbb{Z}^2} |c_{j,k}(f)|^p (1 + (1-\alpha)|j|)^{\frac{s-\alpha(1/p-1/2)}{1-\alpha} p} \right)^{1/p} \;\le\; C_2 \|f\|_{M^{s,\alpha}_{p,p}}.$$

   ii) (Banach Frames) *The set of functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}} := \{T_{x_{j,k}} M_{\omega_j} D_{\beta_\alpha(\omega_j)} \psi\}_{j,k\in\mathbb{Z}^2}$ forms a Banach frame for $M^{s,\alpha}_{p,p}$. In particular, there exist constants $0 < C_1, C_2 < \infty$ (independent of p) such that*

$$C_1 \|f\|_{M^{s,\alpha}_{p,p}} \;\le\; \left(\sum_{(j,k)\in\mathbb{Z}^2} |\langle f, \psi_{j,k}\rangle|^p (1 + (1-\alpha)|j|)^{\frac{s-\alpha(1/p-1/2)}{1-\alpha} p} \right)^{1/p} \;\le\; C_2 \|f\|_{M^{s,\alpha}_{p,p}}.$$

Proof: We start by showing part 1). Let us consider the functions $\psi_{j,k} = T_{x_{j,k}} M_{\omega_j} D_{\beta_\alpha(\omega_j)} \psi$ as in 2) and let us define $\mathcal{H}_0$ as the space of finite linear combinations of these functions. Observe that Theorem 5.12 states that the point set $\{(x_{j,k}, \omega_j)\}$ in 2) is $U(\varepsilon)$-dense and relatively separated. Therefore, the abstract Theorems 4.6 and 4.7 (resp. Remark 4.8 i)) in connection with the Theorem 5.4 about the integrability of the osc_U kernel imply the existence of an atomic decomposition and of Banach frames with respect to the associated $\ell_{p,v_{s-\alpha(1/p-1/2)}}$–spaces. Especially, this means that the functions in $\mathcal{H}_0$ are dense (weak*-dense for $p = \infty$) in $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}}$. Since the space $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}}$ is complete with respect to its norm, this implies $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}} = \overline{\mathcal{H}_0}^{\|\cdot\|_{\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}}}}$. But the bandlimited functions in $\mathcal{H}_0$ are also dense in $M^{s,\alpha}_{p,p}$: In fact one can show that any bandlimited L_p -function f can be expressed as a series of elements in $\mathcal{H}_0$ that is convergent in the $M^{s,\alpha}_{p,p}$ norm. Since L_p bandlimited functions are certainly dense (weak*-dense for $p = \infty$) in $M^{s,\alpha}_{p,p}$ (see [27] and [23]) this shows immediately that $M^{s,\alpha}_{p,p} = \overline{\mathcal{H}_0}^{\|\cdot\|_{M^{s,\alpha}_{p,p}}}$. Moreover $M^{s,\alpha}_{p,p}$ is again a complete space with respect to its norm (6.2) which is equivalent to that of $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}}$ for every function in $\mathcal{H}_0$, i.e., $\|f\|_{M^{s,\alpha}_{p,p}} \asymp \|f\|_{\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)}}}$ for all $f \in \mathcal{H}_0$. This implies that $\mathcal{H}_{p,v_{s+\alpha(1/p-1/2)}} = M^{s,\alpha}_{p,p}$

It remains to prove part 2). Essentially, this part of the theorem follows from the abstract theory outlined above since by 1) we know that the α-modulation spaces can be identified with coorbit spaces. Indeed, the abstract Theorems 4.6 and 4.7 (resp. Remark 4.8 i)) can be applied and yield an atomic decomposition and Banach frames with respect to the associated $\ell_{p,v_{s-\alpha(1/p-1/2)}}$–spaces. We are only left with computing the sequence space norm of $\ell_{p,v_{s-\alpha(1/p-1/2)}}$ explicitly. It is easy to see by Theorem 5.12 that

$$v_s(\omega_j) \;=\; (1 + |p_\alpha(\varepsilon j)|)^s \;=\; (1 + (1-\alpha)|\varepsilon j|)^{\frac{s}{1-\alpha}} \;\asymp\; (1 + (1-\alpha)|j|)^{\frac{s}{1-\alpha}}.$$

Thus, it follows

$$\begin{aligned}\|(c_{j,k})\|_{\ell_{p,v_{s-\alpha(1/p-1/2)}}} &= \left(\sum_{j,k\in\mathbb{Z}} |c_{j,k}|^p v_{s-\alpha(1/p-1/2)}(\omega_j)^p\right)^{1/p} \\ &\asymp \left(\sum_{j,k\in\mathbb{Z}} |c_{j,k}|^p (1+(1-\alpha)|j|)^{\frac{s-\alpha(1/p-1/2)}{1-\alpha}p}\right)^{1/p}.\end{aligned}$$

This concludes the proof. ■

Remark 6.2. i) *This result already appears in a slightly different form in* [23, Theorem 3.2]*, where it has been derived by a combination of decomposition methods* [13, 15, 14]*, a suitable generalization of the theory of instrinsic localization of frames* [24, 25]*, and certain stability/perturbation results for Banach frames and atomic decompositions. In particular, once the characterization Theorems* 6.1 i) *and* ii) *for a bandlimited function* ψ *are established, by an application of the perturbation argument in* [23, Theorem 3.2]*, one can extend the results to atoms that are not necessarily bandlimited, even though at least sufficiently time-frequency localized, e.g., any Schwartz function. Theorem* 6.1 *allows to characterize* α*-modulation spaces by means of frames with different densities* ($\varepsilon > 0$) *even of the (scale-frequency) parameter* j*, while in* [23, Theorem 3.2] *only the density of the (time-shift) paramenter* k *has been considered.*

ii) *Observe that for* $\alpha = 0$ *the* U*-dense sets* $\{(x_{j,k},\omega_j)\} = \{(\varepsilon k, \varepsilon j)\}$ *define a regular lattice in* $X \simeq \mathbb{R}^2$ *with density governed by* $\varepsilon > 0$*. This situation coincides with the well-known case of uniform Gabor frames* [29]*. For* $\alpha \to 1$ *one shows that the points have a limit* $(x_{j,k},\omega_j) = (\varepsilon k e^{-\varepsilon|j|}, \mathrm{sgn}(j)e^{\varepsilon|j|})$*. In particular, if* $\varepsilon = \epsilon\ln(2)$*, for* $\epsilon > 0$ *small enough, then* $(x_{j,k},\omega_j) = (\varepsilon k 2^{-\epsilon|j|}, \mathrm{sgn}(j)2^{\epsilon|j|})$*, which are the typical dyadic sampling points of the classical continuous wavelet transform in order to form wavelet frames* [11]*. Again here the parameter* $\varepsilon > 0$ *is interpreted as governing the density of the set.*

iii) *For* $\alpha = 0$*, it is well known that no frames can be generated by sampling on a set of points* $(x_k,\omega_j) = (ak, bj)$ *for any positive lattice constants* $a, b > 0$ *for which* $ab > 1$ [11, 29]*. In particular for the case* $(x_{j,k},\omega_j) = (\varepsilon k, \varepsilon j)$ *one cannot expect that frames can be derived for* $\varepsilon > 1$*. Then one may investigate for* $\alpha \in (0,1)$ *the shape of the set* $\Omega_\alpha \subset \mathbb{R}^2_+$ *of all parameters* (a,b) *such that no frames can be derived by sampling on* $(x_{j,k},\omega_j) = (a\beta(p_\alpha(bj))k, p_\alpha(bj))$*, see also [8]. Of course, by Theorem 6.1 we know already that* $\mathbb{R}^2_+\backslash\Omega_\alpha$ *contains at least an open neighborhood of* $(0,0)$ *in* $\mathbb{R}^2_+$*.*

iv) *For* $\alpha \to 1$ *one has formally* $(x_{j,k},\omega_j) = (\varepsilon k e^{-\varepsilon|j|}, \mathrm{sgn}(j)e^{\varepsilon|j|})$ *and*

$$\begin{aligned}& T_{x_{j,k}} M_{\omega_j} D_{(1+|\omega_j|)^{-1}}\psi(t) \\ =\ & (1+|\omega_j|)^{1/2} e^{2\pi i\omega_j(t-\varepsilon(1+|\omega_j|)^{-1}k)}\psi\left((1+|\omega_j|)(t-\varepsilon(1+|\omega_j|)^{-1}k)\right) \\ =\ & (1+|\omega_j|)^{1/2} e^{-2\pi i\,\mathrm{sgn}(\omega_j)(t-\varepsilon(1+|\omega_j|)^{-1}k)} e^{2\pi i\,\mathrm{sgn}(\omega_j)(1+|\omega_j|)(t-\varepsilon(1+|\omega_j|)^{-1}k)}\psi\left((1+|\omega_j|)t-\varepsilon k\right) \\ =\ & e^{-2\pi i\,\mathrm{sgn}(\omega_j)(t-\varepsilon(1+|\omega_j|)^{-1}k)} D_{(1+|\omega_j|)^{-1}} T_{\varepsilon k}\left(e^{2\pi i\,\mathrm{sgn}(\omega_j)t}\psi(t)\right).\end{aligned}$$

As for classical wavelets, dilations and translations remain the sole relevant operators, while the modulation contribution almost disappears, except for the phase factor in front of the di-

lation. We conjecture that Theorem 6.1 *can be formulated also for the limit case* $\alpha \to 1$ *to characterize inhomogeneous Besov spaces* $B^{s-1/p-1/2}_{p,p}(\mathbb{R})$ *where the discrete weights appearing in Theorem* 6.1 i) and ii) *will be (formally) of the type* $\lim_{\alpha\to 1}(1+(1-\alpha)|j|)^{\frac{s}{1-\alpha}} = e^{s|j|}$. *For the characterization of* $B^{s-1/p-1/2}_{p,p}(\mathbb{R})$ *by pure wavelet expansions in the context of the coorbit space theory we refer to* [16, 19, 25].

v) *The theorem can also be formulated with the discretization of the (continuous) canonical dual frame involving the Fourier multiplier* A_σ^{-1}.

A Appendix

A.1 Proof of Lemma 5.9

We consider m_β in the form (5.14) with G, L given by (5.15). The function g in (5.7) is implicitly given by

$$J(\xi, x, g(\xi, x)) \;=\; 0 \tag{A.1}$$

with

$$J(\xi, x, z) \;=\; xz + z^{1/\alpha} - 1 - \xi.$$

Using (5.7) we obtain

$$\partial_3 J(\xi, x, g(\xi, x)) \;=\; x + \alpha^{-1} g(\xi, x)^{1/\alpha - 1} \;=\; x + \alpha^{-1}\frac{-xg(\xi, x) + 1 + \xi}{g(\xi, x)} \;=\; x - \alpha^{-1}x + \alpha^{-1}\frac{1+\xi}{g(\xi, x)}.$$

Since $g(\xi, x)$ behaves like ξ^α when $\xi \to \infty$ the latter expression is always strictly positive if ξ is large enough and $x \in [-A, A]$. Thus, by the implicit function theorem $g(\xi, x)$ is uniquely determined by (A.1) (or (5.7), respectively). In this case, and since J is C^∞ also g is infinitely differentiable.

Clearly, we have

$$\begin{aligned}
\frac{\partial}{\partial \xi} G(\xi, x) &= -\alpha(1 + \alpha L(\xi, x))^{-2}\frac{\partial}{\partial \xi}L(\xi, x) \;=\; -\alpha G(\xi, x)^2 \frac{\partial}{\partial \xi}L(\xi, x),\\
\frac{\partial^2}{\partial \xi^2} G(\xi, x) &= -\alpha G(\xi, x)^2 \frac{\partial^2}{\partial \xi^2}L(\xi, x) - 2\alpha G(\xi, x)\frac{\partial}{\partial \xi}G(\xi, x)\frac{\partial}{\partial \xi}L(\xi, x)\\
&= \alpha G(\xi, x)^2\left(2\alpha G(\xi, x)\left(\frac{\partial}{\partial \xi}L(\xi, x)\right)^2 - \frac{\partial^2}{\partial \xi^2}L(\xi, x)\right).
\end{aligned}$$

So let us compute the partial derivatives of L with respect to ξ. To do this we will need the derivatives of g. We obtain

$$0 \;=\; \frac{\partial}{\partial \xi}J(\xi, x, g(\xi, x)) \;=\; \partial_1 J(\xi, x, g(\xi, x)) + \partial_3 J(\xi, x, g(\xi, x))\frac{\partial g}{\partial \xi}(x, \xi).$$

This implies that

$$\frac{\partial g}{\partial \xi}(x, \xi) \;=\; -\partial_1 J(\xi, x, g(\xi, x))\,(\partial_3 J(\xi, x, g(\xi, x)))^{-1} \;=\; \left(x + \frac{1}{\alpha}g(\xi, x)^{1/\alpha - 1}\right)^{-1},$$

and

$$\frac{\partial^2 g}{\partial \xi^2}(\xi,x) = -\frac{1}{\alpha}(\frac{1}{\alpha}-1)g(\xi,x)^{1/\alpha-2}\left(x+\frac{1}{\alpha}g(\xi,x)^{1/\alpha-1}\right)^{-3}.$$

For L this yields

$$\begin{aligned}
\frac{\partial L}{\partial \xi}(\xi,x) &= (1-\frac{1}{\alpha})x\frac{\partial g}{\partial \xi}(\xi,x)g(\xi,x)^{-1/\alpha} \;=\; (1-\frac{1}{\alpha})x\left(x+\frac{1}{\alpha}g(\xi,x)^{1/\alpha-1}\right)^{-1}g(\xi,x)^{-1/\alpha}\\
\frac{\partial^2 L}{\partial \xi^2}(\xi,x) &= (1-\frac{1}{\alpha})x\left(\frac{\partial^2 g}{\partial \xi^2}g^{-1/\alpha}-\frac{1}{\alpha}\left(\frac{\partial g}{\partial \xi}\right)^2 g^{-1/\alpha-1}\right)(\xi,x)\\
&= (1-\frac{1}{\alpha})x\left((\frac{1}{\alpha}-\frac{1}{\alpha^2})g(\xi,x)^{-2}(x+\frac{1}{\alpha}g(\xi,x)^{1/\alpha-1})^{-3}\right.\\
&\qquad\left.-\frac{1}{\alpha}g(\xi,x)^{-1/\alpha-1}(x+\frac{1}{\alpha}g(\xi,x)^{1/\alpha-1})^{-2}\right).
\end{aligned}$$

Together with (5.8) we obtain

$$\begin{aligned}
\lim_{\xi\to\infty}\xi^{2-\alpha}\frac{\partial L}{\partial \xi}(\xi,x) &= \lim_{\xi\to\infty}(1-\frac{1}{\alpha})x\left(\xi^{-1+\alpha}x+\frac{1}{\alpha}\xi^{-1+\alpha}g(\xi,x)^{1/\alpha-1}\right)^{-1}\xi g(\xi,x)^{-1/\alpha}\\
&= \lim_{\xi\to\infty}(1-\frac{1}{\alpha})x\left(\xi^{-1+\alpha}x+\frac{1}{\alpha}(\xi^{-\alpha}g(\xi,x))^{1/\alpha-1}\right)^{-1}(\xi^{-\alpha}g(\xi,x))^{-1/\alpha}\\
&= \alpha(1-\frac{1}{\alpha})x \;=\; (\alpha-1)x,
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi\to\infty}\xi^{3-\alpha}\frac{\partial^2 L}{\partial \xi^2}(\xi,x) &= \lim_{\xi\to\infty}(1-\frac{1}{\alpha})x\left(\frac{\alpha-1}{\alpha^2}\xi^{2\alpha}g(\xi,x)^{-2}(\xi^{-1+\alpha}x+\frac{1}{\alpha}\xi^{-1+\alpha}g(\xi,x)^{1/\alpha-1})^{-3}\right.\\
&\qquad\left.-\frac{1}{\alpha}\xi^{1+\alpha}g(\xi,x)^{-1/\alpha-1}(\xi^{-1+\alpha}x+\frac{1}{\alpha}\xi^{-1+\alpha}g(\xi,x)^{1/\alpha-1})^{-2}\right)\\
&= \lim_{\xi\to\infty}(1-\frac{1}{\alpha})x\left(\frac{\alpha-1}{\alpha^2}(\xi^{-\alpha}g(\xi,x))^{-2}(\xi^{-1+\alpha}x+\frac{1}{\alpha}(\xi^{-\alpha}g(\xi,x))^{1/\alpha-1})^{-3}\right.\\
&\qquad\left.-\frac{1}{\alpha}(\xi^{-\alpha}g(\xi,x))^{-1/\alpha-1}(\xi^{-1+\alpha}x+\frac{1}{\alpha}(\xi^{-\alpha}g(\xi,x))^{1/\alpha-1})^{-2}\right)\\
&= (1-\frac{1}{\alpha})((\alpha-1)\alpha-\alpha)x \;=\; (\alpha-1)(\alpha-2)x.
\end{aligned}$$

This gives

$$\begin{aligned}
\lim_{\xi\to\infty}\xi^{2-\alpha}\frac{\partial G}{\partial \xi}(\xi,x) &= \alpha(1-\alpha)x,\\
\lim_{\xi\to\infty}\xi^{3-\alpha}\frac{\partial^2 G}{\partial \xi^2}(\xi,x) &= 2\alpha^2\underbrace{\lim_{\xi\to\infty}\xi^{3-\alpha}\left(\frac{\partial L}{\partial \xi}(\xi,x)\right)^2}_{=0}-\alpha\lim_{\xi\to\infty}\xi^{3-\alpha}\frac{\partial^2 L}{\partial \xi^2}(\xi,x) \;=\; -\alpha(1-\alpha)(2-\alpha)x.
\end{aligned}$$

Moreover, these limits hold uniformly in $x\in[-A,A]$. This implies

$$\lim_{\xi\to\infty}\xi^{2-\alpha}m'_\beta(\xi) \;=\; \lim_{\xi\to\infty}\xi^{2-\alpha}\int_{-A}^{A}|\hat\psi(x)|^2\frac{\partial}{\partial \xi}G(\xi,x)dx \;=\; \alpha(1-\alpha)\int_{-A}^{A}|\hat\psi(x)|^2 x dx.$$

(If $|\hat{\psi}|$ is even then the last integral even vanishes). Since m'_β is continuous we deduce that

$$|m'_\beta(\xi)| \le C\min\{1, |\xi|^{-2+\alpha}\}.$$

In the same way we obtain

$$\lim_{\xi\to\infty} \xi^{3-\alpha} m''_\beta(\xi) = -\alpha(1-\alpha)(2-\alpha)\int_{-A}^{A} |\hat{\psi}(x)|^2 x dx.$$

and

$$|m''_\beta(\xi)| \le C\min\{1, |\xi|^{-3+\alpha}\}.$$

Observe that

$$\begin{aligned} (m_\beta^{-1})'(\xi) &= -m'_\beta(\xi) m_\beta^{-2}(\xi), \\ (m_\beta^{-1})''(\xi) &= 2(m'_\beta(\xi))^2 m_\beta^{-3}(\xi) - m''_\beta(\xi) m_\beta^{-2}(\xi). \end{aligned}$$

Since m_β is bounded away from zero and $\alpha \in [0,1)$, we finally deduce (5.27) and (5.28), and the proof of Lemma 5.9 is completed.

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