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Dedicated to Prof. Dr. R.A. DeVore on the occasion of his 60th birthday

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Abstract

This paper is concerned with the regularity of the solutions to elliptic boundary value problems in polyhedral domains Ω contained in $\mathbf{R}^3$. Especially, we consider the specific scale $B^\alpha_\tau(L_\tau(\Omega))$, $1/\tau = \alpha/3 + 1/2$, of Besov spaces. The regularity of the variational solution in these Besov spaces determines the order of approximation that can be achieved by adaptive and nonlinear numerical schemes. It is well–known that in polyhedral domains different types of singularities according to edges and vertices occur. In this paper, we shall primarily be concerned with the Besov regularity of edge singularities. We show that the corresponding singularity functions are much smoother in the specific Besov scale than in the usual L_2–Sobolov scale which justifies the use of adaptive schemes. The proofs are based on specific representations of the solutions which were, e.g., derived by Grisvard [17], and on characterizations of Besov spaces by wavelet expansions.

Key Words: Elliptic boundary value problems, adaptive methods, Besov spaces, edge singularities, wavelets.

AMS Subject Classification: Primary 35B65, secondary 41A46, 46E35.

1 Introduction

Quite recently, the regularity of the solutions to second order elliptic boundary value problems

$$\begin{aligned} Lu &= f \quad \text{in} \quad \Omega \subset \mathbf{R}^d, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a Lipschitz domain, in specific Besov spaces has been investigated, see, e.g., [4, 5, 6, 10]. The aim was to provide some theoretical foundations for the use of adaptive

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schemes for the numerical treatment of (1). This note can be interpreted as a continuation of these studies. The order of convergence of usual (linear) Galerkin schemes obtained, e.g., by finite element spaces based on uniform grid refinement, is determined by the regularity of the variational solution u to (1) in the usual Sobolev scale $H^s(\Omega), s \geq 1$. Unfortunately, on a general Lipschitz domain, this Sobolev regularity may not be very high, even if the right–hand side f is sufficiently smooth. This fact is caused by singularities near the boundary. Therefore, to increase efficiency, one often uses *adaptive* methods, i.e., the underlying grid is only refined in regions where the solution lacks smoothness. In this case, one does not use the whole linear spaces, hence an adaptive scheme can be interpreted as some kind of *nonlinear approximation.* Then the question arises if nonlinear methods indeed provide some gain of efficiency when compared with linear schemes. So far, the problem is best understood for numerical schemes based on a *wavelet basis* $\Psi = \{\psi_\mu,\ \mu \in \mathcal{J}\}$. (We refer to one of the textbooks [1, 11, 19, 21] for the definition and the basic properties of wavelets). An adaptive wavelet scheme approximates the solution u to (1) by a linear combination of N wavelets. Therefore a natural benchmark for its performance is given by the *best N–term approximation.* Then one approximates a function $F \in L_2(\mathbf{R}^d)$ by the nonlinear manifolds $\mathcal{M}_n$ of all functions

$$G = \sum_{\mu \in \Gamma} a_\mu \psi_\mu$$

with $\Gamma \subset \mathcal{J}$ of cardinality N and studies the error

$$\sigma_N(F)_{L_2(\mathbf{R}^d)} := \inf_{G \in \mathcal{M}_n} \|F - G\|_{L_2(\mathbf{R}^d)}. \tag{2}$$

For the L_2–metric and an orthonormal wavelet basis, the approximation problem (2) has a simple solution. We order the wavelet coefficients by their absolute values and choose Γ corresponding to the N largest values. In contrary to linear schemes, the order of approximation that can be achieved by best N–term approximation is not determined by the Sobolev regularity but by certain *non–classical* scales of function spaces. Indeed, the following characterization has been derived in [14]

$$\sum_{N=1}^{\infty} [N^{s/d} \sigma_N(F)_{L_2(\mathbf{R}^d)}]^\tau \frac{1}{N} < \infty \text{ if and only if } F \in B^s_\tau(L_\tau(\mathbf{R}^d)), \quad \tau = (s/d + 1/2)^{-1}, \tag{3}$$

where the $B^s_\tau(L_\tau(\mathbf{R}^d))$ are the *Besov* spaces (see, e.g., [15, 20] for the definition and the main properties of Besov spaces). Similar results also hold for other norms such as L_p and Sobolev norms, see, e.g., [7, 13] for details.

Of course, best N–term approximation is not directly applicable in our setting for catching the N biggest wavelet coefficients requires knowing *all* coefficients of the unknown solution u. Nevertheless, quite recently, an *implementable* adaptive wavelet scheme has been developed which produces asymptotically the same rate of convergence as the best N–term approximation [2], see also [3, 8, 9]. Having these results and the characterization (3) in mind, it is therefore natural to ask the following question: what is the regularity of the solution u to (1) as measured in the scale $B^s_\tau(L_\tau(\Omega))$, $\tau = (s/d + 1/2)^{-1}$? Especially, does the solution have a higher smoothness order in these spaces compared to the usual Sobolev scale? For then, adaptive wavelet methods would definitely perform better than linear schemes and the use of adaptive schemes is completely justified. The results in [4, 5, 6, 10] indicate that this is indeed the case for many problems. However, most of these investigations were concerned with general Lipschitz domains, i.e., all boundary points are viewed as equally 'bad' which is often not realistic. In practice, one is typically concerned with domains with piecewise analytic boundary, e.g., with polyhedral domains. One would expect that in this case much sharper results are available. Indeed, in [6], a first result in this direction for the Poisson equation in polygonal domains in $\mathbf{R}^2$ has been established. It turned out that the corresponding singularity functions, although not very smooth in the usual Sobolev scale, have *arbitrary* high regularity in the specific Besov scale we are interested in. The aim of this paper is to derive similar results for polyhedral domains in $\mathbf{R}^3$. In this case, the situation is much more complicated since different types of singularities according to edges and vertices occur, see, e.g., [12, 16, 17] for details. In this paper, we shall primarily be concerned with edge singularities. It turns out that in contrary to polygonal domains the singularity functions for the Poisson equation are not arbitrary smooth in the nonlinear approximation scale of Besov spaces. Nevertheless, compared to the general results for Lipschitz domains from [10], there is still some gain of regularity. Especially, the singularity functions have much higher smoothness order in the Besov spaces compared to the usual Sobolev scale.

This paper is organized as follows. In Section 2, we briefly recall some facts from the classical regularity theory for polyhedral domains and state and discuss our main Besov regularity result. Then, in Section 3, we present a detailed proof of this result which is based on wavelet analysis.

2 Main Results

In this section, we want to present a new regularity result for the model problem

$$\begin{aligned} \triangle u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4}$$

where Ω is a simply connected polyhedral domain contained in $\mathbf{R}^3$. It is well–known that the Sobolev regularity of the variational solution to (4) is often diminished by singularities induced by the shape of the domain Ω. For polyhedral domains, two types of singularities according to edges and vertices can occur. The basic setting can be described as follows. First of all, we have to discuss some facts from the regularity theory for simply connected, bounded polygonal domains Υ contained in $\mathbf{R}^2$. The segments of $\partial\Upsilon$ are denoted by $\overline{\Gamma}_l$, Γ_l open, $l = 1, \ldots, \mathcal{N}$, numbered in positive orientation. Furthermore, S_l denotes the endpoint of Γ_l and ω_l denotes the measure of the interior angle at S_l. We consider the auxiliary 2D–model problem

$$\begin{aligned} \triangle v &= g \quad \text{in} \quad \Upsilon, \\ v &= 0 \quad \text{on} \quad \partial\Upsilon. \end{aligned} \tag{5}$$

It is well–known that for $g \in L_2(\Upsilon)$ the variational solution v to (5) can be decomposed into a regular part v_R and a singular part v_S, $v = v_R + v_S$, where $v_R \in H^2(\Omega)$ and u_S only depends on the shape of the domain and can be computed explicitely. Results of this form were first derived by Kondrat'ev [18], however, in this paper, our standard reference will always be the book of Grisvard [17]. We introduce polar coordinates (r_l, θ_l) in the vicinity of each vertex S_l and introduce the functions

$$\mathcal{S}_l(r_l, \theta_l) = \zeta_l(r_l) r_l^{\lambda_l} \sin(\pi\theta_l/\omega_l), \qquad \lambda_l := \pi/\omega_l, \tag{6}$$

where ζ_l denotes a suitable C^∞ truncation function. Then one has the following theorem (see, e.g., [17], Chapter 2.4):

Theorem 2.1 *For given $g \in L_2(\Upsilon)$, the corresponding variational solution to* (5) *has an expansion $v = v_R + v_S$, where $v_R \in H^2(\Upsilon)$ and*

$$v_S = \sum_{j=1}^{\mathcal{N}} \sum_{0<\lambda_l<1} c_l \mathcal{S}_l. \tag{7}$$

In this paper, we are especially interested in the singularity functions according to the edges of a polyhedral domain in $\mathbf{R}^3$. It turns out that these functions can be constructed by means of the functions defined in (6). In fact, the behaviour of the solutions to elliptic boundary value problems in polyhedral domains in $\mathbf{R}^3$ along edges is often studied by considering a corresponding unbounded domain without vertices of the following type. Let $\tilde{\Omega} \subset \mathbf{R}^3$ be of the form $\tilde{\Omega} = \Upsilon \times \mathbf{R}$, where Υ is a bounded polygonal domain in $\mathbf{R}^2$. Then one has the following theorem [17].

Theorem 2.2 *For each $f \in L_2(\tilde{\Omega})$ there exists a unique solution to*

$$\int_{\tilde{\Omega}} \nabla u \cdot \nabla v dx = -\int_{\tilde{\Omega}} f v dx, \tag{8}$$

and in addition there exist unique functions $\xi_l \in H^{1-\lambda_l}(\mathbf{R})$ such that

$$u - \sum_l \sum_{0<\lambda_l<1} (K * \xi_l)\mathcal{S}_l \in H^2(\tilde{\Omega}), \tag{9}$$

where $K := r/(\pi(r^2 + x_3^2))$, $r = (x_1^2 + x_2^2)^{1/2}$ and $\mathcal{S}_l$ denotes one of the usual singularity functions for Υ.

In other words, $K * \xi_l$ means the function

$$x_3 \longrightarrow \frac{r}{\pi} \int_{\mathbf{R}} \xi_l(x_3 - t)(r^2 + t^2)^{-1} dt.$$

The central aim of this paper is to determine the Besov regularity of the singular parts

$$\mathcal{W}_l := (K * \xi_l)\mathcal{S}_l \tag{10}$$

introduced in (9). In the next section, we shall prove the following theorem which is the main result of this paper.

Theorem 2.3 *Each of the functions defined in* (10) *satisfies*

$$\mathcal{W}_l \in B^s_\tau(L_\tau(\hat{\Omega})), \quad \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}, \quad s < s^* := \frac{9 - 3\lambda_l}{2}, \tag{11}$$

where $\hat{\Omega}$ is any domain of the form $\Upsilon \times \mathcal{I}$, $\mathcal{I} \subset \mathbf{R}$ a bounded interval.

Remark 2.1 i) Let us first compare this theorem with the general Besov regularity results for arbitrary Lipschitz domains. As a special case of the analysis in [10], it turns out that for $f \in L_2(\Omega)$ the solution u to the Poisson equation in an arbitrary Lipschitz domain is contained in the spaces $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$ for all $0 < s < 2$. However, since $0 < \lambda_l < 1$, Theorem 2.3 implies the condition $s^* \geq 3$. Therefore we gain smoothness compared with the general result, at least for the singular part of the solution.
ii) By our method, it is not possible to prove a version of Theorem 2.3 for the whole domain $\tilde{\Omega}$. However, the unbounded domain $\tilde{\Omega}$ according to Theorem 2.2 is only an auxiliary construction to treat the edge singularities of the original problem which is defined on the bounded domain Ω. Therefore the regularity on bounded subsets seems to be what really counts in practice.

3 Proof of Theorem 2.3

First of all, let us fix l and set $\mathcal{W} := \mathcal{W}_l$, $\xi := \xi_l$, $\lambda := \lambda_l$, $S := S_l$, $\zeta := \zeta_l$, $\mathcal{S} := \mathcal{S}_l$. The following proof is based on wavelet analysis. We want to use the fact that function spaces such as Besov spaces can be characterized by wavelet expansions. Let us briefly recall the basic facts. For our purposes, it is sufficient to assume that the wavelet basis is constructed by tensor products of the univariate Daubechies wavelets and scaling functions. In [11], a univariate family ${}_m\psi$ of compactly supported wavelets has been constructed. The smoothness of ${}_m\psi$ increases without bound as m increases, as does the support of ${}_m\psi$. Moreover, the wavelet ${}_m\psi$ has m vanishing moments. We fix a value of m and let $\phi = {}_m\phi$ be the univariate scaling function which generates the wavelet $\psi = {}_m\psi$. We define $\psi^0 := \phi$ and $\psi^1 := \psi$. Further, let E denote the nontrivial vertices of the square $[0,1]^d$. Then, the set Ψ of $2^d - 1$ functions

$$\psi^e(x_1, \ldots, x_d) := \prod_{j=1}^{d} \psi^{e_j}(x_j), \quad e \in E, \tag{12}$$

generates by shifts and dilates an orthonormal wavelet basis for $L_2(\mathbf{R}^d)$. Namely, let $\mathcal{D} := \mathcal{D}(\mathbf{R}^d)$ denote the set of dyadic cubes in $\mathbf{R}^d$. Each cube $I \in \mathcal{D}$ is of the form $I = 2^{-j}k + 2^{-j}[0,1]^d$ with $k \in \mathbf{Z}^d$, $j \in \mathbf{Z}$. The functions

$$\eta_I := \eta_{j,k} := 2^{jd/2}\eta(2^j \cdot -k), \ I = 2^{-j}k + 2^{-j}[0,1]^d, \ k \in \mathbf{Z}^d, j \in \mathbf{Z}, \eta \in \Psi, \tag{13}$$

form an orthonormal basis for $L_2(\mathbf{R}^d)$. Consequently, each function $F \in L_2(\mathbf{R}^d)$ has an expansion

$$F = P_0(F) + \sum_{\eta \in \Psi} \sum_{I \in \mathcal{D}^+} \langle F, \eta_I \rangle \eta_I, \tag{14}$$

where $\mathcal{D}^+$ denotes the set of all dyadic cubes of measure < 1 and P_0 is a projector onto the subspace of $L_2(\mathbf{R}^d)$ spanned by the translates of $\phi(x_1)\ldots\phi(x_d)$. By construction, there exists a cube Q satisfying $\mathrm{supp}(\eta) \subset Q$ for all η. Hence one can also find suitable cubes $Q(I)$ satisfying

$$\mathrm{supp}(\eta_I) \subset Q(I), \qquad |Q(I)| \lesssim |I|. \tag{15}$$

(In this paper, '$a \lesssim b$' indicates inequality up to constant factors). Then, if the generator ϕ is chosen sufficiently smooth, a function F is in the Besov space $B^\alpha_\tau(L_\tau(\mathbf{R}^d)), 1/\tau = \alpha/d + 1/2$, if and only if

$$\|P_0(F)\|_{L_\tau(\mathbf{R}^d)} + \left(\sum_{\eta \in \Psi} \sum_{I \in \mathcal{D}+} |\langle F, \eta_I \rangle|^\tau \right)^{1/\tau} < \infty, \tag{16}$$

see, e.g., [19] for details.

The characterization (16) now gives us a hint how to establish Besov regularity. We have to compute the wavelet coefficients of the singular part $\mathcal{W}$ and to check if their ℓ_τ–norm is finite. To this end, the first step is to extend $\mathcal{W}$ to all of $\mathbf{R}^3$. For technical reasons which will become clear later, we proceed as follows. We introduce the distance to the edge $S \times \mathbf{R}$

$$\delta_I := \inf_{x \in Q(I)} r(x). \tag{17}$$

Furthermore, we set $U := \Upsilon \cap B(S,R)$, where supp $\zeta \subset [-R,R]$ and $B(S,R)$ clearly denotes the ball of radius R at S in $\mathbf{R}^2$. There exists a cone $V \subset \mathbf{R}^2$, centered at S and containing U, and an interval $\hat{\mathcal{I}} \supset \mathcal{I}$, such that for some suitable constant C,

$$Q(I) \subset V \times \hat{\mathcal{I}} \quad \text{if} \quad Q(I) \cap (U \times \mathcal{I}) \neq \emptyset, \ |I| = 2^{-3j} \quad \text{and} \quad \delta_I \geq C2^{-j}. \tag{18}$$

Then the explicit expressions (6) and (10) imply that $\mathcal{W}$ has a trivial extension onto $V \times \hat{\mathcal{I}}$ which we also denote by $\mathcal{W}$. It is at least contained in $H^{3/2}$, see again [16], and we may use a Whitney extension to obtain a function on all of $\mathbf{R}^3$ for which we again keep the notation $\mathcal{W}$. Then, on the old domain $\hat{\Omega}$, $\mathcal{W}$ has an expression

$$\mathcal{W} = P_0(\mathcal{W}) + \sum_{(I,\eta)\in\Lambda} \langle \mathcal{W}, \eta_I \rangle \eta_I, \tag{19}$$

where Λ denotes the set of all indices for which $(U \times \mathcal{I}) \cap Q(I) \neq \emptyset$. Therefore the task is to estimate the right–hand side in (19). It can be shown that $P_0(\mathcal{W})$ does not cause any serious trouble, see, e.g., [10] for details. Therefore it remains to establish Besov regularity for the second term in (19). According to (16), we are left with showing that

$$\sum_{(I,\eta)\in\Lambda} |\langle \mathcal{W}, \eta_I \rangle|^\tau < \infty. \tag{20}$$

Let us start by estimating one wavelet coefficient. By the vanishing moment property, each wavelet η_I is orthogonal to any polynomial of total degree $< m$. Hence, for any $P_I \in \mathcal{P}_{m-1}$,

$$|\langle \mathcal{W}, \eta_I \rangle| \leq |\langle \mathcal{W} - P_I, \eta_I \rangle| \leq \|\mathcal{W} - P_I\|_{L_2(Q(I))} \|\eta_I\|_{L_2(\mathbf{R}^3)}.$$

where $Q(I)$ again denotes the support cube of η_I. Combining this formula with a standard Whitney–type estimate yields

$$|\langle \mathcal{W}, \eta_I \rangle| \leq \|\mathcal{W} - P_I\|_{L_2(Q(I))} \|\eta_I\|_{L_2(Q(I))} \lesssim 2^{-4j} |\mathcal{W}|_{W^4(L_2(Q(I)))}. \tag{21}$$

We fix a refinement level j and introduce the sets

$$\begin{aligned}
\Lambda_j &:= \{ (I,\eta) \mid |I| = 2^{-3j} \}, \\
\Lambda_{j,k} &:= \{ (I,\eta) \in \Lambda_j \mid k2^{-j} \leq \delta_I < (k+1)2^{-j} \}, \\
\Lambda_j^o &:= \Lambda_j \backslash \Lambda_{j,C}, \quad \Lambda_{j,C} := \{ (I,\eta) \in \Lambda_j \mid \delta_I < C2^{-j} \}.
\end{aligned}$$

The first step is to estimate the contribution of all wavelets corresponding to a fixed index set $\Lambda_{j,k} \subset \Lambda_j^o$. The Whitney estimate (21) immediately implies

$$\sum_{I\in\Lambda_{j,k}} |\langle \mathcal{W}, \eta_I \rangle|^\tau \lesssim \sum_{I\in\Lambda_{j,k}} 2^{-4j\tau} \left(\int_{Q(I)} \sum_{|\alpha|=4} |\partial^\alpha \mathcal{W}|^2 dx \right)^{\tau/2}.$$

so that, by using Hölders's inequality with $p = 2/\tau$ and $q = 2/(2-\tau)$ and the fact that $|\Lambda_{j,k}| \lesssim 2^j k$, we obtain

$$\sum_{I\in\Lambda_{j,k}} |\langle \mathcal{W}, \eta_I \rangle|^\tau \lesssim 2^{-4j\tau} 2^{j(2-\tau)/2} k^{(2-\tau)/2} \left(\sum_{I\in\Lambda_{j,k}} \int_{Q(I)} \sum_{|\alpha|=4} |\partial^\alpha \mathcal{W}|^2 dx \right)^{\tau/2}.$$

By employing the set $\mathcal{R} := \{x \in \mathbf{R}^2 \mid k2^{-j} \leq r \leq (k+1)2^{-j}\} \cap V$, this expression may be rewritten as

$$\begin{aligned}
\sum_{I\in\Lambda_{j,k}} |\langle \mathcal{W}, \eta_I \rangle|^\tau &\lesssim 2^{(-(9/2)\tau+1)j} k^{(2-\tau)/2} \left(\int_{\hat{\mathcal{I}}} \int_{\mathcal{R}} \sum_{|\alpha|=4} |\partial^\alpha \mathcal{W}|^2 dx_1 dx_2 dx_3 \right)^{\tau/2} \\
&\lesssim 2^{(-(9/2)\tau+1)j} k^{(2-\tau)/2} \left(\int_{\mathcal{R}} \sum_{|\alpha|=4} \int_{\mathbf{R}} |\partial^\alpha \mathcal{W}|^2 dx_3 dx_2 dx_1 \right)^{\tau/2}.
\end{aligned}$$

Therefore, by using the definition of $\mathcal{W}$ and Leibniz rule, we find

$$\begin{aligned}\sum_{I\in\Lambda_{j,k}}|\langle\mathcal{W},\eta_I\rangle|^\tau &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\int_{\mathbf{R}}|\partial^\alpha((K*\xi)\mathcal{S})|^2dx_3dx_2dx_1\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\sum_{\nu\le\alpha}\int_{\mathbf{R}}|\xi*\partial^{\alpha-\nu}K|^2|\partial^\nu\mathcal{S}|^2dx_3dx_2dx_1\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\sum_{\nu\le\alpha}\int_{\mathbf{R}}\left(\int_{\mathbf{R}}|\xi(x_3-t)||\partial^{\alpha-\nu}K(r,t)|dt\right)^2dx_3\right.\\ &\qquad\left.|\partial^\nu\mathcal{S}|^2dx_2dx_1\right)^{\tau/2}\end{aligned}$$

The next step is to employ the Minkowski–inequality. This yields

$$\begin{aligned}\sum_{I\in\Lambda_{j,k}}|\langle\mathcal{W},\eta_I\rangle|^\tau &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\sum_{\nu\le\alpha}\left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}|\xi(x_3-t)|^2dx_3\right)^{1/2}|\partial^{\alpha-\nu}K(r,t)|dt\right)^2\right.\\ &\qquad\left.|\partial^\nu\mathcal{S}|^2dx_2dx_1\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2)}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\sum_{\nu\le\alpha}\left(\int_{\mathbf{R}}|\partial^{\alpha-\nu}K(r,t)|dt\right)^2|\partial^\nu\mathcal{S}|^2dx_2dx_1\right)^{\tau/2}. \qquad (22)\end{aligned}$$

A direct computation shows that

$$\partial^\nu\mathcal{S} \lesssim r^{\lambda-|\nu|} \qquad (23)$$

and

$$\int_{\mathbf{R}}|\partial^{\alpha-\nu}K(r,t)|dt \lesssim r^{-|\alpha-\nu|}. \qquad (24)$$

Consequently, by inserting (23) and (24) into (22) we finally obtain

$$\begin{aligned}\sum_{I\in\Lambda_{j,k}}|\langle\mathcal{W},\eta_I\rangle|^\tau &\lesssim 2^{(-9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}\sum_{|\alpha|=4}\sum_{\nu\le\alpha}r^{-2|\alpha-\nu|}r^{2(\lambda-|\nu|)}dx_2dx_1\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left(\int_{\mathcal{R}}r^{(2\lambda-8)}dx_2dx_1\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}\left((k2^{-j})^{(2\lambda-8)}|\mathcal{R}|\right)^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(2-\tau)/2}(k2^{-j})^{(2\lambda-8)(\tau/2)}(k2^{-2j})^{\tau/2}\\ &\lesssim 2^{(-(9/2)\tau+1)j}k^{(\lambda-4)\tau+1}2^{-j\tau(\lambda-3)}\\ &\lesssim 2^{j(1-\tau(3/2+\lambda))}k^{(\lambda-4)\tau+1}.\end{aligned}$$

The next step is to treat a typical set Λ_j^o. We get

$$\sum_{I\in\Lambda_j^o}|\langle\mathcal{W},\eta_I\rangle|^\tau \lesssim 2^{j(1-\tau(3/2-\lambda))}\sum_{k=k_1}^{\infty}k^{(\lambda-4)\tau+1},$$

and we are in business if

$$\tau(-4+\lambda)<-2, \quad \text{i.e.,} \quad \frac{1}{\tau}<\frac{4-\lambda}{2},$$

which corresponds to

$$s < \frac{9-3\lambda}{2}. \tag{25}$$

We now define $\Lambda^o = \cup_{j=1}^{\infty}\Lambda_j^o$ and sum the last inequality over all refinement levels. This yields

$$\sum_{I\in\Lambda^o} |\langle \mathcal{W}, \eta_I\rangle|^\tau \ \lesssim \ \sum_{j=1}^{\infty} 2^{j(1-\tau(3/2+\lambda))}$$

and the geometric series converges if

$$-(\frac{3}{2}+\lambda)\tau + 1 < 0, \quad \text{i.e.,} \quad s < 3+3\lambda. \tag{26}$$

Since $1/2 < \lambda < 1$, it turns out that $9-3\lambda < 2+3\lambda$, so that in the setting of Theorem 2.3 condition (26) is always satisfied.

It remains to study the sets $\Lambda_{j,C}$. Combining the fact that $|\Lambda_{j,C}| \ \lesssim \ 2^j$ with Hölders's inequality yields

$$\begin{aligned}
\sum_{(I,\eta)\in\Lambda_{j,C}} |\langle \mathcal{W}, \eta_I\rangle|^\tau \quad &\lesssim \quad 2^{j(\frac{2-\tau}{2})} \left(\sum_{(I,\eta)\in\Lambda_{j,C}} |\langle \mathcal{W}, \eta_I\rangle|^2 \right)^{\tau/2} \\
&\lesssim \quad 2^{j(\frac{2-\tau}{2})} 2^{-j(3\tau/2)} \left(\sum_{(I,\eta)\in\Lambda_{j,C}} 2^{3j} |\langle \mathcal{W}, \eta_I\rangle|^2 \right)^{\tau/2}.
\end{aligned}$$

Therefore summing over all refinement levels and using Hölder's inequality once again gives

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,C}} |\langle \mathcal{W}, \eta_I\rangle|^\tau \ \lesssim \ \left(\sum_{j=1}^{\infty} 2^{j(1-\frac{3\tau}{2-\tau})} \right)^{\frac{2-\tau}{2}} \left(\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,C}} 2^{3j} |\langle \mathcal{W}, \eta_I\rangle|^2 \right)^{\tau/2}.$$

Since $\mathcal{W} \in H^{3/2}(\mathbf{R}^3)$, the second sum is finite, see again [19] for details. The first sum is finite if

$$1 - \frac{3\tau}{2-\tau} < 0, \quad \text{i.e.,} \quad s < 9/2.$$

The theorem is proved.

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