Zentrum für Technomathematik
Fachbereich 3 – Mathematik und Informatik

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Stephan Dahlke

Report 01–11

Berichte aus der Technomathematik

Report 01–11 **August 2001**

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Stephan Dahlke*
Fachbereich 3, ZeTeM
Universität Bremen
Postfach 33 04 40
28334 Bremen
Germany

Abstract

This paper is concerned with the regularity of the solutions to the Neumann problem in Lipschitz domains Ω contained in $\mathbf{R}^d$. Especially, we consider the specific scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d+1/p$, of Besov spaces. The regularity of the variational solution in these Besov spaces determines the order of approximation that can be achieved by adaptive and nonlinear numerical schemes. We show that the solution to the Neumann problem is much smoother in the specific Besov scale than in the usual L_p–Sobolov scale which justifies the use of adaptive schemes. The proofs are performed by combining some recent regularity results derived by Zanger [23] with some specific properties of harmonic Besov spaces.

Key Words: Elliptic boundary value problems, adaptive methods, Besov spaces.

AMS Subject Classification: Primary 35B65, secondary 41A46, 46E35.

1 Introduction

Quite recently, the regularity of the solutions to second order elliptic boundary value problems

$$\begin{aligned} Lu &= F \quad \text{in} \quad \Omega \subset \mathbf{R}^d, \\ u &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a Lipschitz domain, in specific Besov spaces has been investigated, see, e.g., [6, 7, 8, 11]. The aim was to provide some theoretical foundations for the use of adaptive schemes for the numerical treatment of (1). The order of convergence as measured in L_p of usual (linear) Galerkin schemes obtained, e.g., by finite element spaces based on uniform grid refinement, is determined by the regularity of the variational solution u to (1) in the Sobolev scale $W^s(L_p(\Omega))$. Unfortunately, on a general Lipschitz domain, this Sobolev regularity may not be very high, even if the right–hand side F is sufficiently smooth. This fact is caused by singularities near the boundary. Therefore, to increase efficiency, one often uses *adaptive* methods, i.e., the underlying grid is only refined in

*This work has been supported by Deutsche Forschungsgemeinschaft, Grant Da 360/4–1.

regions where the solution lacks smoothness. In this case, one does not use the whole linear spaces, hence an adaptive scheme can be interpreted as some kind of *nonlinear approximation*. Then the question arises if nonlinear methods indeed provide some gain of efficiency when compared with linear schemes. So far, the problem is best understood for numerical schemes based on a *wavelet basis* $\Psi = \{\psi_\lambda,\ \lambda \in \mathcal{J}\}$. (We refer to one of the textbooks [2, 12, 19, 22] for the definition and the basic properties of wavelets). An adaptive wavelet scheme approximates the solution u to (1) by a linear combination of n wavelets. Therefore a natural benchmark for its performance is given by the *best n–term approximation*. Then one approximates a function $f \in L_p(\mathbf{R}^d)$ by the nonlinear manifolds $\mathcal{M}_n$ of all functions

$$S = \sum_{\lambda \in \Gamma} a_\lambda \psi_\lambda$$

with $\Gamma \subset \mathcal{J}$ of cardinality n and studies the error

$$\sigma_n(f)_{L_p(\mathbf{R}^d)} := \inf_{S \in \mathcal{M}_n} \|f - S\|_{L_p(\mathbf{R}^d)}. \tag{2}$$

For the L_2–metric and an orthonormal wavelet basis, the approximation problem (2) has a simple solution. We order the wavelet coefficients by their absolute values and choose Γ corresponding to the n largest values. (Similar results also hold for other values of p, see, e.g., [13] for details). In contrary to linear schemes, the order of approximation that can be achieved by best n–term approximation is not determined by the Sobolev regularity but by certain *non–classical* scales of function spaces. Indeed, the following characterization has been derived in [14]

$$\sum_{n=1}^{\infty} [n^{s/d} \sigma_n(f)_{L_p(\mathbf{R}^d)}]^\tau \frac{1}{n} < \infty \text{ if and only if } f \in B^s_\tau(L_\tau(\mathbf{R}^d)), \quad \tau = (s/d + 1/p)^{-1}, \tag{3}$$

where the $B^s_\tau(L_\tau(\mathbf{R}^d))$ are the *Besov* spaces (see, e.g., [15, 20] for the definition and the main properties of Besov spaces).

Of course, best n–term approximation is not directly applicable in our setting for catching the n biggest wavelet coefficients requires knowing *all* coefficients of the unknown solution u. Nevertheless, quite recently, an *implementable* adaptive wavelet scheme has been developed which produces asymptotically the same rate of convergence as the best n–term approximation [3], see also [4, 9, 10]. Having these results and the characterization (3) in mind, it is therefore natural to ask the following question: what is the regularity of the solution u to (1) as measured in the scale $B^s_\tau(L_\tau(\Omega))$, $\tau = (s/d + 1/p)^{-1}$? Especially, does the solution have a higher smoothness order in these spaces compared to the usual Sobolev scale? For then, adaptive wavelet methods would definitely perform better than linear schemes and the use of adaptive schemes is completely justified. The results in [6, 7, 8, 11] indicate that this is indeed the case for many problems. So far, the deepest results were obtained for the classical Dirichlet problem for harmonic functions:

$$\begin{aligned} \triangle v &= 0 \quad \text{in} \quad \Omega, \\ v &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4}$$

see [11] for details. Once these results are established, it is clearly desirable to generalize them also to the Neumann problem

$$\begin{aligned} \triangle v &= 0 \quad \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} &= g \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{5}$$

However, the proofs in [11] made heavy use of a very systematic study of the homogeneous and inhomogeneous Dirichlet problem presented by Jerison and Kenig [17]. Unfortunately, for the Neumann problem, such a systematic study was an open problem for a long time. Consequently, in his famous book [18], C. Kenig presented this problem as a suggestion for further researches. Soon afterwards, D. Jerison gave this problem to his Ph.D. student D. Zanger who solved it completely [23, 24]. Therefore we can now use his results to establish Besov regularity for both, the homogeneous and the inhomogeneous Neumann problem, and this is the main objective of this note.

This paper is organized as follows. In Section 2, we recall Zanger's developments as far as they are needed for our purposes. Then, in Section 3, we explain how these results can be exploited to establish nonclassical Besov regularity.

2 Classical Regularity Results

In this section, we want to summarize some of Zanger's results as far as they are needed for our purposes. The first step is to introduce the space

$$B_p^s(L_p(\partial\Omega))_{1^\perp} := \{h \in B_p^s(L_p(\partial\Omega)) \mid h(1) = 0\}. \tag{6}$$

Then the main result for the Neumann problem reads as follows.

Theorem 2.1 *Consider ϵ such that $0 < \epsilon \leq 1$. Define p_0 and p_0' by $1/p_0 = (1+\epsilon)/2$ and $1/p_0' = (1-\epsilon)/2$. Let α and p be numbers satisfying one of the following:*

(a) $p_0 < p < p_0'$ *and* $0 < \alpha < 1$

(b) $1 < p \leq p_0$ *and* $2/p - 1 - \epsilon < \alpha < 1$

(c) $p_0' \leq p < \infty$ *and* $0 < \alpha < 2/p + \epsilon$.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ for some $d \geq 3$ whose complement is connected. There exists ϵ depending only on the Lipschitz constant of Ω such that for $g \in B_p^{\alpha-1}(L_p(\partial\Omega))_{1^\perp}$ there exists a unique solution to the Neumann problem

$$\begin{aligned} \triangle v &= 0 \quad \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{7}$$

which satisfies $v \in B_p^{\alpha+1/p}(L_p(\Omega))$.

For later use, let us briefly sketch the idea of the proof. It can be performed by combining estimates for the homogeneous Dirichlet problem with those for the operator sending Neumann boundary values to the Dirichlet boundary values of the harmonic function exhibiting those Neumann boundary values, loosely speaking the inverse of the Calderón operator. We start by recalling the classical method of layer potentials to solve the Neumann problem. We define the *nontangential cone* $\Gamma_a(Q)$ for $a > 0$ via

$$\Gamma_a(Q) := \{X \in \Omega \mid |X - Q| < (1+a)\text{dist}(X, \partial\Omega)\}. \tag{8}$$

If u is a function on Ω we may define its *nontangential maximal function* $M(u)$ by setting

$$M(u)(Q) := \sup\{|u(P)| \mid P \in \Gamma_1(Q)\}. \tag{9}$$

We say that u as a *nontangential limit* at $Q \in \partial\Omega$ if there is a finite, well–defined limit as $P \longrightarrow Q$ from within $\Gamma_a(Q)$ for all $a > 0$. Furthermore, given $h \in L_1(\partial\Omega)$, its *single layer potential* is the function defined via

$$Sh(X) := \frac{-1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{h(Q)}{|X-Q|^{(d-2)}} d\sigma(Q), \tag{10}$$

where ω_d is the surface area of the unit sphere in $\mathbf{R}^d$. We also need the operators

$$K^*h(P) \;:=\; \text{p.v.}\frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle P-Q, n(P)\rangle}{|P-Q|^d} h(Q)d\sigma(Q), \tag{11}$$

$$T \;:=\; \frac{1}{2}I - K^*. \tag{12}$$

Here $n(P)$ clearly denotes the outward unit normal vector on $\partial\Omega$. Then the solutions to the Neumann problem can be constructed as follows, see Dahlberg and Kenig [5] and Verchota [21] for details.

Theorem 2.2 *Let $\Omega \subseteq \mathbf{R}^d$ be a bounded Lipschitz domain whose complement is connected. Then there is $\epsilon = \epsilon(\Omega) > 0$ such that, whenever $1 < p < 2 + \delta$, T is an invertible mapping from $L_p(\partial\Omega)_{1^\perp}$ onto $L_p(\partial\Omega)_{1^\perp}$, and S is an invertible mapping from $L_p(\partial\Omega)$ onto $W^1(L_p(\Omega))$. Moreover, given $g \in L_p(\partial\Omega)_{1^\perp}$ with $1 < p < 2+\delta$ and writing $v = ST^{-1}g$, i.e.,*

$$v(X) = \frac{1}{\omega_d(d-2)} \int_{\partial\Omega} |X-Q|^{2-d}(\frac{1}{2}I - K^*)^{-1}(g)(Q)d\sigma(Q), \tag{13}$$

it follows that v is the unique (modulo constants) harmonic function on Ω such that the nontangential maximal function $M(\nabla v)$ is bounded in $L_p(\partial\Omega)$ and $\frac{\partial v}{\partial n} = g$ nontangentially a.e. on $\partial\Omega$. Finally, we have

$$\|M(\nabla u)\|_{L_p(\partial\Omega)} \leq C\|g\|_{L_p(\partial\Omega)}. \tag{14}$$

In order to determine the Dirichlet boundary values of our single layer potentials one has

Proposition 2.1 *If $h \in L_1(\partial\Omega)$ then $Sh(X) \longrightarrow Sh(Q)$ as $X \longrightarrow Q$ nontangentially for a.e. $Q \in \partial\Omega$. In particular, if $1 < p < 2+\delta$, then for all $g \in L_p(\partial\Omega)_{1^\perp}$, $v(X) = ST^{-1}g(X) \longrightarrow ST^{-1}g(Q)$ for a.e. Q.*

For the proof, we refer to [24]. Consequently, for $1 < p < 2+\delta$ we may define the *inverse Calderón or Neumann to Dirichlet operator* $\Upsilon :\ L_p(\partial\Omega)_{1^\perp} \longrightarrow W^1(L_p(\partial\Omega))$ by setting

$$\Upsilon(g) := (ST^{-1}(g))|_{\partial\Omega}. \tag{15}$$

One of the main results in [23] states that the inverse Calderón operator moreover acts as a bounded operator on a whole scale of Besov spaces.

Theorem 2.3 *There exists ϵ with $0 < \epsilon \leq 1$ so that the inverse Calderón operator Υ introduced in (15) satisfies*

$$\|\Upsilon g\|_{B^\alpha_p(L_p(\partial\Omega))} \leq C\|g\|_{B^{\alpha-1}_p(L_p(\partial\Omega))}, \tag{16}$$

provided

(a) $p_0 < p < p_0'$ *and* $0 < \alpha < 1$

(b) $1 < p \leq p_0'$ and $2/p - 1 - \epsilon < \alpha < 1$

(c) $p_0' < p < \infty$ and $0 < \alpha < 2/p + \epsilon$,

wherein $1/p_0 = (1+\epsilon)/2,\ 1/p_0' = (1-\epsilon)/2$.

The proof of Theorem 2.1 now follows by combining Theorem 2.3 with the following fundamental result for the Dirichlet problem which was proved by Jerison and Kenig [17].

Theorem 2.4 *Consider* ϵ *such that* $0 < \epsilon \leq 1$. *Define* p_0 *and* p_0' *by* $1/p_0 = (1+\epsilon)/2$ *and* $1/p_0' = (1-\epsilon)/2$. *Let* α *and* p *be numbers satisfying one of the following:*

(a) $p_0 < p < p_0'$ and $0 < \alpha < 1$

(b) $1 < p \leq p_0$ and $2/p - 1 - \epsilon < \alpha < 1$

(c) $p_0' \leq p < \infty$ and $0 < \alpha < 2/p + \epsilon$.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^d$ *for some* $d \geq 3$. *There exists* ϵ *depending only on the Lipschitz constant of* Ω *such that for every* $g \in B_p^\alpha(L_p(\partial\Omega))$ *there exists a unique harmonic function* v *such that* $Tr\ v = g$ *and* $v \in B_p^{\alpha+1/p}(L_p(\Omega))$. *Moreover,*

$$\|v\|_{B_p^{\alpha+1/p}(L_p(\Omega))} \leq C\|g\|_{B_p^\alpha(L_p(\partial\Omega))}. \tag{17}$$

The Theorems 2.1 and 2.3 can also be used to derive a regularity result for the inhomogeneous Neumann problem

$$\begin{aligned} \triangle w &= F \quad \text{in} \quad \Omega, \\ \frac{\partial w}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{18}$$

Indeed, by a judicious homogenization procedure, (18) can be reduced to a problem of the form (7). For a detailed elaboration of these ideas, we refer again to [23] where the following fundamental result is proved, see also Section 3.

Theorem 2.5 *Let* Ω *be a bounded Lipschitz domain in* $\mathbf{R}^d$, $d \geq 3$, *and let* $1 < p < \infty$ *and* $1/p + 1/p' = 1$. *There is* ϵ, $0 < \epsilon \leq 1$, *depending only on the Lipschitz constant of* Ω, *such that, for every* $F \in (W^{2-\alpha}(L_{p'}(\Omega))^*_{1\perp}$, *there exists a solution* $w \in W^\alpha(L_p(\Omega))$ *to the inhomogeneous Neumann problem* (18) *provided one of the following holds:*

(a) $p_0 < p < p_0'$ and $1/p < \alpha < 1 + 1/p$

(b) $1 < p \leq p_0'$ and $3/p - 1 - \epsilon < \alpha < 1 + 1/p$

(c) $p_0' \leq p < \infty$ and $1/p < \alpha < 3/p + \epsilon$

wherein $1/p_0 = 1/2 + \epsilon/2$ *and* $1/p_0' = 1/2 - \epsilon/2$. *Moreover, for all* $F \in (W^{2-\alpha}(L_{p'}(\Omega)))^*_{1\perp}$ *we have the extimate*

$$\|w\|_{W^\alpha(L_p(\Omega))} \leq C\|F\|_{(W^{2-\alpha}(L_{p'}(\Omega)))^*}. \tag{19}$$

Finally, modulo constants, this solution is unique.

Remark 2.1 *i.) The space* $(W^{2-\alpha}(L_{p'}(\Omega)))^*_{1\perp}$ *is clearly defined analogously to* (6).
ii.) Quite recently, similar results were also derived by Fabes, Mendez, and Mitrea [16].

3 Nonclassical Regularity Results

In this section, we want to derive some nonclassical regularity results for the Neumann problem, i.e., we want to estimate the regularity of the solution as measured in the specific Besov scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/p$, which determines the approximation order of adaptive numerical schemes. Let us first discuss the homogeneous case.

Theorem 3.1 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$, $d \geq 3$, whose complement is connected. Let v be the solution to the Neumann problem*

$$\begin{aligned} \triangle v &= 0 \quad on \quad \Omega \subset \mathbf{R}^d, \\ \frac{\partial v}{\partial n} &= g \quad on \quad \partial\Omega, \end{aligned} \tag{20}$$

where $g \in B^{\alpha-1}_p(L_p(\partial\Omega))$ and α and p satisfy the conditions of Theorem 2.1. Then

$$v \in B^s_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{s}{d} + \frac{1}{p}\right)^{-1}, \quad 0 < s < \frac{(\alpha + 1/p)d}{(d-1)}. \tag{21}$$

Proof: The proof can be performed by combining Theorem 2.1 with the following nonclassical regularity result proved in [11] which states a specific property of harmonic Besov spaces.

Theorem 3.2 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. If v is an harmonic function on Ω which is in the Besov class $B^\mu_p(L_p(\Omega))$, for some $1 < p < \infty$ and $\mu > 0$, then*

$$v \in B^s_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{s}{d} + \frac{1}{p}\right)^{-1}, \quad 0 < s < \frac{\mu d}{(d-1)}. \tag{22}$$

Now Theorem 2.1 implies that $v \in B^{\alpha+1/p}_p(L_p(\Omega))$. However, the solution v to (20) is clearly an harmonic function. Therefore an application of Theorem 3.2 proves the assertion.

Theorem 3.1 says that we indeed gain regularity in the scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/p$, compared with the usual scale $B^s_p(L_p(\Omega))$, $s > 0$, since the maximal smoothness parameter according to Theorem 2.1 is multiplied by $d/(d-1)$. Consequently, the use of adaptive schemes is completely justified. By interpolation and embeddings for Besov spaces, we can moreover conclude that v is in a family of Besov spaces $B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$ for a certain range of the parameters $\tilde{\tau}$ and $\tilde{s}$. This is depicted in Figure 1 for the special case $p = 2$, $d = 3$. If $g \in L_2(\partial\Omega)$, then v is in $B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$ whenever $(1/\tilde{\tau}, \tilde{s})$ is in the interior of the quadrilateral with vertices $(1/2, 0)$, $(1/2, 3/2)$, $(1.25, 0)$, $(1.25, 2.25)$. The heavy line connecting $(1/2, 0)$ to $(1.25, 2.25)$ corresponds to the spaces $B^s_\tau(L_\tau(\Omega))$ of Theorem 3.1.

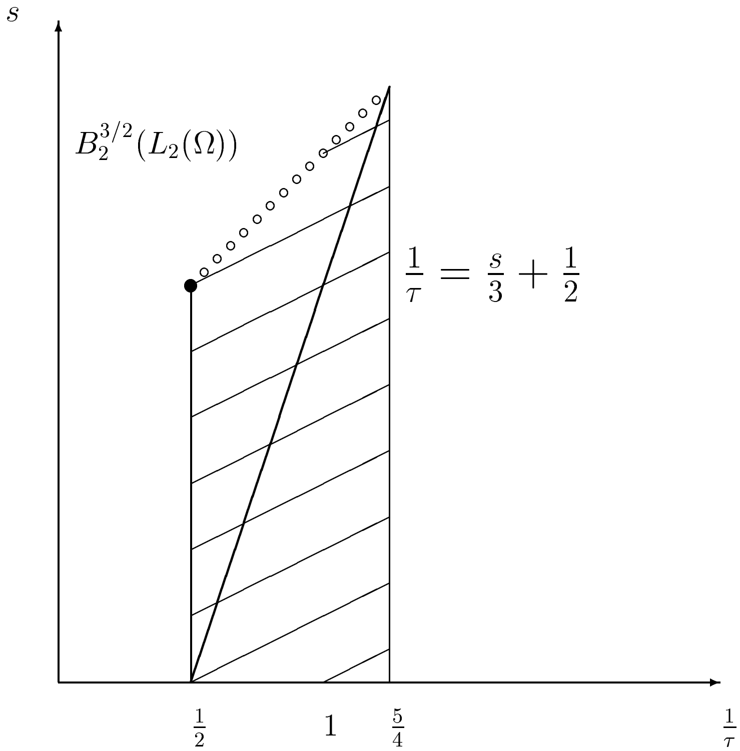

Figure 1: Regularity spaces according to Theorem 3.1, $g \in L_2(\partial\Omega)$.

A further improvement of the smoothness index for v can be obtained by repeatedly applying Theorem 3.2. Indeed, if α and p satisfy the conditions of Theorem 2.1, then we know that

$$v \in B^s_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{s}{d} + \frac{1}{p}\right)^{-1}, \quad 0 < s < \frac{(\alpha + 1/p)d}{d-1},$$

see (21). If

$$\frac{1}{\tau_1} := \frac{\alpha}{d-1} + \frac{1}{p}\left(\frac{d}{d-1}\right) < 1,$$

we may apply Theorem 3.2 for another time which yields

$$v \in B^s_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{s}{d} + \frac{1}{\tau_1}\right)^{-1}, \quad 0 < s < \frac{(\alpha + 1/p)d^2}{(d-1)^2}.$$

We can always keep on going until $\tau \leq 1$. By this kind of bootstrapping arguments, we always obtain a regularity result for $\tau < 1$. A slightly more sophisticated version of the bootstrapping strategy yields the following result.

Theorem 3.3 *Let Ω be a bounded Lipschitz domain. If v is the solution to* (20) *where $g \in B_p^{\alpha-1}(L_p(\Omega))$ and α and p satisfy the conditions of Theorem* 2.1, *then*

$$v \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)), \quad 0 < \tilde{s} < \alpha + \frac{1}{\tilde{\tau}}, \quad p \geq \tilde{\tau} > \tau^*, \quad \tau^* := \left(\frac{\alpha+1}{d-1} + 1\right)^{-1}. \tag{23}$$

Proof: We first observe that the critical value for s in (21) is exactly given by the intersection of the lines

$$s = \alpha + \frac{1}{\tau} \qquad \text{and} \qquad s = \frac{d}{\tau} - \frac{d}{p}.$$

Consequently, if we apply Theorem 3.2 repeatedly and use interpolation and embedding theorems for Besov spaces, we can conclude that

$$v \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)), \quad \tilde{s} < \alpha + \frac{1}{\tilde{\tau}}, \quad p \geq \tilde{\tau} > 1. \tag{24}$$

For any $\tilde{\tau}$ satisfying the condition in (24), we can apply Theorem 3.2 for another time and obtain

$$v \in B^{s}_{\tau}(L_{\tau}(\Omega)), \quad s < \frac{d(\alpha + 1/\tilde{\tau})}{d-1}, \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{\tilde{\tau}}.$$

Since

$$\alpha + \frac{1}{\tau^*} = \frac{d}{d-1}(\alpha + 1),$$

the result follows again by interpolation and embeddings.

As an example, let us again consider the case $p = 2,\ d = 3$, and $g \in L_2(\partial\Omega)$. Theorem 3.3 gives that v is in $B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$ for all $\tilde{s}$ and $\tilde{\tau}$ such that $(1/\tilde{\tau}, \tilde{s})$ is in the shaded region of Figure 2.

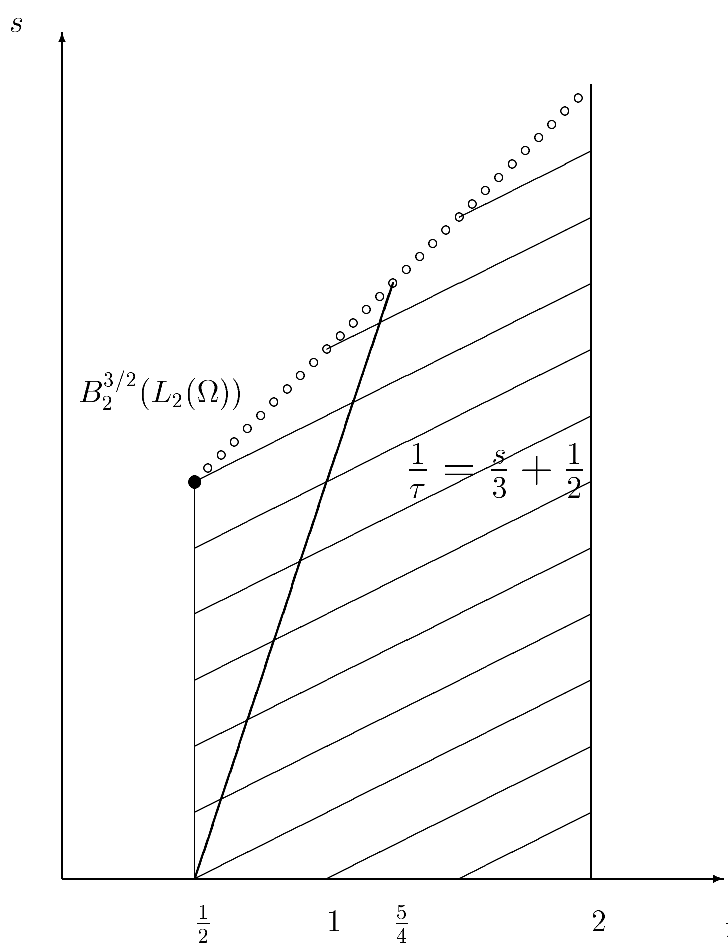

Figure 2: Regularity spaces according to Theorem 3.3, $g \in L_2(\partial\Omega)$.

The results stated in the Theorems 3.1 and 3.3 can also be used to establish Besov regularity for the inhomogeneous Neumann problem.

Theorem 3.4 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Let ϵ and p_0' be defined as in Theorem 2.1. Let w be the solution to*

$$\begin{aligned} \triangle w &= F \quad in \quad \Omega \subset \mathbf{R}^d, \\ \frac{\partial w}{\partial n} &= 0 \quad on \quad \partial\Omega, \end{aligned} \tag{25}$$

with $F \in (W^{2-\mu}(L_{p'}(\Omega)))^*_{1^\perp}$ for some non–integer $\mu > 1/p$.

(a) Suppose that $p_0' > p > 1$. If $\mu \geq 1 + 1/p$, then

$$w \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)), \qquad 0 < \tilde{s} < \min\left(\mu, 1 + \frac{1}{\tilde{\tau}}\right), \qquad p \geq \tilde{\tau} > \frac{d-1}{d+1}.$$

(b) Suppose that $p \geq p_0'$. If $\mu \geq 3/p + \epsilon$, then

$$w \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)), \qquad 0 < \tilde{s} < \min\left(\mu, 2/p + \epsilon + \frac{1}{\tilde{\tau}}\right), \qquad p \geq \tilde{\tau} > \frac{d-1}{2/p + \epsilon + d}.$$

Proof: We shall only prove the first case in detail. The second case can be studied analogously.

Let us first assume that $2 > \mu \geq 1 + 1/p$. Let $R_\Omega(f)$ denote the restriction of a function f on $\mathbf{R}^d$ to Ω. By using the Newtonian potential $N(x) := C_d|x|^{2-d}$, we define

$$\tilde{w} = N * (R^*_\Omega F). \tag{26}$$

Then

$$\triangle \tilde{w} = \triangle N * (R^*_\Omega F) = R^*_\Omega F. \tag{27}$$

Therefore we can write the solution w to (25) as

$$w = \tilde{w} - v \qquad \text{on} \quad \Omega, \tag{28}$$

where v is the solution to the homogeneous Neumann problem

$$\begin{aligned} \triangle v &= 0 \quad \text{in} \quad \Omega \subset \mathbf{R}^d, \\ \frac{\partial v}{\partial n} &= \frac{\partial \tilde{w}}{\partial n} := g \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{29}$$

Therefore we have to establish Besov regularity for both, $\tilde{w}$ and v. Let us start with $\tilde{w}$. It can be shown that $R^*_\Omega F \in W^{\mu-2}(L_p(\mathbf{R}^d))$, see [23] for details. Hence the classical elliptic regularity theory implies that $\tilde{w} \in W^\mu(L_p(\mathbf{R}^d)) = B^\mu_p(L_p(\mathbf{R}^d))$. We refer to [1] for further information. Hence $\tilde{w}|_\Omega \in B^\mu_p(L_p(\Omega))$, and by the embeddings of Besov spaces: $B^\mu_p(L_p(\Omega)) \hookrightarrow B^\mu_p(L_{\tilde{\tau}}(\Omega)) \hookrightarrow B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$, we have $\tilde{w}|_\Omega \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$ for any $\tilde{s}, \tilde{\tau}$ as in the statement (a). It remains to establish Besov regularity for v. It can be shown that $g \in B^{\beta-1}_p(L_p(\Omega))$ for all $\beta < 1$, compare again with [23]. Therefore an application of Theorem 3.3 implies that

$$v \in B^{\tilde{s}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)), \qquad 0 < \tilde{s} < 1 + \frac{1}{\tilde{\tau}}, \qquad p > \tilde{\tau} > \left(\frac{2}{d-1} + 1\right)^{-1} = \frac{d-1}{d+1}$$

and the result follows. The case $\mu > 2$ can be treated analogously by employing a classical extension technique as, e.g., outlined in [11].

Remark 3.1 *We have formulated Theorem 3.4 only for the 'interesting' case of a sufficiently smooth right–hand side. For smaller values of μ, our theory is consistent with Theorem 2.5.*

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Berichte aus der Technomathematik

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— Vertrieb durch den Autor —

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