Adaptive Wavelet Methods for Saddle Point Problems*

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Abstract

Recently, adaptive wavelet strategies for symmetric, positive definite operators have been introduced that were proven to converge. This paper is devoted to the generalization to saddle point problems which are also symmetric, but indefinite.

Firstly, we derive explicit criteria for adaptively refined wavelet spaces in order to fulfill the *Ladyshenskaja–Babuška–Brezzi (LBB)* condition and to be fully equilibrated. Then, we investigate a posteriori error estimates and generalize the known adaptive wavelet strategy to saddle point problems. The convergence of this strategy for elliptic operators essentially relies on the positive definite character of the operator. As an alternative, we introduce an adaptive variant of Uzawa's algorithm and prove its convergence.

Finally, we detail our results for two concrete examples of saddle point problems, namely the mixed formulation of the Stokes problem and second order elliptic boundary value problems where the boundary conditions are appended by Lagrange multipliers.

Keywords: Adaptive schemes, a posteriori error estimates, multiscale methods, wavelets, saddle point problems, Uzawa's algorithm, Stokes problem, Lagrange multipliers.

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1 Introduction

The variational formulation of many problems in mechanics, physics and technology leads to a saddle point problem. For example, mixed methods are widely used in structural and fluid mechanics, [5, 7]. Although significant progress has been made in the numerical treatment of such equations, they still form a class of challenging problems. The indefinite

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character of saddle point problems requires some care in the choice of the discretization in order to obtain a stable numerical method. Moreover, the efficient solvers that are available for symmetric positive definite operators, have to be appropriately modified.

In addition, many saddle point problems show a large scale behaviour in the sense that the solution has some global (low frequency) part and well localized (high frequency) details which may come from singularities of the problem data such as jumping coefficients, non–smooth domains and right–hand sides. These problems demand the use of adaptive strategies in order to resolve the local details of the solution up to a given accuracy while preserving efficiency.

In this paper, we treat these problems by means of wavelet analysis. The first applications of wavelet methods were in image and signal processing. During the last years, they have also been shown to offer some potential for the numerical treatment of partial differential and integral equations, see [13, 21] and the references therein. Among them, the maybe most important features for adaptive solution methods for saddle point problems are:

- Convergent and efficient adaptive wavelet methods for positive definite problems.
- Construction of adapted wavelet bases.

Let us describe this in more detail. Recently, an adaptive wavelet strategy has been introduced for symmetric positive definite operators, [19], see also [18]. It was proven there that this strategy gives rise to a convergent adaptive algorithm. The original method in [19] was somewhat modified in [14] resulting in a strategy that in addition was proven to be asymptotically optimal efficient.

The construction of (biorthogonal) wavelet bases leaves some freedom that can be exploited to fulfill additional requirements that e.g. are forced by the problem to solve. As one example, we mention the construction of divergence- and curl–free wavelets [37, 39, 41] and of wavelet trial spaces for the Stokes problem that fulfill the *Ladyshenskaja–Babuška–Brezzi (LBB)* condition, [23].

From what is said above it seems natural for us to consider the construction of adaptive wavelet strategies for saddle point problems. In this paper, we focus on two main questions:

- Is it possible to derive general and explicit criteria for adaptive wavelet discretizations of saddle point problems in order to fulfill (LBB)? Moreover, is the same possible for the *Full Equilibrium Property (FEP)* (see Definition 2.3 below)?
- Is there an adaptive wavelet strategy for saddle point problems that can be proven to converge?

We answer both questions positively in this paper. After collecting some preliminaries in Section 2, we prove general and explicit criteria for (LBB) and (FEP) in Section 3 in the context of adaptively chosen biorthogonal wavelet bases. We will detail these criteria for two concrete examples. In Section 6, we consider the mixed formulation of the Stokes problem and in Section 7, we treat second order elliptic boundary value problems where boundary conditions are appended by Lagrange multipliers. In both examples we exploit the possibility of adapting wavelet bases to the particular problem.

In order to answer the second question from above, we first introduce an a posteriori error analysis in Section 4 which in fact is a generalization of the result in [19]. Also, the adaptive refinement strategy in [19] can be generalized to saddle point problems. It is still an open problem to prove the convergence of this strategy for saddle point problems. Alternatively, we propose an adaptive variant of Uzawa's algorithm in Section 5 and we prove its convergence. Also this method may be viewed as a generalization of the results in [19] since the adaptive Uzawa algorithm uses a convergent adaptive strategy for the elliptic part as a main ingredient.

2 Preliminaries

In this section, we collect all the auxiliary facts on both, on saddle point problems and on wavelets, that will be needed in the sequel.

2.1 Setting

We consider the following saddle point problem: Given two Hilbert spaces X and M, some continuous bilinear forms

$$a:\ X\times X\to\mathbb{R},\qquad b:\ X\times M\to\mathbb{R}$$

and $f\in X'$ as well as $g\in M'$. Here, Y' denotes the dual space for some Banach space Y. Moreover, we assume $X\subseteq H_X$, $M\subseteq H_M$, where H_X, H_M are Hilbert spaces such that

$$X\hookrightarrow H_X\hookrightarrow X',\qquad M\hookrightarrow H_M\hookrightarrow M'. \tag{2.1}$$

Then, one has to determine a pair $[u,p]\in X\times M$ such that

$$\begin{aligned} a(u,v)+b(v,p) &= \langle f,v\rangle_{X'\times X} \quad \text{for all } v\in X,\\ b(u,q) &= \langle g,q\rangle_{M'\times M} \quad \text{for all } q\in M, \end{aligned} \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_{Y'\times Y}$ denotes the dual pairing. We assume the bilinear form $a(\cdot,\cdot)$ to be elliptic on the subspace

$$V:=\{v\in X:\ b(v,q)=0 \text{ for all } q\in M\}\subset X,$$

i.e., there exists a constant $\alpha>0$ such that

$$a(v,v)\geq\alpha\|v\|_X^2 \tag{2.3}$$

holds for all $v\in V$. Since we are ultimately interested in problems of the kind (2.2) that are uniquely solvable, we finally assume that X and M fulfill the *inf–sup condition*:

$$\inf_{q\in M}\sup_{v\in X}\frac{b(v,q)}{\|v\|_X\,\|q\|_M}\geq\beta \tag{2.4}$$

for some constant $\beta > 0$.

The following equivalent formulation will be very useful for our analysis. Defining the operators

$$\begin{array}{lll} A : X \to X', & \langle Au, v\rangle_{X'\times X} := a(u,v), & v \in X, \\ B : X \to M', & \langle Bu, q\rangle_{M'\times M} := b(u,q), & q \in M, \\ B' : M \to X', & \langle B'p, v\rangle_{X'\times X} := b(v,p), & v \in X, \end{array}$$

the problem (2.2) is equivalent to find $[u,p] \in X \times M =: \mathcal{H}$ such that

$$\begin{array}{lll} Au + B'p & = f & \text{in } X', \\ Bu & = g & \text{in } M'. \end{array} \tag{2.5}$$

If (2.2) is well–posed, the operator

$$\mathcal{A} := \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \tag{2.6}$$

is boundedly invertible with respect to the usual graph norm, i.e., there exist constants $c_{\mathcal{A}}, C_{\mathcal{A}}$ such that

$$c_{\mathcal{A}} \|\mathcal{A}([u,p])\|_{\mathcal{H}'} \leq \|[u,p]\|_{\mathcal{H}} \leq C_{\mathcal{A}} \|\mathcal{A}([u,p])\|_{\mathcal{H}'}, \tag{2.7}$$

where $\|[u,p]\|_{\mathcal{H}}^2 := \|u\|_X^2 + \|p\|_M^2$. We will often use the notation $A \lesssim B$ to abbreviate $A \leq cB$ with some constant $c > 0$, and for $A \lesssim B \lesssim A$, we write $A \sim B$. Hence, (2.7) may also be expressed by $\|\mathcal{A}([u,p])\|_{\mathcal{H}'} \sim \|[u,p]\|_{\mathcal{H}}$.

The Schur complement. In many cases of interest, the operator A is invertible and it makes sense to consider the operator $S := BA^{-1}B'$, which is known as the *Schur complement*. Then, we define the energy norm on $X \times M$ for the operator $\mathcal{A}$ by

$$\|[v,q]\|_{\mathcal{A}}^2 := \|v\|_A^2 + \|q\|_S^2, \qquad [v,q] \in X \times M, \tag{2.8}$$

where $\|\cdot\|_A$, $\|\cdot\|_S$ denote the energy norm corresponding to A and the Schur complement S, respectively. We include the proof of the following fact for completeness and convenience.

Proposition 2.1 *If A is boundedly invertible on X, i.e.,*

$$\|Av\|_{X'} \sim \|v\|_X, \qquad v \in X, \tag{2.9}$$

B' is bounded and the inf–sup condition (2.4) *holds, then*

$$\|q\|_S \sim \|q\|_M, \qquad q \in M. \tag{2.10}$$

Proof. We first establish the upper estimate. Due to the boundedness of B' and of A^{-1}, we have

$$\|q\|_S^2 = \langle Sq, q\rangle_{M'\times M} = \langle A^{-1}B'q, B'q\rangle_{X\times X'} \leq \|A^{-1}B'q\|_X \|B'q\|_{X'} \lesssim \|B'q\|_{X'}^2 \lesssim \|q\|_M^2.$$

To show the lower estimate, we use the boundedly invertibility and the inf–sup condition:

$$\|q\|_S^2 \;=\; \langle A^{-1}B'q, B'q\rangle_{X\times X'} \;\gtrsim\; \|B'q\|_{X'}^2 \;\gtrsim\; \|q\|_M^2.$$

This completes the proof. □

This result shows the equivalence of $\|\cdot\|_{\mathcal{A}}$ to the graph norm $\|\cdot\|_{\mathcal{H}}$ Since we always assume that (2.2) is well posed, we also have that $\mathcal{A}$ is boundedly invertible with respect to $\|\cdot\|_{\mathcal{A}}$, i.e.,

$$\|\mathcal{A}([v,q])\|_{X'\times M'} \sim \|[v,q]\|_{X\times M} \sim \|[v,q]\|_{\mathcal{A}},$$

which is an immediate consequence of Proposition 2.1 and the well–posedness of (2.2).

An equivalent formulation. An equivalent formulation of (2.2) can be introduced by the bilinear form

$$\mathcal{L}([u,p],[v,q]) := a(u,v) + b(v,p) + b(u,q), \tag{2.11}$$

which is defined for $[u,p],[v,q] \in X \times M$. Now, (2.2) can be rewritten in terms of the bilinear form $\mathcal{L}(\cdot,\cdot)$: Given $f \in X$, $g \in M'$ find a pair $[u,p] \in X \times M$ such that

$$\mathcal{L}([u,p],[v,q]) = \langle f, v\rangle_{X'\times X} + \langle g, q\rangle_{M'\times M}, \qquad [v,q] \in X \times M. \tag{2.12}$$

A short reflection shows that (2.2) and (2.12) are indeed equivalent: The conclusion from (2.2) to (2.12) can be made by adding up the two equations in (2.2). For the other direction one can take test functions with $q = 0$ and $v = 0$, respectively, to obtain (2.2). The following lemma shows that $\mathcal{L}$ also fulfills an inf–sup condition. We will use this fact later on.

Lemma 2.2 ([34]) *There exists is a constant $\tilde{\beta} \in \mathbb{R}^+$ with*

$$\inf_{[u,p]\in X\times M} \sup_{[v,q]\in X\times M} \frac{\mathcal{L}([u,p],[v,q])}{(\|u\|_X + \|p\|_M)(\|v\|_X + \|q\|_M)} \geq \tilde{\beta} > 0. \qquad \square \tag{2.13}$$

2.2 Multiscale methods and wavelets

Let us now summarize the basic notations for multiscale methods that are needed in this paper. For a survey of multiscale methods and wavelets, we refer to [13, 21].

Given some Hilbert space H, we call a system of functions $\Phi_j := \{\varphi_{j,k} : k \in \Delta_j\}$, $j \geq j_0$, Δ_j some (finite) set of indices, *(primal) single scale system*, if Φ_j is *refinable, i.e.,* there exists a matrix $M_{j,0} \in \mathbb{R}^{|\Delta_{j+1}|\times|\Delta_j|}$ such that

$$\Phi_j = M_{j,0}^T \Phi_{j+1}. \tag{2.14}$$

Here, $j_0 \in \mathbb{N}$ denotes some coarse level. Equation (2.14) in particular implies that the induced spaces

$$S_j := S(\Phi_j) := \mathrm{span}(\Phi_j)$$

are *nested*: $S_j \subset S_{j+1}$. We always assume that the union of all S_j is dense in H. Moreover, we assume the existence of a *dual single scale system* $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k};\ k \in \Delta_j\}$, such that

$$\langle \Phi_j, \tilde{\Phi}_j \rangle := \Big((\varphi_{j,k}, \tilde{\varphi}_{j,k'})_H \Big)_{k,k' \in \Delta_j} = I, \tag{2.15}$$

where I denotes the identity matrix of corresponding size.

Biorthogonal wavelet spaces $W_j, \tilde{W}_j$ are then defined by

$$W_j := S_{j+1} \ominus S_j,\ \tilde{W}_j := \tilde{S}_{j+1} \ominus \tilde{S}_j,\ S_j \perp \tilde{W}_j,\ \tilde{S}_j \perp W_j, \tag{2.16}$$

where the orthogonality is to be understood with respect to the H–inner product. Constructing *biorthogonal wavelets* then amounts finding bases

$$\Psi_j := \{\psi_{j,k} :\ k \in \nabla_j\},\ \tilde{\Psi}_j := \{\tilde{\psi}_{j,k} :\ k \in \nabla_j\}, \quad (\nabla_j := \Delta_{j+1} \setminus \Delta_j) \tag{2.17}$$

of W_j, $\tilde{W}_j$, respectively, such that

$$\langle \Psi_j, \tilde{\Psi}_j \rangle = I \tag{2.18}$$

and the collections

$$\Psi := \{\psi_\lambda :\ \lambda \in \nabla\}, \qquad \tilde{\Psi} := \{\tilde{\psi}_\lambda :\ \lambda \in \nabla\}, \qquad \nabla := \{\lambda = (j,k) :\ j \geq j_0 - 1,\ k \in \nabla_j\} \tag{2.19}$$

($\Psi_{j_0-1} := \Phi_{j_0}, \tilde{\Psi}_{j_0-1} := \tilde{\Phi}_{j_0}$) form *Riesz bases* for H, i.e., they form a basis for H and the following norm equivalence holds

$$\|\boldsymbol{d}^T \Psi\|_H = \Big\| \sum_{\lambda \in \nabla} d_\lambda \psi_\lambda \Big\|_H \sim \|\boldsymbol{d}\|_{\ell_2(\nabla)} = \Big(\sum_{\lambda \in \nabla} |d_\lambda|^2 \Big)^{1/2}. \tag{2.20}$$

Often, Ψ, $\tilde{\Psi}$ will be termed *biorthogonal wavelet system* or simply *multiscale basis*. In many cases, an equation similar to (2.20) also holds for a whole range of Sobolev or Besov spaces including H (see [20] and also (2.22), (2.23) below).

For any subset $\Lambda \subset \nabla$, we define the corresponding set of wavelets by

$$\Psi_\Lambda := \{\psi_\lambda :\ \lambda \in \Lambda\}, \qquad \tilde{\Psi}_\Lambda := \{\tilde{\psi}_\lambda :\ \lambda \in \Lambda\},$$

and the induced spaces by $S_\Lambda := S(\Psi_\Lambda)$ and $\tilde{S}_\Lambda := S(\tilde{\Psi}_\Lambda)$.

2.3 Multiscale discretization of saddle point problems

In order to discretize (2.2), we want to use trial and test spaces that are induced by multiscale bases. To be more specific, we assume that there exist wavelet bases $\Psi = \{\psi_\lambda : \lambda \in \nabla^X\}$ and $\Theta = \{\vartheta_\mu :\ \mu \in \nabla^M\}$ that form Riesz bases for H_X and H_M, respectively, see (2.1). In the sequel, we shall restrict ourselves mainly to the case that X and M are Sobolev spaces defined on suitable domains or manifolds $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$, i.e.,

$$X = H^t(\Omega_1), \qquad M = H^s(\Omega_2), \quad s,t \in \mathbb{R}. \tag{2.21}$$

Then, we also assume that the Riesz bases give rise to the following norm equivalences:

$$\|\boldsymbol{d}^T\Psi\|^2_{\tau,\Omega_1} \sim \sum_{\lambda\in\nabla^X} 2^{2\tau|\lambda|} d^2_\lambda, \qquad \tau\in[-t,t], \tag{2.22}$$

$$\|\boldsymbol{c}^T\Theta\|^2_{\sigma,\Omega_2} \sim \sum_{\mu\in\nabla^M} 2^{2\sigma|\mu|} c^2_\mu, \qquad \sigma\in[-s,s], \tag{2.23}$$

where $\|\cdot\|_{m,\Omega}$ denotes the norm in the Sobolev space $H^m(\Omega)$, $m\in\mathbb{R}$. Since it should be clear from the context, we will omit the dependencies of the norms on Ω_1 and Ω_2, respectively, in the sequel. Now, the trial spaces $(X_\Lambda, M_\Lambda)\subset(X,M)$ are defined by a pair of index sets

$$\Lambda := (\Lambda^X, \Lambda^M) \subset (\nabla^X, \nabla^M).$$

The LBB condition. It is well–known that trial spaces for the stable numerical solution of (2.2) need to fulfill the *Ladyshenskaja–Babuška–Brezzi (LBB)* condition

$$\inf_{q_\lambda\in M_\Lambda} \sup_{v_\lambda\in X_\Lambda} \frac{b(v_\lambda, q_\lambda)}{\|v_\lambda\|_X\,\|q_\lambda\|_M} \geq \beta \tag{2.24}$$

for some constant $\beta>0$ independent of Λ.

Full equilibrium property. For the numerical treatment of saddle point problems as well as for the analysis of discretizations, the following property is very useful.

Definition 2.3 *A discretization* (X_Λ, M_Λ) *is said to have the* Full Equilibrium Property (FEP) *if for* $u_\Lambda\in X_\Lambda$ *the equality* $b(u_\Lambda, q_\Lambda)=0$ *for all* $q_\Lambda\in M_\Lambda$ *already implies that* $u_\Lambda\in V$, *i.e.,* $b(u_\Lambda, q)=0$ *for all* $q\in M$. *The spaces are also called* equilibrated.

Roughly speaking, this means that $\operatorname{Ker} B_\Lambda \subset \operatorname{Ker} B$, which is, of course a very strong property. There are many different names in the literature for this property. We choose (FEP), which is used in applications of mixed methods in structural mechanics.

3 Multiscale bases, the LBB and FEP condition

This section is devoted to conditions on the particular choice of multiscale trial spaces in order to fulfill (LBB) and (FEP). It will turn out that biorthogonality is the main technical tool to derive explicit criteria.

3.1 The LBB condition

The LBB condition has already been studied in the wavelet context in [4, 23]. Both papers are however restricted to the Stokes problem. While [23] does not consider adaptively refined spaces, in [4] this problem is treated using ideas from [23]. But the conditions derived in [4] are still somewhat implicit. Here, we will deal with arbitrary saddle point problems and we derive explicit criteria for the adaptively refined spaces in order to fulfill

(LBB). The basic idea, namely to use biorthogonality and the following well–known result by M. Fortin can already be found in [23].

Proposition 3.1 ([29]) *The LBB condition holds if and only if there exists an operator* $Q_\Lambda \in \mathcal{L}(X, X_\Lambda)$ *satisfying*

$$\begin{aligned} b(v - Q_\Lambda v, q_\Lambda) &= 0 \quad \text{for all } v \in X,\ q_\Lambda \in M_\Lambda, \text{ and} \qquad (3.1)\\ \|Q_\Lambda\|_{\mathcal{L}(X,X)} &\lesssim 1, \qquad (3.2)\end{aligned}$$

independent of Λ. □

For the spaces X_Λ defined above, there is a natural choice for the operator Q_Λ given by

$$Q_\Lambda v := \sum_{\lambda \in \Lambda^X} \langle v, \tilde{\psi}_\lambda \rangle_{X \times X'} \, \psi_\lambda. \qquad (3.3)$$

Due to the norm equivalences (2.22), condition (3.2) is always fulfilled:

$$\|Q_\Lambda v\|_X^2 \sim \sum_{\lambda \in \Lambda^X} 2^{2t|\lambda|} |\langle v, \tilde{\psi}_\lambda \rangle_{X \times X'}|^2 \le \sum_{\lambda \in \nabla} 2^{2t|\lambda|} |\langle v, \tilde{\psi}_\lambda \rangle_{X \times X'}|^2 \lesssim \|v\|_X^2.$$

For any subset $\bar{X} \subseteq X$ we will use the notations

$$\bar{X}^{\perp_b} := \{q \in M : b(v, q) = 0 \quad \text{for all } v \in \bar{X}\}, \qquad (3.4)$$

and similar for $\bar{M} \subseteq M$

$$\bar{M}^{\perp_b} := \{v \in X : b(v, q) = 0 \quad \text{for all } q \in \bar{M}\}. \qquad (3.5)$$

Moreover, we use the standard definition of the polar space for any subset $\bar{X} \subseteq X$

$$\bar{X}^0 := \{x' \in X' : \langle x', v \rangle_{X' \times X} = 0 \quad \text{for all } v \in \bar{X}\}, \qquad (3.6)$$

and similarly for subsets in M. By definition, we have

$$X_\Lambda^{\perp_b} = B(X_\Lambda)^0 \quad \text{and} \quad M_\Lambda^{\perp_b} = B'(M_\Lambda)^0. \qquad (3.7)$$

Now, we obtain the desired result.

Theorem 3.2 *The multiscale spaces* X_Λ, M_Λ *defined above fulfill the LBB condition* (2.24) *provided that one of the following equivalent conditions holds:*

(a) $M_\Lambda \subseteq (X \ominus X_\Lambda)^{\perp_b}$,

(b) $B'(M_\Lambda) \subseteq \tilde{X}_\Lambda$,

(c) $B(X \ominus X_\Lambda) \subseteq M' \ominus \tilde{M}_\Lambda$.

Proof. Due to the Riesz basis property, we have for $v \in X$ and $q_\Lambda \in M_\Lambda$

$$b(v - Q_\Lambda v, q_\Lambda) = \sum_{\lambda \in \nabla \setminus \Lambda} \langle v, \tilde{\psi}_\lambda \rangle_{X \times X'} \, b(\psi_\lambda, q_\Lambda) = 0 \tag{3.8}$$

if and only if

$$b(v_\mu, q_\Lambda) = 0 \qquad \text{for all } v_\mu \in X \ominus X_\Lambda, \; q_\Lambda \in M_\Lambda,$$

which is equivalent to (a). It remains to verify the equivalence of (a)–(c). In fact, using $M_\Lambda^0 = M' \ominus \tilde{M}_\Lambda$ and $\tilde{X}_\Lambda^0 = X \ominus X_\Lambda$, the assertion is an immediate consequence of the well–known equivalences

$$Y \subseteq B(L)^0 \iff B'(Y) \subseteq L^0 \iff B(L) \subseteq Y^0 \tag{3.9}$$

for any subset $Y \subseteq M$ and $L \subseteq X$. □

3.2 Full equilibrium

It is obvious that equilibrated discretizations allow the use of more powerful analytical tools for studying the approximation of saddle point problems. For instance, one may obtain error estimates only for the variable u without using the graph norm, see e.g. [5] and (4.24) below. However, it is in general a non trivial task to realize equilibrated discretizations. Hence, we investe in the development of sufficient criteria when using multiscale bases. Again, it turns out that biorthogonality is a useful tool.

Theorem 3.3 *If a multiscale discretization given by the set of indices $\Lambda = (\Lambda^X, \Lambda^M)$ fulfills one of the following equivalent conditions*

$$B(X_\Lambda) \subseteq \tilde{M}_\Lambda, \tag{3.10}$$

$$B'(M \ominus M_\Lambda) \subseteq X' \ominus \tilde{X}_\Lambda, \tag{3.11}$$

then the discretization is equilibrated.

Proof. Indeed, if (3.10) holds, this means that $\langle Bv_\Lambda, q \rangle_{M' \times M} = 0$ for all $q \in M \ominus M_\Lambda$ for some $v_\Lambda \in X_\Lambda$. This shows that $\langle Bv_\Lambda, q_\Lambda \rangle_{M' \times M} = 0$ for all $q_\Lambda \in M_\Lambda$ already implies $Bv_\Lambda = 0$. The stated equivalence of (3.10) and (3.11) follows by (3.9). □

We may combine Theorem 3.2 and 3.3, so that we easily obtain the following result.

Corollary 3.4 *If the spaces X_Λ and M_Λ fulfill $B'(M_\Lambda) = \tilde{X}_\Lambda$, or equivalently, $B(X_\Lambda) = \tilde{M}_\Lambda$, both (LBB) and (FEP) are valid.* □

One example. Let us illustrate the above conditions by one simple example. Let us assume that we have two basis functions $\psi_{\lambda_1}, \psi_{\lambda_2} \in X$, such that $B\psi_{\lambda_1} = B\psi_{\lambda_2} = c\tilde{\vartheta}_\mu$ for some dual basis function $\tilde{\vartheta}_\mu \in M'$ and some $c \neq 0$. Let us now assume that

$$\psi_{\lambda_1} \in X_\Lambda, \qquad \psi_{\lambda_2} \notin X_\Lambda. \tag{3.12}$$

Using condition (c) in Theorem 3.2 applied to ψ_{λ_2}, it follows that $\vartheta_\mu \notin M_\Lambda$ for ensuring the LBB condition. On the other hand, using (3.10) applied to ψ_{λ_1} for checking (FEP) , one would obtain $\vartheta_\mu \in M_\Lambda$ which contradicts the condition (c) in Theorem 3.2. This shows that (3.12) is not possible for a stable *and* equilibrated discretization. The 'inverse' images with respect to B of a certain basis function ϑ_μ either *all* have to belong to X_Λ or *none* of them.

4 A posteriori error estimates and a refinement strategy

As already mentioned, a convergent adaptive wavelet strategy for symmetric, positive definite operators has been introduced in [19]. However, a closer look to the proofs in [19] shows that the results concerning the construction of an adaptive refinement strategy (without proof of convergence) can easily be generalized to a more general setting (including saddle point problems). In this section, we will therefore briefly review the relevant results but we omit the proofs since they can easily be deduced from [19]. Finally, we describe the application of these results to saddle point problems.

4.1 The general setting

Let us now describe the setting that we consider in this section. Let $L : H \to H'$ be a linear boundedly invertible operator, i.e.,

$$c_L \|Lx\|_{H'} \leq \|x\|_H \leq C_L \|Lx\|_{H'}, \qquad x \in H, \tag{4.1}$$

where $0 < c_L \leq C_L$ are absolute constants and H is some Hilbert space. We consider the problem

$$Lx = z \tag{4.2}$$

for a given $z \in H'$. Moreover, we assume the existence of biorthogonal wavelet bases $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$, $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \nabla\}$, such that $\tilde{\Psi}$ characterizes H', i.e., there exist absolut constants $0 < c_\Psi \leq C_\Psi$, such that

$$c_\Psi \Big(\sum_{\lambda \in \nabla} |\gamma_\lambda \langle y, \psi_\lambda \rangle|^2 \Big)^{1/2} \leq \|y\|_{H'} \leq C_\Psi \Big(\sum_{\lambda \in \nabla} |\gamma_\lambda \langle y, \psi_\lambda \rangle|^2 \Big)^{1/2}, \tag{4.3}$$

for any $y \in H'$, where γ_λ are suitable weight factors and $\langle \cdot, \cdot \rangle$ denotes the dual pairing of H and H'.

Now, we consider the Galerkin approximation of (4.2), i.e., we look for some $x_\Lambda \in S_\Lambda := S(\Psi_\Lambda)$ such that

$$L_\Lambda x_\Lambda = z_\Lambda, \tag{4.4}$$

where L_Λ, z_Λ are the usual Galerkin projections of L and z, i.e., $L_\Lambda = (\langle L\psi_{\lambda'}, \psi_\lambda\rangle)_{\lambda,\lambda'\in\Lambda}$ and $z_\Lambda = (\langle z, \psi_\lambda\rangle)_{\lambda\in\Lambda}$, respectively. Note that x_Λ is indeed the Galerkin solution, i.e., we have the *Galerkin orthogonality*

$$\langle L(x - x_\Lambda), y_\Lambda\rangle = 0 \quad \text{for all} \quad y_\Lambda \in S_\Lambda. \tag{4.5}$$

4.2 A posteriori error estimates

Now, using (4.1), (4.3) and the Galerkin orthogonality already gives rise to an a posteriori error estimate:

Proposition 4.1 *Under the above assumptions, one has for*

$$\delta_\lambda := |\gamma_\lambda \langle L(x - x_\Lambda), \psi_\lambda\rangle|, \qquad \lambda \in \nabla, \tag{4.6}$$

the estimate

$$c_L c_\Psi \Big(\sum_{\lambda\in\nabla\setminus\Lambda} \delta_\lambda^2 \Big)^{1/2} \le \|x - x_\Lambda\|_H \le C_L C_\Psi \Big(\sum_{\lambda\in\nabla\setminus\Lambda} \delta_\lambda^2 \Big)^{1/2}. \qquad \Box \tag{4.7}$$

Equation (4.7) states that we already have an *efficient* and *reliable* error estimator. However, this is numerically useless, since $\nabla \setminus \Lambda$ is a set of infinite cardinality so that the estimator is not accessible. Hence, the idea is to reduce the infinte sums in (4.7) to finite ones allowing some additional error that is under control. In order to do so, we have to pose one more assumption on L and Ψ. To be precise, we call L *quasi sparse* w.r.t. Ψ, if

$$2^{-(|\lambda'|+|\lambda|)t} |\langle L\psi_{\lambda'}, \psi_\lambda\rangle| \;\lesssim\; \frac{2^{-\sigma \big| |\lambda| - |\lambda'| \big|}}{(1 + 2^{\min(|\lambda|,|\lambda'|)}) \mathrm{dist}(\Box_\lambda, \Box_{\lambda'})^\tau}, \tag{4.8}$$

where $\Box_\lambda := \operatorname{supp} \psi_\lambda$ and the constants t, σ and τ depend on L and Ψ (see [19], Section 4.3, for details). Let us remark that (4.8) is in fact valid for a wide class of differential and integral operators. Then, one can show ([19], Lemma 4.2 and Remark 4.2) that for each $\lambda \in \nabla$ and a given tolerance $\varepsilon > 0$ there exists a finite *influence set* $\mathcal{J}_{\lambda,\varepsilon} \subset \nabla$ such that the quantities

$$e_\lambda := \sum_{\lambda'\in\Lambda\setminus\mathcal{J}_{\lambda,\varepsilon}} \langle L\psi_{\lambda'}, \psi_\lambda\rangle\, x_{\lambda'}, \qquad x_{\lambda'} := \langle x_\Lambda, \tilde\psi_{\lambda'}\rangle, \tag{4.9}$$

satisfy

$$\left(\sum_{\lambda\in\nabla\setminus\Lambda} |\gamma_\lambda\, e_\lambda|^2 \right)^{1/2} \le C_\varepsilon\, \varepsilon\, \|x_\Lambda\|_H \tag{4.10}$$

for some constant $C_\varepsilon > 0$.

Now, we may define the finite index set, which will reduce the infinite sum in (4.7). To be specific, let

$$N_{\Lambda,\varepsilon} := \{\lambda \in \nabla \setminus \Lambda : \Lambda \cap \mathcal{J}_{\lambda,\varepsilon} \neq \emptyset\}. \tag{4.11}$$

Thus, setting

$$Z_\Lambda := \Big(\sum_{\lambda \in \nabla\setminus\Lambda} |\gamma_\lambda z_\lambda|^2 \Big)^{1/2}, \tag{4.12}$$

one can prove (see [19], Theorem 4.1):

Proposition 4.2 *Defining*

$$g_\lambda(\Lambda,\varepsilon) := \Big|\gamma_\lambda \sum_{\lambda' \in \Lambda\cap\mathcal{J}_{\lambda,\varepsilon}} \langle L\psi_{\lambda'}, \psi_\lambda\rangle\, x_{\lambda'}\Big|, \qquad \lambda \in \nabla \setminus \Lambda, \tag{4.13}$$

the following estimates hold:

$$\|x - x_\Lambda\|_H \le C_L C_\Psi \Big[\Big(\sum_{\lambda \in N_{\Lambda,\varepsilon}} g_\lambda(\Lambda,\varepsilon)^2 \Big)^{1/2} + Z_\Lambda + \varepsilon C_\varepsilon \, \|x_\Lambda\|_H \Big] \tag{4.14}$$

and

$$\Big(\sum_{\lambda \in N_{\Lambda,\varepsilon}} g_\lambda(\Lambda,\varepsilon)^2 \Big)^{1/2} \le \frac{1}{c_L c_\Psi} \|x - x_\Lambda\|_H + Z_\Lambda + \varepsilon C_\varepsilon \, \|x_\Lambda\|_H. \quad \Box \tag{4.15}$$

Now, (4.14) and (4.15) state that the sum over $g_\lambda(\Lambda,\varepsilon)$ is an efficient and reliable error estimator up to a fixed tolerance. On the other hand, in contrary to (4.7), it is now reduced to a finite sum over $N_{\Lambda,\varepsilon}$, so that it is in fact numerically accessible.

4.3 An adaptive refinement strategy

Now, we may use the latter proposition to formulate a refinement strategy (see [19], Theorem 4.2).

Proposition 4.3 *Under the above assumptions, we have: Let* eps > 0 *be a given tolerance and fix any* $\vartheta^* \in (0,1)$. *Then, defining* $C_e := \frac{1}{c_L c_\Psi} + \frac{1-\vartheta^*}{2C_L C_\Psi}$ *and choosing* $\mu^* > 0$ *such that* $\mu^* C_e \le \frac{1-\vartheta^*}{2(2-\vartheta^*)C_L C_\Psi}$, *we set*

$$\varepsilon := \frac{\mu^* \mathrm{eps}}{2C_\varepsilon \|u_\Lambda\|_H}. \tag{4.16}$$

Suppose that $\Lambda \subset \nabla$ *is chosen so that*

$$Z_\Lambda < \frac{1}{2}\mu^* \mathrm{eps}. \tag{4.17}$$

Then whenever $\tilde{\Lambda} \subset \nabla$, $\Lambda \subset \tilde{\Lambda}$ *is chosen so that*

$$\Big(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\varepsilon}} g_\lambda(\Lambda, \varepsilon)^2 \Big)^{1/2} \geq (1 - \vartheta^*) \Big(\sum_{\lambda \in N_{\Lambda,\varepsilon}} g_\lambda(\Lambda, \varepsilon)^2 \Big)^{1/2}, \tag{4.18}$$

there exists a constant $\kappa \in (0, 1)$ *such that either*

$$\| x_\Lambda - x_{\tilde{\Lambda}} \|_H \geq \kappa \, \| x - x_\Lambda \|_H \tag{4.19}$$

or

$$\Big(\sum_{\lambda \in N_{\Lambda,\varepsilon}} g_\lambda(\Lambda, \varepsilon)^2 \Big)^{1/2} = \Big(\sum_{\lambda \in \nabla \setminus \Lambda} g_\lambda(\Lambda, \varepsilon)^2 \Big)^{1/2} < \mathrm{eps}. \qquad \Box \tag{4.20}$$

In [19], the *distance property* (4.19) is used to prove the convergence of the above strategy for symmetric positive definite operators (see also Section 5 below). In our general setting, a convergence result can not be expected.

4.4 Saddle point problems

Now, we apply the above results to the saddle point operator $\mathcal{A}$ in (2.6). Obviously, assumption (4.1) is fulfilled by (2.7). As a wavelet basis on $H = \mathcal{H}$, we choose $\Psi \times \Theta$ and then (2.22) and (2.23) imply (4.3) for the weight factors

$$\gamma_{[\lambda,\mu]} := \begin{pmatrix} 2^{-t|\lambda|} \\ 2^{-s|\mu|} \end{pmatrix}, \qquad [\lambda, \mu] \in \nabla^X \times \nabla^M,$$

and $y = [v, q] \in X \times M = H$. In fact, for $[v', q'] \in X' \times M'$, we have

$$\begin{aligned} \|[v', q']\|_{H'}^2 &= \|v'\|_{X'}^2 + \|q'\|_{M'}^2 \\ &\sim \sum_{\lambda \in \nabla^X} 2^{-2t|\lambda|} |\langle v', \psi_\lambda \rangle_{X' \times X}|^2 + \sum_{\mu \in \nabla^M} 2^{-2s|\mu|} |\langle q', \vartheta_\mu \rangle_{M' \times M}|^2 \\ &= \sum_{[\lambda,\mu] \in \nabla^X \times \nabla^M} \left| \begin{pmatrix} 2^{-t|\lambda|} \langle v', \psi_\lambda \rangle_{X' \times X} \\ 2^{-s|\mu|} \langle q', \vartheta_\mu \rangle_{M' \times M} \end{pmatrix} \right|^2 \\ &= \sum_{[\lambda,\mu] \in \nabla^X \times \nabla^M} |\gamma_{[\lambda,\mu]} \langle [v', q'], [\psi_\lambda, \vartheta_\mu] \rangle_{\mathcal{H}' \times \mathcal{H}}|^2. \end{aligned}$$

Now, we define the *residual*

$$R_\Lambda := \begin{pmatrix} r_\Lambda \\ \rho_\Lambda \end{pmatrix} := \mathcal{A} \begin{pmatrix} u_\Lambda - u \\ p_\Lambda - p \end{pmatrix} = \mathcal{A} \begin{pmatrix} u_\Lambda \\ p_\Lambda \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} A u_\Lambda + B' p_\Lambda - f \\ B u_\Lambda - g \end{pmatrix}, \tag{4.21}$$

and the quantities

$$\varrho_\lambda := |\langle r_\Lambda, \psi_\lambda \rangle_{X' \times X}|, \quad \lambda \in \nabla^X, \qquad \zeta_\mu := |\langle \rho_\Lambda, \vartheta_\mu \rangle_{M' \times M}|, \quad \mu \in \nabla^M. \tag{4.22}$$

Theorem 4.4 *For the above discretization* (X_Λ, M_Λ)*, the following error estimate is valid:*

$$\|u - u_\Lambda\|_X + \|p - p_\Lambda\|_M \sim \Big(\sum_{\lambda \in \nabla^X \setminus \Lambda^X} 2^{-2t|\lambda|} \varrho_\lambda^2 \Big)^{1/2} + \Big(\sum_{\mu \in \nabla^M \setminus \Lambda^M} 2^{-2s|\mu|} \zeta_\mu^2 \Big)^{1/2}. \qquad \Box \quad (4.23)$$

As a consequence, we get the following result:

Corollary 4.5 *For equilibrated discretizations, the following equivalence holds for* $g \equiv 0$*:*

$$\|u - u_\Lambda\|_X \sim \Big(\sum_{\lambda \in \nabla^X \setminus \Lambda^X} 2^{-2t|\lambda|} \varrho_\lambda^2 \Big)^{1/2}. \qquad \Box \qquad (4.24)$$

Remark 4.6 In view of Proposition 2.1, the estimates (4.23), (4.24) also hold if we replace $\|u - u_\Lambda\|_X + \|p - p_\Lambda\|_M$ by the energy norm $\|[u - u_\Lambda, p - p_\Lambda]\|_{\mathcal{A}}$ and $\|u - u_\Lambda\|_X$ by $\|u - u_\Lambda\|_A$, respectively, provided that the assumptions of Proposition 2.1 are fulfilled.

Finally, we apply the results concerning the adaptive strategy to saddle point problems. Firstly, we have to assume that both A and B are quasi sparse in the sense of (4.8), where for B we have to replace t in (4.8) by s. Then, for $u_{\lambda'} := \langle u, \tilde{\psi}_{\lambda'} \rangle_{X' \times X}$, $\lambda' \in \nabla^X$, and $p_{\mu'} := \langle p, \tilde{\vartheta}_{\mu'} \rangle_{M' \times M}$, $\mu' \in \nabla^M$, the quantity in (4.9) becomes

$$e_{[\lambda,\mu]} = \sum_{[\lambda',\mu'] \in (\Lambda^X \setminus \mathcal{J}^X_{\lambda,\varepsilon}) \times (\Lambda^M \setminus \mathcal{J}^M_{\mu,\varepsilon})} \begin{pmatrix} u_{\lambda'} \langle A\psi_{\lambda'}, \psi_\lambda \rangle_{X' \times X} + p_{\mu'} \langle B' \vartheta_{\mu'}, \psi_\lambda \rangle_{X' \times X} \\ u_{\lambda'} \langle B\psi_{\lambda'}, \vartheta_\mu \rangle_{M' \times M} \end{pmatrix},$$

where $\mathcal{J}^X_{\lambda,\varepsilon} \subset \nabla^X$ and $\mathcal{J}^M_{\mu,\varepsilon} \subset \nabla^M$ are suitable finite influence sets. Next, setting $f_\lambda := \langle f, \psi_\lambda \rangle_{X' \times X}$, $\lambda \in \nabla^X$ and $g_\mu := \langle g, \vartheta_\mu \rangle_{M' \times M}$, $\mu \in \nabla^M$, equation (4.12) reads

$$Z_\Lambda = \Big(\sum_{[\lambda,\mu] \in (\nabla^X \setminus \Lambda^X) \times (\nabla^M \setminus \Lambda^M)} 2^{-2t|\lambda|} |f_\lambda|^2 + 2^{-2s|\mu|} |g_\mu|^2 \Big)^{1/2}.$$

Finally, the error quantities $g_{[\lambda,\mu]}(\Lambda, \varepsilon)$ defined in (4.13) take the form

$$g_{[\lambda,\mu]}(\Lambda, \varepsilon) = \Big| \sum_{[\lambda',\mu'] \in (\Lambda^X \cap \mathcal{J}^X_{\lambda,\varepsilon}) \times (\Lambda^M \cap \mathcal{J}^M_{\mu,\varepsilon})} \Big(2^{-t|\lambda|} u_{\lambda'} \langle A\psi_{\lambda'}, \psi_\lambda \rangle_{X' \times X} + 2^{-t|\lambda|} p_{\mu'} \langle B' \vartheta_{\mu'}, \psi_\lambda \rangle_{X' \times X} + 2^{-s|\mu|} u_{\lambda'} \langle B\psi_{\lambda'}, \vartheta_\mu \rangle_{M' \times M} \Big) \Big|.$$

With these definitions, Proposition 4.3 easily applies.

5 Convergence of adaptive schemes

So far, we have set up an a posteriori error analysis for adaptively refined wavelet spaces and introduced explicit conditions for the crucial properties (LBB) and (FEP). However, it remains to study the convergence of such an adaptive algorithm. In [19], the above described adaptive wavelet strategy for *positive definite* operators was proven to converge. Now, one might think that the generalization to saddle point problems is an easy task.

Unfortunately, we did not succeed in adapting the arguments used in [19] to saddle point problems. Let us briefly point out the main differences when going from a positive definite to an indefinite problem. We consider the problem $Au = f$, where $A : H_0^t(\Omega) \to H^{-t}(\Omega)$ is some positive definite, boundedly invertible operator and $f \in H^{-t}(\Omega)$ are the given data, while the function $u \in H_0^t(\Omega)$ has to be seeked. Taking as above a wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\} \subset H_0^t(\Omega)$, we denote by u_Λ the Galerkin solution w.r.t. a (finite) set $\Lambda \subset \nabla$.

In the previous section, we have described a strategy how to enlarge Λ to some $\tilde{\Lambda} \supset \Lambda$ such that the distance property holds, i.e., there exists some $0 < \kappa < 1$ such that

$$\|u_\Lambda - u_{\tilde{\Lambda}}\|_A \geq \kappa \, \|u - u_\Lambda\|_A,$$

see Proposition 4.3, (4.19). Note that here the energy norm $\|\cdot\|_A$ is used, which already assumes that A is positive definite. Now, one proceeds using Galerkin orthogonality

$$a(u_\Lambda - u_{\tilde{\Lambda}}, u - u_{\tilde{\Lambda}}) = 0 \tag{5.1}$$

to conclude

$$\|u - u_{\tilde{\Lambda}}\|_A^2 = \|u - u_\Lambda\|_A^2 - \|u_\Lambda - u_{\tilde{\Lambda}}\|_A^2 \leq (1 - \kappa^2)\|u - u_\Lambda\|_A^2,$$

which proves the *saturation property*, i.e., a strict error reduction since $0 < 1 - \kappa^2 < 1$.

Unfortunately, (5.1) is no longer true when A is replaced by the operator $\mathcal{A}$ in (2.6) which represents a saddle point operator. Now, one could try to use that $\mathcal{A}$ is positive definite on $V = \mathrm{Ker}(B)$. This approach in fact gives rise to a convergent adaptive algorithm (which can be numerically performed) for computing u provided that a basis for V is *explicitly* available. This of course contradicts the philosophy of the saddle point approach and is not what we aimed at.

As a second approach, one could consider the reduced problem for p, i.e.,

$$Sp = BA^{-1}f - g,$$

involving the Schur complement S. But also this approach seems to have some ultimate obstacles. Firstly, due to the presence of A^{-1}, the entries of the corresponding stiffness matrix $\langle S\vartheta_\mu, \vartheta_{\mu'}\rangle_{M'\times M}$ can not be easily computed (and the same is true for the right hand side). Now, one could approximate A^{-1} by some A_Λ^{-1}. But then one ends up with the problem that the discretization of S is not the same as discretizing the three factors separately, i.e.,

$$B_\Lambda A_\Lambda^{-1} B'_\Lambda \neq S_\Lambda. \tag{5.2}$$

This means that the computed solution does not correspond to the Galerkin solution w.r.t. Λ. This, however, is essential for (5.1). Note that the non–equality in (5.2) still holds if one could replace B_Λ, B'_Λ by B and B', respectively, which is indeed possible if (FEP) holds.

Next, one could try to study the error between the real Galerkin solution p_Λ w.r.t. S_Λ and the perturbed one p^*_Λ w.r.t. $B_\Lambda A_\Lambda^{-1} B'_\Lambda$ (which is available for example by Uzawa's algorithm). We have not been able to give meaningful quantitative criteria for the index set Λ such that this error is below some given tolerance.

Another approach is to make use of (FEP) which is a very strong property so that one could hope to derive a convergent strategy at least for u. Indeed, exploiting (LBB), we obtain for the operator $Q_\Lambda \in \mathcal{L}(X, X_\Lambda)$ in Proposition 3.1

$$b(Q_\Lambda u - u_\Lambda, q_\Lambda) = b(Q_\Lambda u, q_\Lambda) - \langle g, q_\Lambda\rangle_{M'\times M} = b(u, q_\Lambda) - \langle g, q_\Lambda\rangle_{M'\times M} = 0 \tag{5.3}$$

for all $q_\Lambda \in M_\Lambda$, where u, u_Λ are the solutions of the continuous and discrete problem (2.5), respectively. Hence, (FEP) implies

$$b(Q_\Lambda u - u_\Lambda, q) = 0 \qquad \text{for all} \quad q \in M. \tag{5.4}$$

Since $Q_\Lambda u - u_\Lambda \in X_\Lambda$, we obtain $a(Q_\Lambda u - u_\Lambda, u - u_\Lambda) = 0$, which, in turn, implies

$$\|Q_\Lambda u - u\|_A^2 = \|Q_\Lambda u - u_\Lambda\|_A^2 + \|u - u_\Lambda\|_A^2. \tag{5.5}$$

This latter equation immediately implies

$$\|u - u_\Lambda\|_A \;\lesssim\; \|u - Q_\Lambda u\|_A \tag{5.6}$$

and since Q_Λ was nothing but the biorthogonal projector on X_Λ, the Riesz basis property ensures that u_Λ converges to u. However, this is also not what we really want to achieve due to two reasons. Firstly, the right hand side of (5.6) contains quantities depending on the unknown solution u. This means, the choice of the index sets Λ depend directly on u, which is not available in numerical calculations. Secondly, (5.6) gives no quantitative estimate which allows to predict the number of iterations an adaptive algorithm has to perform at most to reach a prescribed error tolerance.

Hence, we looked for a new approach that circumvents all the above listed problems and drawbacks. In this section, we introduce an alternative, namely an adaptive version of Uzawa's algorithm. The analysis of this method leads us to the desired result, namely a convergent adaptive refinement strategy for saddle point problems.

5.1 An adaptive Uzawa algorithm

The Uzawa algorithm is a well–known iterative solver for saddle point problems, [2]. We aim at using this algorithm as an outer iteration for an adaptive method. To this end, we consider the Uzawa algorithm for (infinite dimensional) Hilbert spaces. In a second step, we formulate our adaptive version of Uzawa's algorithm. This adaptive version creates some additional errors that need to be controlled over the iteration.

5.1.1 Uzawa algorithm in Hilbert spaces

Originally, the Uzawa algorithm was formulated for saddle point problems involving matrices of finite dimension, [2]. Here, we consider its formulation in infinite dimensional Hilbert spaces X and M. Given any bounded linear operator $R : M' \to M$ (whose role will be discussed later) and $\alpha \in \mathbb{R}$, we consider the following variant of the Uzawa algorithm:

Algorithm 5.1 *Given any $p^{(0)} \in M$, we compute $u^{(i)}$ and $p^{(i)}$ for $i = 1, 2, \ldots$, by*

$$\begin{aligned} Au^{(i)} &= f - B'p^{(i-1)}, &(5.7)\\ p^{(i)} &= p^{(i-1)} + \alpha R B u^{(i)}. &(5.8) \end{aligned}$$

The convergence of this algorithm is well–known if R is the Riesz operator (see [7] and the references therein). However, since for the subsequent error analysis it will be important to keep track of the influence of the data to the error, we need an explicit error estimate here. Hence, we state the following result and include also the proof for completeness and convenience.

Theorem 5.2 *Suppose that RS is selfadjoint and positive definite and $0 < \alpha < 2\|RS\|_{[M]}^{-1}$ (S again being the Schur complement). Then the Algorithm 5.1 converges. To be precise, for $p^{(0)} := 0$ and setting $q := \|Id - \alpha RS\|_{[M]}$, we obtain the following error estimate*

$$\|p - p^{(i)}\|_M \leq \|A^{-1}f\|_X \, \|\alpha RB\|_{[X,M]} \frac{q^i}{1-q}, \qquad q < 1. \tag{5.9}$$

Proof. By induction, it is easy to show that

$$p^{(i)} = (Id - \alpha RS)^i p^{(0)} + \Big(\sum_{k=0}^{i-1} (Id - \alpha RS)^k \Big) \alpha RBA^{-1} f. \tag{5.10}$$

Now, let H be a Hilbert space. Then, for any linear bounded and selfadjoint operator $T : H \to H$, the following equation is well–known [1]

$$\sup\{|\lambda| : \lambda \in \sigma(T)\} = \|T\|_{[H]}, \tag{5.11}$$

where the norm is the operator norm induced by the norm in H and $\sigma(T)$ denotes the spectrum of T. Now, we use the fact that RS is selfadjoint and hence

$$\sup\{|\lambda| : \lambda \in \sigma(\alpha RS)\} = \|\alpha RS\|_{[M]} = \alpha \|RS\|_{[M]} < 2. \tag{5.12}$$

The assumptions on α and on RS imply that $\sigma(Id - \alpha RS) \subset (-1, 1)$, since

$$\sup\{\lambda : \lambda \in \sigma(Id - \alpha RS)\} = 1 - \alpha \inf\{\lambda : \lambda \in \sigma(RS)\} < 1$$

and

$$\inf\{\lambda : \lambda \in \sigma(Id - \alpha RS)\} = 1 - \alpha \sup\{|\lambda| : \lambda \in \sigma(RS)\} = 1 - \alpha \|RS\|_{[M]} > -1.$$

Consequently, we obtain

$$q = \|Id - \alpha RS\|_{[M]} = \sup\{|\lambda| : \lambda \in \sigma(Id - \alpha RS)\} < 1.$$

This finally implies, using $q < 1$ and $p^{(0)} = 0$

$$\begin{aligned} \|p - p^{(i)}\|_M &= \Big\| S^{-1}BA^{-1}f - \sum_{k=0}^{i-1}(Id - \alpha RS)^k \alpha RBA^{-1}f \Big\|_M \\ &\leq \|A^{-1}f\|_X \, \Big\| \sum_{k=i}^{\infty}(Id - \alpha RS)^k \, \alpha RB \Big\|_{[X,M]} \\ &\leq \|A^{-1}f\|_X \, \|\alpha RB\|_{[X,M]} \frac{q^i}{1-q}, \end{aligned}$$

which proves (5.9). □

Let us add some comments on the role of the operator R in (5.8). One natural choice is the Riesz operator. However, we do not want to restrict Algorithm 5.1 to this case only. The reason for this is the fact that by Theorem 3.2 and 3.3 we have explicit conditions at hand to check (LBB) and (FEP). In performing (5.8) for some discretization, one has to guarantee that this discretization fulfills (LBB). Hence, the freedom in the choice of R may also be used to ensure (LBB) and also (FEP).

5.1.2 Adaptive version

Now, in general, we cannot compute $u^{(i)}$ and $p^{(i)}$ in each step *exactly* but only with some approximations. Note that $u^{(i)}$ and $p^{(i)}$ are elements of infinite dimensional spaces. We in fact compute approximations $u^{(i)}_{\Lambda_i}$, $p^{(i)}_{\Lambda_i}$ with respect to finite dimensional subsets $\Lambda_i = (\Lambda_i^X, \Lambda_i^M) \subset \nabla^X \times \nabla^M$. The aim of this subsection is to study the overall error in the Uzawa iteration introduced by this approximation, where Λ_i will be chosen adaptively.

To be precise, we set

$$\tilde{u}^{(i)} := A^{-1}(f - B'p^{(i-1)}_{\Lambda_{i-1}})$$

(which is not computable) and we assume that we approximate $\tilde{u}^{(i)}$ by

$$u^{(i)}_{\Lambda_i} = A^{-1}_{\Lambda_i} Q'_{\Lambda_i}(f - B'p^{(i-1)}_{\Lambda_{i-1}})$$

up to a certain error, i.e.,

$$\epsilon_i := u^{(i)}_{\Lambda_i} - \tilde{u}^{(i)}, \qquad \|\epsilon_i\|_X < q^i \varepsilon_i, \tag{5.13}$$

where we may choose ε_i. Here, Q'_{Λ_i} denotes the adjoint of the projector $Q_{\Lambda_i} : X \to X_{\Lambda_i}$. Now, we can formulate our adaptive Uzawa iteration:

Algorithm 5.3 *Let $\Lambda_0^M = \emptyset$ and $p^{(0)}_{\Lambda_0} = p^{(0)} = 0$. Then, for $i = 1, 2, \ldots,$ and given Λ^M_{i-1}, proceed as follows:*

1. *Determine by an adaptive algorithm a set of indices* Λ_i^X *such that for* $u_{\Lambda_i}^{(i)}$ *determined by*

$$A_{\Lambda_i} u_{\Lambda_i}^{(i)} = Q'_{\Lambda_i}(f - B' p_{\Lambda_{i-1}}^{(i-1)}), \tag{5.14}$$

one has $\|\epsilon_i\|_X < q^i \varepsilon_i$.

2. *Determine an index set* Λ_i^M *such that* $RB(X_{\Lambda_i}) \subseteq M_{\Lambda_i}$ *and such that the LBB condition holds. Then, set*

$$p_{\Lambda_i}^{(i)} = p_{\Lambda_{i-1}}^{(i-1)} + \alpha RB u_{\Lambda_i}^{(i)}. \tag{5.15}$$

Theorem 5.4 *Assume that* ε_i *are chosen such that*

$$\sum_{i=0}^{\infty} \varepsilon_i \leq C < \infty$$

for some constant $C > 0$. *Then, we have*

$$\|p^{(i)} - p_{\Lambda_i}^{(i)}\|_M \leq C \|\alpha RB\|_{[X,M]}\, q^i, \tag{5.16}$$

where $p^{(0)} = p_{\Lambda_0}^{(0)}$ *and* $p^{(i)}$ *is defined by* (5.10).

Proof. As above, it is readily seen that

$$p^{(i)} - p_{\Lambda_i}^{(i)} = (Id - \alpha RS)(p^{(i-1)} - p_{\Lambda_{i-1}}^{(i-1)}) - \alpha RB\epsilon_i. \tag{5.17}$$

By iteration and assuming that $p^{(0)} = p_{\Lambda_0}^{(0)}$, we obtain

$$p_{\Lambda_i}^{(i)} - p^{(i)} = \sum_{k=0}^{i-1} (Id - \alpha RS)^k \alpha RB \epsilon_{i-k}. \tag{5.18}$$

Inserting our assumption on $\|\epsilon_i\|_X$, we conclude that

$$\|p^{(i)} - p_{\Lambda_i}^{(i)}\|_M \;\leq\; \|\alpha RB\|_{[X,M]}\, q^i \sum_{k=0}^{i-1} \varepsilon_k \;\leq\; C \|\alpha RB\|_{[X,M]} q^i,$$

which proves the result. □

Finally we obtain our desired result:

Theorem 5.5 *Under the above assumptions, we obtain the following error estimates for the adaptive Uzawa Algorithm 5.3:*

(a) The Algorithm 5.3 converges, i.e., we have

$$\|p - p_{\Lambda_i}^{(i)}\|_M \;\lesssim\; q^i.$$

(b) *The solution of the saddle point problem can be approximated with any desired accuracy:*

$$\|u - u^{(i+1)}_{\Lambda_{i+1}}\|_X + \|p - p^{(i)}_{\Lambda_i}\|_M \lesssim q^i.$$

Proof. Using the triangle inequality and the Theorems 5.2 and 5.4 gives

$$\begin{aligned}
\|p - p^{(i)}_{\Lambda_i}\|_M &\leq \|p - p^{(i)}\|_M + \|p^{(i)} - p^{(i)}_{\Lambda_i}\|_M \\
&\leq \|A^{-1}f\|_X \, \|\alpha RB\|_{[X,M]} \frac{q^i}{1-q} + C\|\alpha RB\|_{[X,M]} q^i \\
&= q^i \, \|\alpha RB\|_{[X,M]} \Big(\|A^{-1}f\|_X \frac{1}{1-q} + C \Big),
\end{aligned}$$

which proves part (a) of the claim. For proving (b), we use standard arguments to obtain

$$\|u - \tilde{u}^{(i+1)}\|_X \lesssim \|B' p^{(i)}_{\Lambda_i} - B'p\|_{X'} \lesssim q^i,$$

where we have used (a) in the last step. Finally, using triangle inequality and (5.13) yields

$$\|u - u^{(i+1)}_{\Lambda_{i+1}}\|_X \leq \|u - \tilde{u}^{(i+1)}\|_X + \|\tilde{u}^{(i+1)} - u^{(i+1)}_{\Lambda_{i+1}}\|_X \lesssim q^i,$$

which proves the desired result. □

Now, several remarks on the above results are in order:

- As can be seen in (5.14), a convergent adaptive strategy for the positive definite operator A builds the kernel of our method. By assuming that this algorithm reduces the error to $\|\epsilon_i\|_X < q^i \varepsilon_i$, we implicitly assumed the convergence of the inner iteration, i.e., we assume that there exists a strategy to build Λ_i^X which allows this error reduction. As already pointed out, the algorithm in [19] meets this requirement. Also the question arises how large the set Λ_i^X is, i.e., how many degrees of freedom are neccessary to reach the desired accuracy. This is a property of the adaptive strategy used for (5.14) and the possible fill–ins due to (LBB). For example, the method introduced in [14] for positive definite operators was proven to have asymptotically optimal complexity. However, we will not study the complexity of our adaptive Uzawa algorithm here and devote this to a forthcoming paper.

- Clearly, the essential quantity $q = \|Id - \alpha RS\|_{[X,M]} < 1$ determining the speed of convergence will often not be available exactly. One could however estimate q in order to obtain a priori a maximum number of (outer) Uzawa iterations to reach the desired accuracy. Of course, an estimate for q depends on the various data for a particular saddle point problem.

- At a first look one might get the impression that the performance of Algorithm 5.3 depends only on p whereas the choice of the adaptive index sets Λ_i depends only on u. However, the situation is somewhat more involved. Since the behaviour of the right hand side influences the choice of Λ_i, it can be seen by (5.14) that p in fact effects the adaptive refinement. On the other hand, u influences also the Uzawa algorithm since in (5.8) we have to make sure that (LBB) is valid, i.e., we have to determine Λ_i^M for a given Λ_i^X such that $(\Lambda_i^X, \Lambda_i^M)$ gives rise to spaces fulfilling (LBB).

- Finally, we comment on the relationship of the above algorithm to the *inexact Uzawa* algorithm. The latter one has recently been studied e.g. in [6, 28], where an error analysis is given if the elliptic subproblem corresponding to (5.7) is only solved up to some tolerance (in this sense *inexact*). Moreover, the preconditioning of this method is considered there. One might think that our algorithm is simply a variant of such an inexact Uzawa iteration. However, again, we point out that our method works in infinite dimensional Hilbert spaces and the error analysis considers the inexact solution of the continuous elliptic problem in (5.7). To our knowledge, inexact Uzawa iterations are based on finite dimensional spaces.

6 Application No. 1: The Stokes problem

In this section, we detail our general criteria for (LBB) and (FEP) for the mixed formulation of the Stokes problem. Let us first apply Theorem 3.2 and 3.3 to this special case. We will use the particular multiscale discretization introduced in [23] fulfilling the LBB condition. Firstly, we will review this construction and then we apply Theorem 3.2 and 3.3 to obtain concrete conditions for this discretization.

Let us start by reviewing the Stokes problem and its mixed formulation. For simplicity, we assume homogeneous boundary conditions, but the theory is of course not restricted to this special case.

Problem 6.1 *Given a vector field $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)^n$, one has to determine the velocity $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)^n$ and the pressure $p \in L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q(x)\, dx = 0\}$ such that*

$$\begin{array}{rcll} -\Delta \boldsymbol{u} + \nabla p & = & f & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} & = & 0 & \text{in } \Omega, \end{array} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ is the bounded Lipschitz domain of interest.

Thus, its mixed formulation is given by (2.2) for the particular choice

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{0;\Omega} = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)\, dx \\ b(\boldsymbol{v}, q) &:= -(\nabla \cdot \boldsymbol{v}, q)_{0;\Omega} = \sum_{i=1}^{n} \int_\Omega q(x) \frac{\partial v_i}{\partial x_i}(x)\, dx, \end{aligned}$$

for $\boldsymbol{u} = (u_1, \ldots, u_n)^T$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^T$. For simplicity, let us restrict ourselves to the case $\Omega = (0,1)^n$. More general domains may be treated by domain decomposition approaches using the cube as a reference domain, see [9, 10, 11, 17, 25, 27]. Hence, wavelet bases on cubes also serve as reference basis elements in this approach.

6.1 Stable multiscale spaces for the Stokes problem

In this section, we briefly review the construction of biorthogonal wavelet bases in $\boldsymbol{H}(\mathrm{div};\Omega)$ which can be used to generate multiscale bases for velocities and pressures, respectively. For the sake of simplicity, we again only describe this construction for $\Omega = (0,1)^n$ here and refer to [41] for extensions to more general domains.

Derivatives and primitives. The key for the subsequent construction is the following result for wavelet bases on $(0,1)$, which can be found e.g. in [26, 38, 40]. Roughly speaking, it states that certain wavelet systems on the interval are linked by derivatives and primitives. This powerful mechanism is the key ingredient not only for the construction of wavelet bases for the mixed formulation of the Stokes problem, which we will review now, but also for the construction of divergence free wavelets, [37, 39]. To our knowledge, the result has so far been proven for three examples of wavelet systems on $(0,1)$:

(a) Orthonormal wavelets on $(0,1)$, [16]. In this case, all what is said below, holds for $\tilde{\Upsilon} = \Upsilon$.

(b) Systems arising by iteratively applying Theorem 6.2 below to the systems in (a) and the arising results. I.e., these are biorthogonal systems arising from orthonormal ones by differentiation and integration.

(c) Biorthogonal spline wavelets on $(0,1)$, [24, 26].

Theorem 6.2 *Let $\Xi_j^{(1)} := \{\xi_{j,k}^{(1)} : k \in \mathcal{I}_j\}$, $\tilde{\Xi}_j^{(1)} := \{\tilde{\xi}_{j,k}^{(1)} : k \in \mathcal{I}_j\}$ be one of the above listed systems of univariate scaling functions and $\Upsilon_j^{(1)} := \{\eta_{j,k}^{(1)} : k \in \mathcal{J}_j\}$, $\tilde{\Upsilon}_j^{(1)} := \{\tilde{\eta}_{j,k}^{(1)} : k \in \mathcal{J}_j\}$ be the induced biorthogonal wavelet system on $(0,1)$ such that $\tilde{\Xi}_j^{(1)} \subset H_0^1(0,1)$. Then, there exists a second system of dual scaling functions $\Xi_j^{(0)}$, $\tilde{\Xi}_j^{(0)}$ and induced biorthogonal wavelets $\Upsilon_j^{(0)}$, $\tilde{\Upsilon}_j^{(0)}$ (w.r.t. the* same *set of indices $\mathcal{I}_j$, $\mathcal{J}_j$, respectively) such that*

$$\begin{array}{rclcrcl} \frac{d}{dx}\tilde{\Xi}_j^{(1)} &=& D_{j,0}\,\tilde{\Xi}_j^{(0)}, & & \frac{d}{dx}\Xi_j^{(0)} &=& -D_{j,0}^T\,\Xi_j^{(1)}, \\ \frac{d}{dx}\tilde{\Upsilon}_j^{(1)} &=& D_{j,1}\,\tilde{\Upsilon}_j^{(0)}, & & \frac{d}{dx}\Upsilon_j^{(0)} &=& -D_{j,1}^T\,\Upsilon_j^{(1)}, \end{array} \tag{6.2}$$

where $D_{j,e} \in GL(|\mathcal{J}_{j,e}| \times |\mathcal{J}_{j,e}|)$, $e = 0,1$, are sparse, regular matrices and

$$\mathcal{J}_{j,e} := \begin{cases} \mathcal{I}_j, & e = 0, \\ \mathcal{J}_j & e = 1. \end{cases} \qquad \square$$

Here and in the sequel, we use the short hand notation $\frac{d}{dx}\Xi_j^{(0)} := (\frac{d}{dx}\xi_{j,k}^{(0)})_{k\in\mathcal{I}_j}$ and similar for all other systems of functions. It will be necessary to detail the first equation in (6.2), namely

$$\frac{d}{dx}\tilde{\vartheta}_\lambda^{(1)} = \frac{d}{dx}\tilde{\vartheta}_{j,e,k}^{(1)} = \sum_{k'\in\nabla(\lambda)} d_{k,k'}^{j,e}\,\tilde{\vartheta}_{j,e,k'}^{(0)}, \qquad k \in \mathcal{J}_{j,e},\ e = 0,1, \tag{6.3}$$

with some (small) set of indices $\nabla(\lambda) \subset \mathcal{J}_{j,e}$ and

$$\tilde{\vartheta}_{j,e,k} := \begin{cases} \tilde{\xi}_{j,k}, & e = 0, \\ \tilde{\eta}_{j,k} & e = 1. \end{cases}$$

Systems on the cube. The next step is to use tensor products to construct biorthogonal wavelet systems on the unit cube $(0,1)^n$. The aim is to use the systems described above in an appropriate way so that a formula similar to (6.2) holds for the partial derivatives. Hence, we use the systems induced by $\Xi^{(1)}$, $\tilde{\Xi}^{(1)}$ as well as $\Xi^{(0)}$, $\tilde{\Xi}^{(0)}$ within a tensor product framework. For $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)^T \in \{0,1\}^n =: E^n$, we define

$$\psi_\lambda^{(\boldsymbol{\gamma})}(x) := \psi_{(j,e,k)}^{(\boldsymbol{\gamma})}(x) := \prod_{\nu=1}^{n} \vartheta_{j,e_\nu,k_\nu}^{\gamma_\nu}(x_\nu), \qquad \vartheta_{j,e_\nu,k_\nu}^{\gamma_\nu} := \begin{cases} \eta_{j,k_\nu}^{(1)}, & \gamma_\nu = 1, \quad e_\nu = 1, \\ \eta_{j,k_\nu}^{(0)}, & \gamma_\nu = 0, \quad e_\nu = 1, \\ \xi_{j,k_\nu}^{(1)}, & \gamma_\nu = 1, \quad e_\nu = 0, \\ \xi_{j,k_\nu}^{(0)}, & \gamma_\nu = 0, \quad e_\nu = 0, \end{cases} \tag{6.4}$$

and $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, $\lambda_\nu = (j, e_\nu, k_\nu)$. All systems of functions $\Psi_j^{(\boldsymbol{\gamma})}$ as well as its duals are defined in a straightforward manner. The corresponding set of indices is given by

$$\nabla := \bigcup_{j \geq j_0 - 1} \nabla_j, \tag{6.5}$$

where

$$\nabla_{j_0-1} := \{\lambda = (j_0, 0, k) : \ k \in \mathcal{I}_{j_0}^n\},$$

and for $j \geq j_0$, we set

$$\nabla_j := \{\lambda = (\lambda_1, \ldots, \lambda_n)^T, \lambda_\nu = (j, e_\nu, k_\nu) : \ e = (e_1, \ldots, e_n)^T \in E^n \setminus \{0\}, k_\nu \in \mathcal{J}_{j,e_\nu}\}.$$

Now, Theorem 6.2 implies:

Corollary 6.3 *For the above defined systems of wavelets, we obtain*

$$\frac{\partial}{\partial x_i} \Psi^{(\boldsymbol{\gamma})} = D^{(i)} \Psi^{(\boldsymbol{\gamma} - \boldsymbol{\delta}_i)}, \qquad \frac{\partial}{\partial x_i} \tilde{\Psi}^{(\boldsymbol{\gamma} - \boldsymbol{\delta}_i)} = -(D^{(i)})^T \tilde{\Psi}^{(\boldsymbol{\gamma})}, \tag{6.6}$$

where $\boldsymbol{\delta}_i := (\delta_{1,i}, \ldots, \delta_{n,i})^T$, $1 \leq i \leq n$ *denotes the canonical unit vector in* $\mathbb{R}^n$. *The matrices* $D^{(i)}$ *can be obtained by* $D_{j,0}$ *and* $D_{j,1}$ *in* (6.2) *in a straightforward way.* □

In view of (6.3), we can express (6.6) as

$$\begin{aligned} \frac{\partial}{\partial x_i} \tilde{\psi}_\lambda^{(\boldsymbol{\gamma})} &= \frac{\partial}{\partial x_i} \Big(\bigotimes_{\nu=1}^{n} \tilde{\vartheta}_{\lambda_\nu}^{(\gamma_\nu)} \Big) = \Big(\bigotimes_{\nu \neq i} \tilde{\vartheta}_{\lambda_\nu}^{(\gamma_\nu)} \Big) \Big(\sum_{k_i' \in \nabla(\lambda_i)} d_{k_i,k_i'}^{j,e_i} \, \tilde{\vartheta}_{j,e_i,k_i'}^{(\gamma_i - 1)} \Big) \\ &= \sum_{k_i' \in \nabla(\lambda_i)} d_{k_i,k_i'}^{j,e_i} \, \tilde{\psi}_{(j,e,k+\boldsymbol{\delta}_i(k_i'-k_i))}^{(\boldsymbol{\gamma}-\boldsymbol{\delta}_i)}. \end{aligned} \tag{6.7}$$

Vector fields. For the space $\boldsymbol{L}^2(\Omega)$ of square integrable vector fields, we denote wavelet systems by boldface characters, i.e., by $\boldsymbol{\Psi}$ (and similar for all other vector valued function spaces). Moreover, we have to equip the index $\lambda \in \nabla$ labeling the scalar wavelets with some additional index indicating the component of the vector field. For example, let $\Psi^{[\nu]} := \{\psi_\lambda^{[\nu]} : \lambda \in \nabla^{[\nu]}\}$, $\tilde{\Psi}^{[\nu]} := \{\tilde{\psi}_\lambda^{[\nu]} : \lambda \in \nabla^{[\nu]}\}$, $1 \le \nu \le n$, be (possibly different) biorthogonal systems in $L^2(\Omega)$. Then, the vector fields

$$\boldsymbol{\psi}_{(i,\lambda)} := \psi_\lambda^{[i]}\,\boldsymbol{\delta}_i, \qquad \tilde{\boldsymbol{\psi}}_{(i,\lambda)} := \tilde{\psi}_\lambda^{[i]}\,\boldsymbol{\delta}_i, \qquad \lambda \in \nabla^{[i]},\ 1 \le i \le n,$$

obviously form a biorthogonal wavelet basis for $\boldsymbol{L}^2(\Omega)$. Denoting by

$$\boldsymbol{\nabla} := \bigcup_{i=1}^{n} \bigcup_{\lambda \in \nabla^{[i]}} (i,\lambda), \qquad \boldsymbol{\lambda} := (i,\lambda),$$

the corresponding set of indices, we have

$$\boldsymbol{\Psi} = \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}, \qquad \tilde{\boldsymbol{\Psi}} = \{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}.$$

Now, in view of Theorem 6.2, setting $\boldsymbol{\nabla} := \{(i,\lambda) : i = 1,\dots,n;\ \lambda \in \nabla\}$ with ∇ defined by (6.5) and using the above mentioned systems adapted to differentiation and integration, we define

$$\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{div}} := \psi_\lambda^{(\boldsymbol{\delta}_i)}\,\boldsymbol{\delta}_i, \quad \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\mathrm{div}} := \tilde{\psi}_\lambda^{(\boldsymbol{\delta}_i)}\,\boldsymbol{\delta}_i, \qquad \boldsymbol{\lambda} \in \boldsymbol{\nabla}. \tag{6.8}$$

It was proven that the wavelet systems $\boldsymbol{\Psi}^{\mathrm{div}}$, $\tilde{\boldsymbol{\Psi}}^{\mathrm{div}}$ defined by (6.8) in a straightforward way, form a biorthogonal basis for $\boldsymbol{H}(\mathrm{div};\Omega)$, [23, 40, 41].

6.2 Mixed wavelet discretizations

Now, we want to use the above described wavelet bases to obtain a mixed wavelet discretization of the Stokes problem. In [23] it was shown that the 'full' spaces

$$M_j := S(\Psi_{\Delta_j}^{(\mathbf{0})}), \qquad \text{and} \qquad X_j := S(\tilde{\boldsymbol{\Psi}}_{\boldsymbol{\Delta}_j}^{\mathrm{div}}) \tag{6.9}$$

indeed satisfy the LBB condition, where

$$\Delta_j := \{\lambda \in \nabla : |\lambda| \le j\}, \quad \boldsymbol{\Delta}_j := \{\boldsymbol{\lambda} \in \boldsymbol{\nabla} : |\boldsymbol{\lambda}| \le j\},$$

and $|\boldsymbol{\lambda}| := j$ for $\boldsymbol{\lambda} = (i,\lambda)$, $\lambda = (j,k)$. Moreover, in order to use $\tilde{\boldsymbol{\Psi}}^{\mathrm{div}}$ and $\Psi^{(\mathbf{0})}$ to discretize (6.1), these functions do not only have to give rise to spaces that fulfill (LBB). They also have to fulfill appropriate boundary conditions. Since $X = \boldsymbol{H}_0^1(\Omega)$, the velocities need to have vanishing traces at $\Gamma := \partial\Omega$, which leads to the demand that the univariate systems $\tilde{\Xi}^{(\mathbf{0})}$ and $\tilde{\Xi}^{(\mathbf{1})}$ need to fulfill homogeneous Dirichlet boundary conditions at $x = 0, 1$. On the other hand, the pressures are functions in $M = L_0^2(\Omega)$, so that no boundary conditions have to be prescribed. Again, this implies that $\Xi^{(\mathbf{0})}$ has to span *all* of $L^2(\Omega)$. However, it was shown in [26] how to modify the construction of biorthogonal wavelets on the interval from [24] in order to fulfill these *complementary* boundary conditions. Finally, the pressure is forced to have a vanishing integral in a postprocessing.

Norm equivalences. Let us finally recall the norm equivalences for the above trial bases. It was shown in [23] that (2.22) and (2.23) are indeed satisfied for at least $t = 1$ and $s = 0$.

6.3 Adaptivity and the LBB condition

So far, the spaces in (6.9) are defined w.r.t. a full level, i.e., they are *not* adaptively chosen. As already mentioned, the validity of (LBB) for these spaces was shown in [23]. In [4], this result was extended to the adaptive case resulting in some condition on the set of indices. However, the condition in [4] is still somewhat implicit since one has to check if certain inner products vanish. Here, we give an explicit condition on the corresponding set of indices.

Corollary 6.4 *Let $M_\Lambda \subset L_0^2(\Omega)$ be given in terms of some set of indices $\Lambda^M \subset \nabla^M = \nabla$. Then, the LBB condition is satisfied provided that*

$$\Lambda^X \supseteq \mathcal{B}'(\Lambda^M) := \{\boldsymbol{\lambda} = (i,\lambda) \in \boldsymbol{\nabla},\ \lambda = (j,e,k) : \quad \begin{array}{l} \exists\, \mu = (j,e,k') \in \Lambda^M : \\ (1) k_i' \in \Delta(\lambda_i) \\ (2) k_{i'}' = k_{i'},\ \textit{for all } i' \neq i\}. \end{array} \tag{6.10}$$

Proof. We have to check the condition (a) in Theorem 3.2. Now, for $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$ and $\mu \in \nabla$, we have

$$\begin{aligned} b(\tilde{\boldsymbol{\psi}}^{\mathrm{div}}_{\boldsymbol{\lambda}}, \psi^{(\mathbf{0})}_\mu) &= (\nabla \cdot \tilde{\boldsymbol{\psi}}^{\mathrm{div}}_{\boldsymbol{\lambda}}, \psi^{(\mathbf{0})}_\mu)_{0;\Omega} = \left(\frac{\partial}{\partial x_i}\tilde{\psi}^{(\boldsymbol{\delta}_i)}_\lambda, \psi^{(\mathbf{0})}_\mu\right)_{0;\Omega} \\ &= \sum_{k_i' \in \Delta(\lambda_i)} d^{j,e_i}_{k_i,k_i'} (\tilde{\psi}^{(\mathbf{0})}_{(j,e,k+\boldsymbol{\delta}_i(k_i'-k_i))}, \psi^{(\mathbf{0})}_\mu)_{0;\Omega} \\ &= d^{j,e_i}_{k_i,k_i'} \prod_{i' \neq i} \delta_{k_{i'},k_{i'}'}, \end{aligned}$$

if $k_i' \in \Delta(\lambda_i)$ and $b(\tilde{\boldsymbol{\psi}}^{\mathrm{div}}_{\boldsymbol{\lambda}}, \psi^{(\mathbf{0})}_\mu) = 0$ else. Consequently, we obtain $b(\boldsymbol{\psi}^{\mathrm{div}}_{\boldsymbol{\lambda}}, \psi^{(\mathbf{0})}_\mu) = 0$ for all $\mu \in \Lambda^M$ if $\boldsymbol{\lambda} \in \boldsymbol{\nabla} \setminus \mathcal{B}'(\Lambda^M)$. This, in particular, implies that for $\mu \in \Lambda^M$ one has $\psi^{(\mathbf{0})}_\mu \in (X \ominus X_\Lambda)^{\perp_b}$, which, in view of condition (a) in Theorem 3.2 proves our assertion. □

6.4 Adaptivity and the equilibrium condition

Applying our general criteria for (FEP), we obtain

Corollary 6.5 *The discretization induced by $(\Lambda^X, \Lambda^M) \subset \nabla^X \times \nabla^M = \boldsymbol{\nabla} \times \nabla$ fulfills (FEP) provided that*

$$\Lambda^M \supseteq \mathcal{B}(\Lambda^X) := \bigcup_{(j,e,k) \in \Lambda^X} \bigcup_{k_i' \in \Delta(\lambda_i)} (j,e,k+\boldsymbol{\delta}_i(k_i'-k_i)). \tag{6.11}$$

Proof. We have already seen that

$$\nabla \cdot \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{\mathrm{div}} = \sum_{k_i' \in \Delta(\lambda_i)} d_{k_i,k_i'}^{j,e_i} \, \tilde{\psi}^{(\mathbf{0})}_{(j,e,k+\boldsymbol{\delta}_i(k_i'-k_i))}$$

holds for any $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$. Hence, (6.11) implies $B(X_\Lambda) \subseteq \tilde{M}_\Lambda$, which, by Theorem 3.3 proves the assertion. □

Finally, we combine the above results and obtain

Corollary 6.6 *If $\Lambda^M = \mathcal{B}(\Lambda^X)$, then both (LBB) and (FEP) are vaild.*

Proof. Obviously, we only have to check (LBB). Now, given any $\boldsymbol{\lambda} = (i, \lambda) \in \mathcal{B}(\Lambda^M)$, $\lambda = (j, e, k)$. Then, by assumption, we have that $\mu := (j, e, k+\boldsymbol{\delta}_i(k_i'-k_i)) \in \Lambda^M = \mathcal{B}'(\Lambda^M)$ for some $k_i' \in \Delta(\lambda_i)$. Hence, by definition of $\mathcal{B}(\Lambda^X)$, we have that $\boldsymbol{\lambda} \in \Lambda^X$ which proves (LBB). □

We see that the two conditions (LBB) and (FEP) are in fact contrary in nature. While the condition for (LBB) determines Λ^X for a given Λ^M by the condition $\Lambda^X \supseteq \mathcal{B}'(\Lambda^M)$, the condition for (FEP) acts in the opposite way. In fact, given Λ^X, one can determine Λ^M by $\mathcal{B}(\Lambda^X) \subseteq \Lambda^M$.

7 Application No. 2: Appending boundary conditions by Lagrange multipliers

As a second example, we consider the inhomogeneous Dirichlet problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary $\Gamma := \partial\Omega$. In particular, Γ is assumed to be Lipschitzian and hence there exists a continuous trace operator $\gamma_0 : u \mapsto \gamma_0 u = u|_\Gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$, $s \in (1/2, 1]$, with a continuous right inverse γ_0^-. By $(\cdot,\cdot)$ we denote the dual pairing between $H^s(\Omega)$ and $H^{-s}(\Omega) := (H^s(\Omega))'$, $s \geq 0$, such that $(u, v) = \int_\Omega u(x)v(x)\,dx$ for smooth functions u and v. Analogously, $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, such that $\langle p, q\rangle = \int_\Gamma p(x)\,q(x)\,d\sigma_x$ for functions p and q in $L^2(\Gamma)$. Within this setting, we consider the following:

Problem 7.1 *Given two functions $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$, determine $u \in H^1(\Omega)$ such that*

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

holds.

The 'standard' weak formulation of this boundary value problem is formulated in affine subspaces of $H^1(\Omega)$ related to the boundary condition posed by the function g. In certain applications it turns out that it is advantageous to consider the weak mixed formulation arising from appending the boundary conditions by Lagrange multipliers. This has also been studied in the wavelet context for example in [22, 34, 36].

A mixed formulation for the inhomogeneous Dirichlet problem reads as follows:

Problem 7.2 *For given* $[f,g] \in H^{-1}(\Omega) \times H^{1/2}(\Gamma)$ *find functions* $[u,p] \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *such that*

$$\begin{array}{rcll} a(u,v)+b(v,p) & = & (f,v), & v \in H^1(\Omega), \\ b(u,q) & = & -\langle q,g\rangle, & q \in H^{-1/2}(\Gamma), \end{array} \tag{7.1}$$

where $a(u,v) := (\nabla u, \nabla v)_{0;\Omega} + (u,v)_{0;\Omega}$ *and* $b(v,q) := -\langle q, \gamma_0 v\rangle$ *for* $u,v \in H^1(\Omega)$, $q \in H^{-1/2}(\Gamma)$.

It is well–known that appending boundary conditions by Lagrange multipliers as in the above mixed formulation gives rise to a uniquely solvable problem which fits to our abstract setting introduced in Section 2.1. In particular, one has $X = H^1(\Omega)$, $M = H^{-1/2}(\Gamma)$, $H_X := L^2(\Omega)$, $H_M := L^2(\Gamma)$ and $V := H^1_0(\Omega)$.

For the sake of simplicity, we restrict ourselves to domains $\Omega \subset \mathbb{R}^2$. However, everything what will be stated in the sequel easily generalizes to higher dimensions. Firstly, we describe a discretization of Problem 7.2 for $\hat\Omega := (0,1)^2$. Secondly, we consider distorted domains $\Omega \subset \mathbb{R}^2$ which are isomorphic to $\hat\Omega$. Finally, we use these domains in a domain decomposition context to derive analogous results for more general domains.

7.1 Wavelet discretization on $\hat\Omega = (0,1)^2$

Let us assume that $\hat\Omega$ and the four patches of the boundary $\hat\Gamma_i$, $i = 1,\dots,4$, are oriented as shown in the left part of Figure 1 below. Moreover, we choose the following parametrizations $\hat\rho_i : [0,1] \to \hat\Gamma_i$ by

$$\hat\rho_1(t) := (t,1), \quad \hat\rho_2(t) := (0,t), \quad \hat\rho_3(t) := (t,0), \quad \hat\rho_4(t) := (1,t), \qquad t \in [0,1].$$

On a first view, the orientation of the latter mappings might seem a little curious. Its usefulness will become clear later (see (7.6) below).

The simplest wavelet discretization for $X = H^1(\hat\Omega)$ probably consists of tensor products of univariate wavelet and scaling functions on $[0,1]$. Examples of such wavelet bases can be found in the literature, e.g., in [16, 24, 26, 33], and we will not go into the technical details here. Another possibility would be to choose the so called *hyperbolic bases*, which are advantageous w.r.t. nonlinear approximation of functions with anisotropic smoothness, [35].

In any case, it remains to construct a suitable wavelet basis on $\hat\Gamma$. During the past years, several constructions of wavelets on general domains and manifolds have been introduced [9, 10, 11, 17, 25, 27]. In this section, we will follow the approach in [9, 10] to obtain a suitable wavelet basis on $\hat\Gamma$. Let us briefly review the main ingredients.

One starts by any biorthogonal wavelet basis on $[0,1]$. The next step is to modify this basis such that in addition it has the property that on a single level only *one* scaling function and *one* wavelet does *not* vanish at $x=0$ and $x=1$, respectively. Let us denote by

$$\Xi^I_j := \{\xi_{j,k} : k \in \mathcal{I}^I_j\}, \qquad \Upsilon^I_j := \{\eta_{j,k} : k \in \mathcal{J}^I_j\}$$

those scaling functions and wavelets that vanish at the end points, i.e.,

$$\xi_{j,k}(0) = \xi_{j,k}(1) = 0,\ k \in \mathcal{I}_j^I, \qquad \eta_{j,k}(0) = \eta_{j,k}(1) = 0,\ k \in \mathcal{J}_j^I,$$

while $\mathcal{I}_j^L$, $\mathcal{I}_j^R$, $\mathcal{J}_j^L$ and $\mathcal{J}_j^R$ denote the indices of those scaling functions and wavelets which do not vanish at the left and right end point, respectively. The 'inner' functions are simply mapped to $\hat{\Gamma}_i$, i.e.,

$$\vartheta_{j,k}^{(i)}(\hat{\rho}_i(t)) := \begin{cases} \xi_{j,k}(t), & \text{if } k \in \mathcal{I}_j^I, \\ \eta_{j,k}(t), & \text{if } k \in \mathcal{J}_j^I. \end{cases} \tag{7.2}$$

A similar definition is given for the dual functions. For those functions that do not vanish on the boundary, a matching is performed. To be precise, the functions $\xi_{j,k}$, $\tilde{\xi}_{j,k}$, $k \in \mathcal{I}_j^L, \mathcal{I}_j^R$ as well as $\eta_{j,k}$, $\tilde{\eta}_{j,k}$, $k \in \mathcal{J}_j^L, \mathcal{J}_j^R$ are mapped to $\hat{\Gamma}_i$ in the same way as (7.2). Then, these mapped functions are matched by building suitable linear combinations (note that only two functions per patch $\hat{\Gamma}_i$ enter these linear combination, i.e., four such functions per corner of $\hat{\Omega}$). The coefficients are chosen in such a way that the matched functions are continuous and biorthogonal. The corresponding matching coefficients can e.g. be found in [10]. The resulting functions will be labeled by the set $\mathcal{I}^c$ for the four scaling functions corresponding to the four corners and by $\mathcal{J}_j^L$ and $\mathcal{J}_j^R$ for the matched wavelets. We set $\mathcal{J}_{j_0-1} := \mathcal{I}_{j_0}^I \cup \mathcal{I}^c$ and $\mathcal{J}_j = \mathcal{J}_j^L \cup \mathcal{J}_j^I \cup \mathcal{J}_j^R$, $j \geq j_0$. Then, we end up with a wavelet basis

$$\Theta := \{\vartheta_{\hat{\mu}} : \hat{\mu} \in \hat{\nabla}^\Gamma\}, \qquad \tilde{\Theta} := \{\tilde{\vartheta}_{\hat{\mu}} : \hat{\mu} \in \hat{\nabla}^\Gamma\},$$

where $\hat{\nabla}^\Gamma := \hat{\nabla}_I^\Gamma \cup \hat{\nabla}_C^\Gamma$ and

$$\hat{\nabla}_I^\Gamma := \{\hat{\mu} := (i,j,k) : i = 1, \ldots, 4,\ k \in \mathcal{I}_{j_0}^I \text{ or } k \in \mathcal{J}_j^I \text{ for } j \geq j_0, \text{ resp.}\}$$

denotes those functions vanishing at the four corners and

$$\hat{\nabla}_C^\Gamma := \mathcal{I}^c \cup \mathcal{J}^L \cup \mathcal{J}^R, \qquad \mathcal{J}^K := \bigcup_{j \geq j_0} \mathcal{J}_j^K, \quad K \in \{L, R\},$$

indicate the matched fucntions around the corners. Finally, we introduce the notation

$$\#\hat{\mu} := \{i \in \{1, \ldots, 4\} : \vartheta_{\hat{\mu}|\hat{\Gamma}_i} \not\equiv 0\}$$

indicating the set of the particular patches $\hat{\Gamma}_i$ the corresponding function $\vartheta_{\hat{\mu}}$ is defined on. The wavelet bases Θ, $\tilde{\Theta}$ indeed fulfill the norm equivalences (2.23) for $s = 1/2$, [9].

Now, it remains to choose a wavelet basis for $H^1(\hat{\Omega})$ which fits to Θ, $\tilde{\Theta}$ on $\hat{\Gamma}$ in the sense of (LBB) and (FEP) . For $e = (e_1, e_2) \in \{0,1\}^2 \setminus \{0\}$ and $\mathcal{J}_{j,e_i} := \begin{cases} \mathcal{I}_j, & \text{if } e_i = 0, \\ \mathcal{J}_j, & \text{if } e_i = 1, \end{cases}$ we set

$$\hat{\nabla}^\Omega := \{\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2), \hat{\lambda}_i = (j, e_i, k_i) : \ k_i \in \mathcal{J}_{j,e_i}, i = 1, 2\},$$

as well as

$$\hat{\psi}_{\hat{\lambda}}(\hat{x}) := \theta_{\hat{\lambda}_1}(\hat{x}_1)\, \theta_{\hat{\lambda}_2}(\hat{x}_2), \quad \hat{\lambda}_i = (j, e_i, k_i), \qquad \theta_{\hat{\lambda}_i} := \begin{cases} \xi_{j,k_i}, & \text{if } e_i = 0, \\ \eta_{j,k_i}, & \text{if } e_1 = 1. \end{cases}$$

Finally, for $\hat{\Lambda}^X \subset \hat{\nabla}^\Omega$ and $\hat{\Lambda}^M \subset \hat{\nabla}^\Gamma$, we define

$$\hat{X}_\Lambda := S(\hat{\Psi}_{\hat{\Lambda}^X}), \qquad \hat{M}_\Lambda := S(\tilde{\Theta}_{\hat{\Lambda}^M}).$$

With these definitions at hand, we obtain for $\hat{\lambda} \in \hat{\nabla}^\Omega$

$$\gamma_0 \hat{\psi}_{\hat{\lambda}}(\hat{x}) = \begin{cases} \theta_{\hat{\lambda}_1}(\hat{x}_1)\, \theta_{\hat{\lambda}_2}(1), & \text{if } \hat{x} \in \hat{\Gamma}_1, \\ \theta_{\hat{\lambda}_1}(0)\, \theta_{\hat{\lambda}_2}(\hat{x}_2), & \text{if } \hat{x} \in \hat{\Gamma}_2, \\ \theta_{\hat{\lambda}_1}(\hat{x}_1)\, \theta_{\hat{\lambda}_2}(0), & \text{if } \hat{x} \in \hat{\Gamma}_3, \\ \theta_{\hat{\lambda}_1}(1)\, \theta_{\hat{\lambda}_2}(\hat{x}_2), & \text{if } \hat{x} \in \hat{\Gamma}_4. \end{cases} \tag{7.3}$$

The LBB condition on $\hat{\Omega}$. With all the above preparations, it can easily be seen that the 'full' spaces induced by the set of indices $\hat{\nabla}^\Omega_j := \{\hat{\lambda} \in \hat{\nabla}^\Omega : |\hat{\lambda}| \le j\}$ and $\hat{\nabla}^\Gamma_j := \{\hat{\mu} \in \hat{\nabla}^\Gamma : |\hat{\mu}| \le j\}$ (with obvious definitions of $|\hat{\lambda}|$ and $|\hat{\mu}|$) fulfill (LBB). For the adaptive case, we obtain

Corollary 7.3 *The spaces $\hat{X}_\Lambda$ and $\hat{M}_\Lambda$ fulfill (LBB), if*

$$\hat{\Lambda}^X \supseteq \mathcal{B}'(\hat{\Lambda}^M),$$

where

$$\mathcal{B}'(\hat{\Lambda}^M) := \{\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2) \in \hat{\nabla}^\Omega : \exists \nu = 1,2 : \hat{\lambda}_\nu \in \hat{\Lambda}^M \text{ and } \hat{\psi}_{\hat{\lambda}|\hat{\Gamma}_{\#\hat{\lambda}_\nu}} \neq 0\}.$$

Proof. In view of (7.3), we have for $\hat{\lambda} \in \hat{\nabla}^\Omega$ and $\hat{\mu} \in \hat{\nabla}^\Gamma_I$, $\{i\} = \#\hat{\mu}$

$$b(\hat{\psi}_{\hat{\lambda}}, \tilde{\vartheta}_{\hat{\mu}}) = c(\hat{\lambda}, i) \int_0^1 \theta_{\hat{\lambda}_i}(t)\, \tilde{\theta}_{\hat{\mu}}(t)\, dt, \tag{7.4}$$

where

$$c(\hat{\lambda}, i) := \begin{cases} \theta_{\hat{\lambda}_2}(1), & \text{if } i = 1, \\ \theta_{\hat{\lambda}_1}(0), & \text{if } i = 2, \\ \theta_{\hat{\lambda}_2}(0), & \text{if } i = 3, \\ \theta_{\hat{\lambda}_1}(1), & \text{if } i = 4. \end{cases}$$

Obviously, (7.4) vanishes for all $\hat{\mu} \in \hat{\nabla}^\Gamma_I$ provided that $\hat{\lambda} \in \hat{\nabla}^\Omega \setminus \mathcal{B}'(\hat{\Lambda}^M)$. For $\hat{\mu} \in \hat{\nabla}^\Gamma_C$, we obtain

$$b(\hat{\psi}_{\hat{\lambda}}, \tilde{\vartheta}_{\hat{\mu}}) = \sum_{i \in \#\hat{\mu}} c(\hat{\lambda}, i) \int_0^1 \theta_{\hat{\lambda}_i}(t)\, \tilde{\theta}_{\hat{\mu}}(t)\, dt, \tag{7.5}$$

which also vanishes provided that $\hat{\lambda} \notin \mathcal{B}'(\hat{\Lambda}^M)$. This proves the claim. □

Full equilibrium on $\hat{\Omega}$. In a similar manner, we obtain

Corollary 7.4 *If $\hat{\Lambda}^X$ and $\hat{\Lambda}^M$ fulfill $\mathcal{B}(\hat{\Lambda}^X) \subseteq \hat{\Lambda}^M$, then the generated spaces are equilibrated, where*

$$\mathcal{B}(\hat{\Lambda}^X) := \bigcup_{\substack{\hat{\lambda}\in\hat{\Lambda}^X \\ \gamma_0\psi_{\hat{\lambda}}\neq 0}} \bigcup_{\nu=1,2} \{\hat{\mu} \in \hat{\nabla}^\Gamma : \hat{\lambda}_\nu = \hat{\mu}\}.$$

Proof. The equations (7.4) and (7.5) in the proof of the above Corollary 7.3 and the biorthogonality on $[0, 1]$ show that the assumption indeed implies $B(\hat{X}_\Lambda) \subseteq \hat{\tilde{M}}_\Lambda$, which proves the claim. Note that the trace of $\hat{\psi}_{\hat{\lambda}}$ is continuous at the corners, so that $\tilde{\Theta}$ in fact is a dual basis to $B\hat{\Psi}$. □

Finally, putting everything together leads to the following result:

Corollary 7.5 *The assumption $\mathcal{B}(\hat{\Lambda}^X) = \hat{\Lambda}^M$ implies (LBB) and (FEP) .*

Proof. We only have to prove (LBB). To this end, let $\hat{\lambda} \in \mathcal{B}'(\hat{\Lambda}^M)$. Then, $\hat{\lambda}_\nu \in \hat{\Lambda}^M = \mathcal{B}(\hat{\Lambda}^X)$ for some $\nu = 1, 2$. Since $\mathcal{B}(\hat{\Lambda}^X)$ consists of the union over all $\hat{\lambda} \in \hat{\Lambda}^X$, we obtain $\hat{\lambda} \in \hat{\Lambda}^X$ which proves (LBB). □

7.2 Distorted domains $\Omega \subset \mathbb{R}^2$

Now, we consider domains $\Omega \subset \mathbb{R}^2$, that are the parametric image of the reference domain $\hat{\Omega}$, i.e., there exists a function $G \in C^1(\overline{\hat{\Omega}})$ such that $\overline{\Omega} = G(\overline{\hat{\Omega}})$ and $|JG(\hat{x})| > 0$ for $\hat{x} \in \overline{\hat{\Omega}}$. In particular, the parametric mapping $G : \overline{\hat{\Omega}} \to \overline{\Omega}$ is constructed with the aid of a method introduced by Gordon and Hall, [31, 32] using transfinite interpolation, see also [8]. Given any parametric mappings $\pi_i : \hat{\Gamma}_i \to \Gamma_i$, $i = 1, \ldots, 4$ (recall their orientation as indicated in Figure 1), the mapping is given by

$$\begin{aligned} G(\hat{x}_1, \hat{x}_2) \;=\; & \hat{x}_2\, \pi_1(\hat{x}_1) + (1 - \hat{x}_2)\, \pi_3(\hat{x}_1) \\ & + (1 - \hat{x}_1) \Big[\pi_2(\hat{x}_2) - \hat{x}_2\, \pi_2(1) - (1 - \hat{x}_2)\, \pi_2(0)\Big] \\ & + \hat{x}_1 \Big[\pi_4(\hat{x}_2) - \hat{x}_2 \pi_4(1) - (1 - \hat{x}_2)\, \pi_4(0)\Big]. \end{aligned} \tag{7.6}$$

An analogous 3d–formulation can be found in [31, 32]. The advantage of this approach for our example is obvious, namely that the mappings of the pieces of the boundary enter in a natural and easy way into the mapping of the domain. Hence, we can restrict ourselves to the consideration of the image of Γ.

Let us now assume that $\hat{\Gamma}_i$ is parametrized by $\hat{\gamma}_i : [(i - 1), i] \to \hat{\Gamma}_i$ in a straightforward manner so that $\hat{\Gamma}$ is parametrized over $I := [0, 4]$ with some canonical mapping $\hat{\gamma} : I \to \hat{\Gamma}$ satisfying $|\hat{\gamma}'(t)| = 1$ (piecewise) for all $t \in I$. Then, we obtain for any integrable function $\hat{g}$ on $\hat{\Gamma}$

$$\int_{\hat{\Gamma}} \hat{g}(\hat{x})\, d\sigma_{\hat{x}} = \int_I \hat{g}(\hat{\gamma}(t))\, dt. \tag{7.7}$$

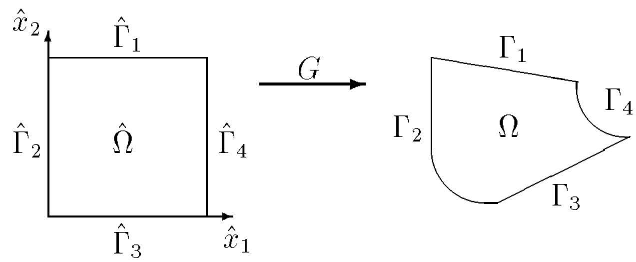

Figure 1: Mapping of the rectangle $\hat{\Omega} = (0,1)^2$ into a quadrilateral Ω with curved boundaries.

Adapting the bilinear form. First, we follow the ideas in [9, 10, 11, 17, 25], that (for this example) may be sketched as follows: one builds biorthogonal wavelet systems $\hat{\Psi}$, $\tilde{\hat{\Psi}}$ on the reference cube $\hat{\Omega}$ as above and then simply defines systems Ψ, $\tilde{\Psi}$ on Ω by mapping:

$$\psi(x) := \hat{\psi}(G^{-1}(x)),\ x \in \Omega, \qquad \hat{\psi} \in \hat{\Psi}.$$

Defining the dual system $\tilde{\Psi}$ in the same way gives rise to a system Ψ, $\tilde{\Psi}$ on Ω which is biorthogonal w.r.t. a modified inner product, namely $[u,v]_\Omega := \int_\Omega u(x)\, v(x)\, |JG^{-1}(x)|\, dx$. It can be shown that the norm induced by this latter inner product is in fact equivalent to the usual $L^2(\Omega)$–norm.

However, this has a drawback for the example treated in this section. Since our conditions for checking (LBB) as well as (FEP) are based on biorthogonality w.r.t. the usual $L^2(\Omega)$–inner product $(\cdot,\cdot)$, we cannot directly apply our conditions. Let us make this a little bit more precise. Using similar arguments as for obtaining (7.7), we conclude for any integrable function g on Γ with $\gamma := G \circ \hat{\gamma}$ that

$$\int_\Gamma g(x)\, d\sigma_x = \int_I g(\gamma(t)) |\gamma'(t)|\, dt.$$

Setting as above $\hat{g}(\hat{x}) := g(G(\hat{x}))$ and taking into account that $g(\gamma(t)) = \hat{g}(G^{-1}(\gamma(t))) = \hat{g}(\hat{\gamma}(t))$ leads to

$$\int_\Gamma g(x)\, d\sigma_x \quad = \quad \int_{\hat{\Gamma}} \hat{g}(\hat{x})\, |JG(\hat{x})|\, d\sigma_{\hat{x}}.$$

Hence, biorthogonality on $\hat{\Gamma}$ implies biorthogonality on Γ only for linear mappings G which represent of course only a very limited number of domains Ω, namely parallelepipeds.

However, this problem can be solved by adapting the bilinear form $b(\cdot,\cdot)$ as follows: To begin with, we replace $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle_\Gamma$ defined by

$$\langle p, q\rangle_\Gamma := \int_\Gamma p(x)\, q(x)\, |\gamma'(\gamma^{-1}(x))|^{-1}\, d\sigma_x \qquad (7.8)$$

for piecewise smooth functions p and q on Γ. Analogously we introduce $b_\Omega(\cdot,\cdot) : H^{-1/2}(\Gamma)\times H^1(\Omega) \to \mathbb{R}$ by

$$b_\Omega(v,q) := -\int_\Gamma q(x)\,(\gamma_0 v)(x)|\gamma'(\gamma^{-1}(x))|^{-1}\,d\sigma_x \tag{7.9}$$

for piecewise smooth functions q on Γ and v on Ω. Formally, we could introduce $\langle\cdot,\cdot\rangle_{\hat\Gamma}$ and $b_{\hat\Omega}(\cdot,\cdot)$ using the definitions (7.8) and (7.9), respectively. However, since $|\hat\gamma'(t)| = 1$, these forms coincide with the original ones.

Now, we note that the biorthogonality relations with respect to $b(\cdot,\cdot)$ on $\hat\Omega$ imply those on Ω with respect to $b_\Omega(\cdot,\cdot)$, since

$$\begin{aligned} b_\Omega(v,q) &= -\int_\Gamma q(x)\,v(x)|\gamma'(\gamma^{-1}(x))|^{-1}d\sigma_x \\ &= -\int_{\hat\Gamma} q(G(\hat x))\,v(G(\hat x))\,|\gamma'(\gamma^{-1}(G(\hat x)))|^{-1}\,|\gamma'(\gamma^{-1}(G(\hat x)))|\,d\sigma_{\hat x} \\ &= -\langle \hat q,\gamma_0\hat v\rangle_{\hat\Gamma} = -\langle \hat q,\gamma_0\hat v\rangle = b(\hat v,\hat q) \end{aligned} \tag{7.10}$$

for $v\in H^1(\Omega)$ and $q\in H^{-1/2}(\Gamma)$

Next, we introduce an adapted mixed formulation with respect to $\langle\cdot,\cdot\rangle_\Gamma$ and $b_\Omega(\cdot,\cdot)$

Problem 7.6 *For given $[f,g]\in H^{-1}(\Omega)\times H^{1/2}(\Gamma)$ find functions $[u,p]\in H^1(\Omega)\times H^{-1/2}(\Gamma)$ such that*

$$\begin{aligned} a(u,v)+b_\Omega(v,p) &= (f,v), & v\in H^1(\Omega),\\ b_\Omega(u,q) &= -\langle q,g\rangle_\Gamma, & q\in H^{-1/2}(\Gamma). \end{aligned} \tag{7.11}$$

Clearly, both mixed problems are equivalent in the sense that they provide the same solution $u\in H^1(\Omega)$. But one should notice that the interpretation of the Lagrange multiplier is changed, since the bilinear form $b(\cdot,\cdot)$ is changed to $b_\Omega(\cdot,\cdot)$. To be precise, it is well–known that for smooth data f, the Langrange multiplier p in Problem 7.2 can be interpreted as the normal derivative of the solution u on the boundary Γ. On the other hand, in Problem 7.6 the situation is as follows: for $f\in L^2(\Omega)$ one obtains for the solution $[u,p]\in H^1(\Omega)\times H^{-1/2}(\Gamma)$ the identity

$$-(\Delta u,\zeta)+(u,\zeta)=(f,\zeta),\quad \zeta\in C_0^\infty(\Omega),$$

i.e., in particular $\Delta u = -f+u\in L^2(\Omega)$. Thus, we have for $v\in H^1(\Omega)$ by (7.11)

$$\langle p,\gamma_0 v\rangle_\Gamma = -b_\Omega(v,p) = (\nabla u,\nabla v)+(u,v)-(f,v) = (\Delta u,v)+(\nabla u,\nabla v),$$

which implies by the Green formula

$$p = |\gamma'(\gamma^{-1}(\cdot))|^{-1}\frac{\partial u}{\partial n}\in H^{-1/2}(\Gamma),$$

i.e., one has to multiply the normal derivative of the solution u by the factor $|\gamma'(\gamma^{-1}(\cdot))|$. Now we have to check whether the inf–sup condition still holds with respect to $b_\Omega(\cdot,\cdot)$.

Theorem 7.7 *Under the above assumptions, there exists a constant $\beta > 0$ such that*

$$\inf_{q\in H^{-1/2}(\Gamma)} \sup_{v\in H^1(\Omega)} \frac{b_\Omega(v,q)}{\|q\|_{H^{-1/2}(\Gamma)}\|v\|_{H^1(\Omega)}} \geq \beta. \tag{7.12}$$

Proof. Let $v \in H^1(\Omega)$ be the variational solution of the Neumann problem

$$\begin{aligned} -\Delta v + v &= 0, && \text{in } \Omega, \\ \partial_n v &= q \cdot |\gamma'(\gamma^{-1}(\cdot))|, && \text{on } \Gamma. \end{aligned} \tag{7.13}$$

Then, one has by definition

$$b_\Omega(v,q) = -\langle q, \gamma_0 v\rangle_\Gamma = \|v\|^2_{H^1(\Omega)} \tag{7.14}$$

as well as

$$\|v\|_{H^1(\Omega)} \geq \|q \cdot |\gamma'(\gamma^{-1}(\cdot))|\|_{H^{-1/2}(\Gamma)}.$$

On the other hand, one obtains by

$$\begin{aligned} \|q\|_{H^{-1/2}(\Gamma)} &= \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle q,\varphi\rangle_\Omega}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\ &\leq \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\|q(\cdot)\cdot|\gamma'(\gamma^{-1}(\cdot))|\|_{H^{-1/2}(\Gamma)}\,\|\varphi\cdot|\gamma'(\gamma^{-1}(\cdot))|^{-1}\|_{H^{1/2}(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\ &\lesssim \|q(\cdot)\cdot|\gamma'(\gamma^{-1}(\cdot))|\|_{H^{-1/2}(\Gamma)}, \end{aligned}$$

the estimate

$$\|v\|_{H^1(\Omega)} \gtrsim \|q\|_{H^{-1/2}(\Gamma)}. \tag{7.15}$$

The identity (7.14) and the estimate (7.15) imply the inf–sup condition (7.12). □

Now, in view of (7.10), all results in the Corollaries 7.3, 7.4 and 7.5 remain valid for the Problem 7.6. Finally, the norm equivalences in Ω and on Γ allow to use the a posteriori error estimates to construct an adaptive wavelet strategy.

7.3 More general domains

Let us briefly indicate some generalizations of the above presented results to more general domains Ω.

7.3.1 Domain decomposition

The results in the latter two sections can easily be used to obtain pairs of wavelet spaces that fulfill (LBB) and (FEP) also for more general domains Ω. In fact, one may use one of the constructions of wavelets on domains and manifolds in [9, 10, 17, 25] featuring domain decomposition approaches. The basic idea is very similar to the above construction of wavelets on $\hat{\Gamma}$, namely mapping and matching. For (LBB) and (FEP), we only have to consider functions on Ω that have a non–trivial trace on $\Gamma = \partial\Omega$. As long as this trace vanishes at the corners of $\bar{\Omega}$, the same reasoning as in the previous section applies. Accross the interelement boundaries a matching is performed so that the resulting functions (and hence also their traces) are globally continuous. Since both (LBB) and (FEP) only reflect subsets where $b(\cdot,\cdot)$ vanishes, the discussion indeed reduces to the single subdomains, which have been considered in the previous section. We will not formulate the corresponding results in detail here, since this would force us to introduce some additional technicalities whereas the above guidelines should be sufficiently clear.

7.3.2 Biorthogonality on Ω

Recently, a new approach for constructing wavelets on domains and manifolds has been introduced, [27]. This method differs from those in [9, 10, 17, 25] that have already been discussed above. The advantage is that the wavelet bases in [27] are constructed such that they are biorthogonal w.r.t. the L^2–inner product on the domain Ω. Hence, one may consider (LBB) and (FEP) directly w.r.t. $b(\cdot,\cdot)$ without modifying this bilinear form.

However, the relationship between X_Λ and its trace space is not so easy as in the above presented case. This connection turns out to be more complicated. Since this would go beyond the scope of the present paper this subject will be treated elsewhere.

7.3.3 Imbedding strategies

Another approach to deal with wavelet methods on complex domains is to imbed the domain Ω into a larger but simple domain $\Box \supset \Omega$ (such as a cube). Then, one uses a wavelet basis on $\Box$ and is left to find appropriate bases on Γ in order to fulfill (LBB) and (FEP). Uniform wavelet spaces in $H^{1/2}(\Gamma)$ and $H^1(\Box)$ have e.g. been studied in [22] resulting in general criteria for (LBB) for the 'full' spaces. Now, using tensor product bases on $\Box$, the spaces on Γ and the trace space of $H^1(\Box)$ may *not* be related in such a nice way as described above. As a consequence, our above criteria do not apply directly. Hence, the question arises if the (technical) construction of wavelet bases on Ω pays compared with the (less technical) construction of such a basis on $\Box$. It was shown above that we obtain *explicit* criteria (in terms of single indices) for (LBB) and (FEP) for adaptively refined wavelet spaces. This is of course very helpful for implementations and in our opinion justifies the above presented approach even though the preprocessing might be more involved.

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