MULTILEVEL CHARACTERIZATIONS OF FUNCTION SPACES ON SKELETONS

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ABSTRACT

Function spaces on skeletons arise in the numerical treatment of elliptic boundary value problems by certain domain decomposition methods. In this note, we discuss such function spaces, which can be interpreted as trace spaces. We construct a well–defined trace operator and present a characterization of the trace spaces by means of multilevel expansions.

Key Words: Elliptic boundary value problems, domain decomposition, skeleton, d-set, characterizations of smoothness spaces, wavelets.

1. INTRODUCTION

Much effort is spent to design efficient numerical schemes to treat boundary value problems in the weak formulation

find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)_{L_2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega), \tag{1}$$

where $a(\cdot,\cdot)$ is a bilinear form stemming from a second order elliptic differential operator and $\Omega \subset \mathbb{R}^n$ is a bounded polyhedral domain with boundary $\partial\Omega$. In addition to the standard notation for Sobolev spaces and their norms, $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of all $\mathcal{C}^\infty$ functions vanishing on $\partial\Omega$ or, equivalently, the set of all $v \in H^1(\Omega)$ for which also $\chi_\Omega v \in H^1(\mathbb{R}^n)$. Here χ_G is the characteristic function on G.

Many domain decomposition approaches aim at solving (1) on nonoverlapping subdomains Ω^k into which Ω is decomposed,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}^k, \qquad \Omega^k \cap \Omega^m = \emptyset,\ k \neq m. \tag{2}$$

An important case is when one wants to use grids with possibly different grid sizes on the different subdomains suggested by physical or geometrical reasons without adaptation of the grids at the interface boundaries. A very general class of domain decomposition methods with such *non-matching grids* is provided by the *Three-Field-Formulation* [1, 3]. It stems from appending the boundary conditions on the boundary of each subdomain, $\partial\Omega^k$, by Lagrange multipliers, and then requiring in addition smoothness of these multipliers in a weak sense. In this context, a particular role is played by a Sobolev space of positive order defined on the *skeleton*

$$\Sigma = \overline{\left(\bigcup_{k=1}^{K} \partial\Omega^k\right) \setminus \partial\Omega} \tag{3}$$

of the decomposition (2). In fact, the space used in [1, 3] is

$$X(\Sigma) := H_0^1(\Omega)|_\Sigma \tag{4}$$

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which is defined as $\{t \in L_2(\Sigma) :$ there exists some $v \in H_0^1(\Omega)$ such that $t = v|_\Sigma\}$ with norm

$$\|t\|_{X(\Sigma)} := \inf_{v \in H_0^1(\Omega):\ v|_\Sigma = t} \|v\|_{H^1(\Omega)}. \tag{5}$$

At this point it is not clear yet that the 'trace operator' $|_\Sigma$ is indeed well-defined since Σ is *not* the boundary of a Lipschitz domain. Recall that only for $n < 2$, by the Sobolev embedding theorem, all functions in $H^1(\Omega)$ are guaranteed to be continuous. In any case, whether one wants to use finite elements or wavelets in the discretization scheme, in order to prove results for preconditioning or stability, or error estimates, the question arises how $|_\Sigma$ could be defined and how to characterize $X(\Sigma)$ for such an Σ. Recall that for Lipschitz domains, one always has the classical trace operator

$$Tr:\ H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$$

which coincides with point evaluations on $\partial\Omega$ for continuous functions, see [19] for details. Correspondingly, one may view $X(\Sigma)$ as above, namely, by taking the usual trace with respect to Σ on a dense subset of $H_0^1(\Omega)$ and then consider its closure with respect to a certain norm, and denote this by $H^{1/2}(\Sigma)$. Or one may define it directly on Σ only by means of certain differences of function values in the sense of [14], see Section 2 below. Moreover, motivated by the domain decomposition approach, we want to investigate whether such a space can be characterized by weighted sequence norms induced by multilevel expansions as it is known for general Besov spaces on Lipschitz domains or manifolds, see e.g. [7, 8, 9, 13].

Thus, the purpose of this paper is the following:

(i) to construct a well–defined trace operator $Tr : H^1(\Omega) \longrightarrow X(\Sigma)$ into some suitable smoothness space $X(\Sigma)$;

(ii) to describe $X(\Sigma)$ and its norm(s) and provide equivalent definitions;

(iii) to characterize $X(\Sigma)$ in terms of weighted sequence norms of coefficients with respect to a stable multiscale basis on Σ.

In the following, we will assume that Ω is an n-dimensional *cube* decomposed into smaller n-dimensional cubes Ω^k, see Figure 1. In this case, the skeleton only contains faces which are parallel to the coordinate axes. For more general domains, see Remark 3.5 below.

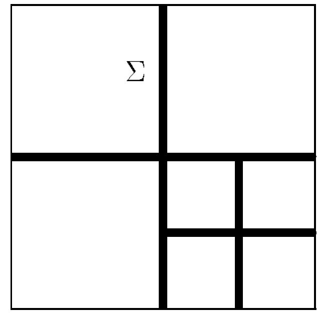

Figure 1: Example of a domain $\Omega \subset \mathbb{R}^2$ with its skeleton Σ

This note is organized as follows. In Section 2 we apply some results from [14] to the situation at hand. First we see that Σ is a d-set for which, following [14], Besov spaces can be defined. If we then identify the space $H^1(\Omega)$ with the Besov space $B_2^1(L_2(\Omega))$, then there exists a suitable trace operator, and the trace space $X(\Sigma)$ can be described as a specific Besov space on Σ. In Section 3 we derive a multilevel characterization of $X(\Sigma)$ by using techniques similar to those in [15]. This note cloeses with Section 4 with remarks how wavelet bases on Σ can be constructed.

2. TRACE SPACES ON d–SETS

In this section, we briefly recall some results from [14] to apply them to problem (i). Let F be a closed non–empty subset of $\mathbb{R}^n$ and d a real number satisfying $0 < d \le n$. We denote by $B(x,\rho)$ the closed ball with center x and radius ρ. A positive Borel measure μ with support F is called *d–measure on F* if

$$\mu(B(x,\rho)) \sim \rho^d, \quad \text{for } x \in F,\ 0 < \rho \le 1. \tag{6}$$

In this paper, '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, '$a \lesssim b$' indicates inequality up to constant factors.

Then a closed and non–empty subset of $\mathbb{R}^n$ is called a *d–set* if there exists a d–measure on F. It is possible to define Besov spaces on d–sets. Let $0<\alpha<1$ and $1\le p\le\infty$. A function f belongs to the *Besov space* $B_p^\alpha(L_p(F))$ if and only if it has finite norm

$$\begin{aligned}\|f\|_{B_p^\alpha(L_p(F))} &= \|f\|_{L_p(d\mu)}\\ &+ \left(\int\int_{|x-y|<1}\frac{|f(x)-f(y)|^p}{|x-y|^{d+\alpha p}}d\mu(x)d\mu(y)\right)^{1/p}.\end{aligned}\tag{7}$$

There exist trace theorems with respect to Besov spaces on d–sets. The following version is proved in [14].

Theorem 2.1 *Let F be a d–set, $0<d\le n$, $0<\beta=\alpha-(n-d)/p<1$, and $1\le p\le\infty$. Then one has*

$$B_p^\alpha(L_p(I\!R^n))\Big|_F = B_p^\beta(L_p(F)),$$

i.e., there exists a linear operator

$$Tr_F : B_p^\alpha(L_p(I\!R^n)) \longrightarrow B_p^\beta(L_p(F))$$

such that

$$\|Tr_F(f)\|_{B_p^\beta(L_p(F))} \lesssim \|f\|_{B_p^\alpha(L_p(I\!R^n))}\tag{8}$$

for all $f\in B_p^\alpha(L_p(I\!R^n))$. Conversely, for any $h\in B_p^\beta(L_p(F))$, there exists some $f\in B_p^\alpha(L_p(I\!R^n))$ such that $h=Tr_F(f)$ and

$$\|f\|_{B_p^\alpha(L_p(I\!R^n))} \lesssim \|h\|_{B_p^\beta(L_p(F))}.\tag{9}$$

Using the previous theorem and the fact that $B_2^1(L_2(\Omega)) = H^1(\Omega)$ with equivalent norms [19], we can establish the following result.

Theorem 2.2 *There exists a well–defined continuous trace operator*

$$\mathcal{T}:\ H^1(\Omega)\longrightarrow B_2^{1/2}(L_2(\Sigma)) =: H^{1/2}(\Sigma)\tag{10}$$

such that

$$\mathcal{T}(H^1(\Omega)) = H^{1/2}(\Sigma).\tag{11}$$

Proof: First we observe that the set Σ is in fact a d–set. In a measurable sense, Σ may be identified with a set Γ which is the union of half–open intervals or cubes, respectively. We then define the Borel measure μ as the induced Lebesgue measure on the set Γ. Let r denote the minimal side length of all cubes under consideration. Without loss of generality, we may assume that $r>2$. Then the relations

$$\begin{aligned}\rho &\le \mu(B(x),\rho) \le 4\rho, && \text{if } n=2,\ 0<\rho\le 1,\\ \tfrac{\pi}{4}\rho^2 &\le \mu(B(x),\rho) \le 4\pi\rho^2, && \text{if } n=3,\ 0<\rho\le 1,\end{aligned}\tag{12}$$

(and correspondingly for $n>3$) can be easily verified so that Σ is an $(n-1)$–set. On Ω, we have the identity $H^1(\Omega) = B_2^1(L_2(\Omega))$. Moreover, since Ω is clearly minimally smooth in the sense of Stein [18], there exists a continuous Whitney extension operator $E:\ H^1(\Omega)\longrightarrow H^1(I\!R^n)$. By applying Theorem 2.1 with $p=2$, $\alpha=1$, $d=n-1$ and $F=\Sigma$, there exists a linear operator $Tr_\Sigma : H^1(I\!R^n)\to H^{1/2}(\Sigma)$. Defining

$$\mathcal{T} := Tr_\Sigma\circ E,\tag{13}$$

$\mathcal{T}$ is a well–defined linear operator into $H^{1/2}(\Sigma)$. Moreover, $\mathcal{T}$ is continuous,

$$\begin{aligned}\|\mathcal{T}(f)\|_{H^{1/2}(\Sigma)} &= \|(Tr_\Sigma\circ E)(f)\|_{H^{1/2}(\Sigma)}\\ &\le \|Tr_\Sigma\|\|E\|\|f\|_{H^1(\Omega)}.\end{aligned}$$

It remains to check that $\mathcal{T}$ is onto and that a converse estimate corresponding to (9) holds. We have to show that

$$Tr_\Sigma(f) = Tr_\Sigma(E(\chi_\Omega f))\tag{14}$$

holds for all $f\in H^1(I\!R^n)$. For then, writing $\tilde f = \chi_\Omega f$, $\mathcal{T}$ is onto and we obtain by (9) for any $h\in H^{1/2}(\Sigma)$ such that $h=Tr_\Sigma f$ where $f\in H^1(I\!R^n)$

$$\begin{aligned}\|\tilde f\|_{H^1(\Omega)} &= \|\chi_\Omega f\|_{H^1(\Omega)} \le \|f\|_{H^1(I\!R^n)}\\ &\lesssim \|Tr_\Sigma(f)\|_{H^{1/2}(\Sigma)}\\ &\lesssim \|Tr_\Sigma(E(\tilde f))\|_{H^{1/2}(\Sigma)}\\ &= \|\mathcal{T}(\tilde f)\|_{H^{1/2}(\Sigma)}.\end{aligned}\tag{15}$$

By using Sobolev embeddings and the fact that the Whitney extension operator E extends simultaneously all orders of differentiability, it is easy to check that (14) holds for all $f\in\mathcal{C}_0^\infty(I\!R^n)$. Then the result follows by density. $\diamond$

Remark 2.3 *Since, by Theorem 2.2, $\mathcal{T}$ is now well–defined, one immediately has equivalence of the two norms defined on Σ,*

$$\|h\|_{X(\Sigma)} \sim \|h\|_{H^{1/2}(\Sigma)} \qquad \textit{for any } h\in H^{1/2}(\Sigma).\tag{16}$$

3. MULTILEVEL CHARACTERIZATION OF $H^{1/2}(\Sigma)$

3.1. Multiresolution for $H_0^1(\Omega)$

We now apply similar techniques as in [15] to derive a multilevel characterization for $H^{1/2}(\Sigma)$.

First note that we can obtain from a biorthogonal multiresolution for $H^1(I\!R)$ [5] a multiresolution for $H^1([0,1])$ by applying the results from [10] for this particular case. Clearly, one can get from this by tensor product standard techniques multiresolution spaces and (biorthogonal) wavelets for $H^1(\Omega)$. Furthermore, using the results in [12], it is known how to construct corresponding wavelets with homogeneous boundary conditions. Thus, in the following, we can assume that we have a multiresolution for $H_0^1(\Omega)$ with a corresponding (biorthogonal) generator basis with the following properties:

(P1) multiresolution for $H_0^1(\Omega)$, starting from a small fixed level $j_0 \in I\!N$:

$$\begin{array}{lll} S_{j_0} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots \subset & H_0^1(\Omega), \\ \tilde{S}_{j_0} \subset \ldots \subset \tilde{S}_j \subset \tilde{S}_{j+1} \subset \ldots \subset & L_2(\Omega), \end{array} \tag{17}$$

such that

$$\mathrm{clos}_{H^1}(\cup_{j\geq j_0} S_j) = H_0^1(\Omega) \tag{18}$$

and

$$\mathrm{clos}_{L_2}(\cup_{j\geq j_0} \tilde{S}_j) = L_2(\Omega); \tag{19}$$

(P2) generator bases for S_j, $\tilde{S}_j$:

$$S_j = \mathrm{span}\ \Phi_j, \quad \tilde{S}_j = \mathrm{span}\ \tilde{\Phi}_j, \tag{20}$$

where for some finite index set Δ_j, $\#\Delta_j \sim 2^{jn}$,

$$\Phi_j = \{\phi_{j,k} : k \in \Delta_j\}, \tilde{\Phi}_j = \{\tilde{\phi}_{j,k} : k \in \Delta_j\}, \tag{21}$$

consisting of functions with local support,

$$\mathrm{diam}(\mathrm{supp}\,\phi_{j,k}) \sim \mathrm{diam}(\mathrm{supp}\,\tilde{\phi}_{j,k}) \sim 2^{-j}, \tag{22}$$

which are L_2-stable;

(P3) projectors $Q_j : L_2(\Omega) \longrightarrow S_j$ defined as

$$\begin{aligned} Q_j v &= (v, \tilde{\Phi}_j)\Phi_j \\ &:= \sum_{k\in\Delta_j} (v, \tilde{\phi}_{j,k})_{L_2(\Omega)} \phi_{j,k}, \end{aligned} \tag{23}$$

which are uniformly bounded on $L_2(\Omega)$;

(P4) a Jackson estimate for S_j:

$$\inf_{v_j\in S_j} \|v - v_j\|_{L_2(\Omega)} \lesssim 2^{-j}\|v\|_{H^1(\Omega)} \tag{24}$$

for any $v \in H^1(\Omega)$;

(P5) a Bernstein inequality for S_j:

$$\|v_j\|_{H^1(\Omega)} \lesssim 2^j \|v_j\|_{L_2(\Omega)} \tag{25}$$

for any $v_j \in S_j$.

It was shown in [7, 9] how the following result follows from (P3)–(P5).

Theorem 3.1 *Let $\{S_j\}_{j\geq j_0}$ be a multiresolution analysis for $H_0^1(\Omega)$ having properties (P1)–(P5). Then for any $f \in H_0^1(\Omega)$ the norm equivalence*

$$\|f\|^2_{H^1(\Omega)} \sim \sum_{j\geq j_0} 2^{2j} \|(Q_j - Q_{j-1})f\|^2_{L_2(\Omega)} \tag{26}$$

holds.

3.2. Norm Equivalences for $H^{1/2}(\Sigma)$

Let the discretization spaces on the skeleton Σ be defined by

$$T_j := S_j|_\Sigma \ \subset H^{1/2}(\Sigma) \tag{27}$$

such that for any $h_j \in T_j$ the *localization property*

$$\inf_{f_j\in S_j,\ f_j|_\Sigma = h_j} \|f_j\|_{L_2(\Omega)} \sim 2^{-j/2}\,\|h_j\|_{L_2(\Sigma)} \tag{28}$$

holds. Using the Bernstein and Jackson inequalities on Ω, we can now prove their counterparts on Σ.

Lemma 3.2 *For any $h_j \in T_j$, we have the Bernstein estimate*

$$\|h_j\|_{H^{1/2}(\Sigma)} \lesssim 2^{j/2} \|h_j\|_{L_2(\Sigma)}. \tag{29}$$

Proof: By definition of the trace spaces (27), combined with (5) and (16), we have for any $h_j \in T_j$

$$\|h_j\|_{H^{1/2}(\Sigma)} \lesssim \inf_{f_j\in S_j:\ f_j|_\Sigma = h_j} \|f_j\|_{H^1(\Omega)}.$$

Applying the Bernstein estimate on Ω, (25), and the localization property (28) yields

$$\begin{aligned} \|h_j\|_{H^{1/2}(\Sigma)} &\lesssim 2^j \inf_{f_j\in S_j:\ f_j|_\Sigma = h_j} \|f_j\|_{L_2(\Omega)} \\ &\lesssim 2^{j/2}\|h_j\|_{L_2(\Sigma)}. \end{aligned}$$

$\diamond$

Lemma 3.3 *On Σ, the Jackson estimate*

$$\inf_{h_j \in T_j} \|h - h_j\|_{L_2(\Sigma)} \lesssim 2^{-j/2} \|h\|_{H^{1/2}(\Sigma)} \qquad (30)$$

holds for any $h \in H^{1/2}(\Sigma)$.

Proof: By the Trace Theorem 2.2, there exists for any $h \in H^{1/2}(\Sigma)$ some $f \in H^1(\Omega)$ satisfying (15) such that $h = f|_\Sigma$ and thus

$$\begin{aligned}\inf_{h_j \in T_j} & \|h - h_j\|_{L_2(\Sigma)} \\ &\leq \|h - (Q_j f)|_\Sigma\|_{L_2(\Sigma)} \\ &= \|(f - Q_j f)|_\Sigma\|_{L_2(\Sigma)} \\ &= \|\sum_{m \geq j+1} ((Q_m - Q_{m-1})f)|_\Sigma\|_{L_2(\Sigma)} \\ &\leq \sum_{m \geq j+1} \|((Q_m - Q_{m-1})f)|_\Sigma\|_{L_2(\Sigma)}.\end{aligned}$$

Applying now the localization property and using Cauchy-Schwartz' inequality gives

$$\begin{aligned}\inf_{h_j \in T_j} & \|h - h_j\|_{L_2(\Sigma)} \\ &\lesssim \sum_{m \geq j+1} 2^{m/2} \|(Q_m - Q_{m-1})f\|_{L_2(\Omega)} \\ &= \sum_{m \geq j+1} 2^{m/2} 2^{-m}\, 2^m \\ &\qquad \cdot \|(Q_m - Q_{m-1})f\|_{L_2(\Omega)} \\ &\lesssim \left(\sum_{m \geq j+1} 2^{-m} \right)^{-1/2} \\ &\qquad \cdot \left(\sum_{m \geq j+1} 2^{2m} \|(Q_m - Q_{m-1})f\|^2_{L_2(\Omega)} \right)^{1/2}.\end{aligned}$$

The first term on the right hand side is equal to $2^{-j/2}$, and the second term can be estimated using the norm equivalence (26) such that we obtain

$$\inf_{h_j \in T_j} \|h - h_j\|_{L_2(\Sigma)} \lesssim 2^{-j/2} \|f\|_{H^1(\Omega)}.$$

Applying again the trace estimate (15) yields the assertion. $\diamond$

Let now

$$P_j : H^{1/2}(\Sigma) \longrightarrow T_j \qquad (31)$$

be uniformly bounded projectors onto T_j. Combining this with the Jackson and Bernstein inequalities on Σ, we obtain as in [7] the following result.

Theorem 3.4 *For any $h \in H^{1/2}(\Sigma)$, we have the norm equivalence*

$$\|h\|^2_{H^{1/2}(\Sigma)} \sim \sum_{j \geq j_0} 2^j \|(P_j - P_{j-1})h\|^2_{L_2(\Sigma)}. \qquad (32)$$

Of course, the techniques used here apply also to the case where Σ contains the boundary of $\partial\Omega$, see Figure 2, or part of it, since the crucial cross points are already present in Σ.

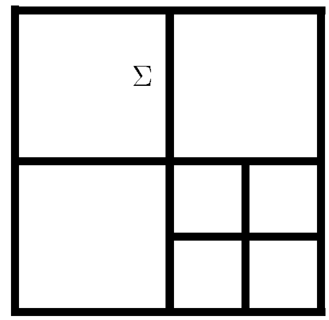

Figure 2: 2-dimensional example where Σ also contains $\partial\Omega$

Remark 3.5 *The techniques used here apply to more general domains and skeletons as long as the corresponding spaces satisfy the localness property (28). Note also that the generalization for other smoothness spaces $H^s(\Sigma)$ is obvious for $s > 0$. However, for the case $s < 0$, this is not so apparent since one cannot immediately apply duality techniques.*

The results derived here could have also been obtained by using techniques from [17].

4. WAVELET BASES ON Σ

Recall that the results in the previous section have been derived under the assumptions (27) and (28), i.e., that approximation spaces on the skeleton are traces of the ones on Ω. In order to carry out such a construction, one could apply the ideas from [2] or [6]. There, for the boundary $\partial\Omega$ of a Lipschitz domain Ω, approximation spaces satisfying (27) and (28) with uniformly bounded projectors (31) are constructed. It seems that this can be generalized to Σ. From this, wavelet bases for the complement spaces $(P_j - P_{j-1})S_j$ could be constructed which then yields from (32) a norm equivalence in terms of the wavelet coefficients of $h \in H^{1/2}(\Sigma)$.

However, the approach for the construction of wavelets in [11] (see also [4]) seems to be simpler to realize. Starting from generator and wavelet bases on an $(n-1)$-dimensional unit cube, one can define composite bases on Σ which is a union of parametric images of the unit cube by gluing their end points together. This construction would be directly defined on Σ only, without assuming it to be embedded in a domain.

In order to be able to apply the construction from [11], we first need approximation spaces on Σ which are traces of approximation spaces on Ω according to (27). We briefly indicate a possible construction for the simple example of Ω being the unit square which is subdivided into four equal squares of side length 1/2. Let Φ_j be generator bases for S_j defined on Ω consisting of tensor products of the hat function. Clearly, their restrictions to Σ will be again piecewise linear functions which are continuous at any crosspoints of Σ, see Figure 3.

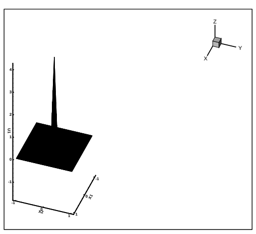

Figure 3: text

It has been shown (see [17]) that (28) holds for this case. Now the construction from [11] can be applied to obtain wavelets on Σ. A detailed description will be given elsewhere.

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