

# Optimal Approximation of Elliptic Problems II: Wavelet Methods

Stephan Dahlke, Erich Novak, Winfried Sickel  
Philipps–Universität Marburg, Friedrich–Schiller–Universität Jena

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# 1. Introduction

- optimal approximation of the solution to

$$\mathcal{A}(u) = f$$

$$\mathcal{A} : H \longrightarrow G, \quad H = H^t \text{ Sobolev space,} \quad G = H^{-t} \text{ dual space}$$

# 1. Introduction

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$$\mathcal{A}(u) = f$$

$$\mathcal{A} : H \longrightarrow G, \quad H = H^t \text{ Sobolev space,} \quad G = H^{-t} \text{ dual space}$$

- elliptic

$$\|\mathcal{A}(u)\|_{H^{-t}} \sim \|u\|_{H^t}, \quad u \in H^t$$

example:

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d \quad \text{Lipschitz}$$

$$u = 0 \quad \text{on } \partial\Omega$$

$$\mathcal{A} = \Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

## Questions:

- How to measure optimality?
- What happens in the special case of elliptic PDEs?
- Do there exist optimal bases/methods?

## 2. Basic Concepts

$$\mathcal{A}(u) = f, \quad \mathcal{A}: H \rightarrow G, \quad H, G \text{ Hilbert spaces}$$

Riesz basis  $\mathcal{B} = \{h_i \mid i \in \mathcal{N}\}$ :

$$A \left( \sum_k |\alpha_k|^2 \right)^{1/2} \leq \left\| \sum_k \alpha_k h_k \right\|_H \leq B \left( \sum_k |\alpha_k|^2 \right)^{1/2}, \quad \mathcal{B}_C := \left\{ \mathcal{B} : B/A \leq C \right\}$$

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- **nonlinear** mappings  $S_n$

$$\mathcal{N}_n := \left\{ S_N : F \rightarrow H, S_n(f) = u_n = \sum_{k=1}^n c_k h_{i_k} \right\}$$

$$e(S_n, F, H) := \sup_{\|f\|_F \leq 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H$$

$$e_{n,C}^{\text{non}}(S, F, H) = \inf_{\mathcal{B} \in \mathcal{B}_C} \inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, F, H), \quad S := \mathcal{A}^{-1}$$

error of **best  $n$ -term approximation**

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- **linear** mapping

$$\mathcal{L}_n := \left\{ S_n : F \rightarrow H, S_n(f) = \sum_{i=1}^n L_i(f) \cdot \tilde{h}_i \right\}$$

$$e_n^{\text{lin}}(S, F, H) = \inf_{S_n \in \mathcal{L}_n} e(S_n, F, H) = \inf_{S_n \in \mathcal{L}_n} \sup_{\|f\|_F \leq 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H$$

- continuous mapping  $S_n$

$$S_n = \phi_n \circ N_n, \quad N_n : F \rightarrow \mathbb{R}^n, \quad \phi_n : \mathbb{R}^n \rightarrow H \quad \text{continuous}$$

$$e_n^{\text{cont}}(S, F, H) = \inf_{S_n \in \mathcal{C}_n} e(S_n, F, H)$$

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## Theorem 1.

$$e_n^{\text{lin}}(S, F, H) = e_n^{\text{cont}}(S, F, H) \sim e_{n,C}^{\text{non}}(S, F, H)$$

### 3. Elliptic Problems

$$\mathcal{A}(u) = f, \quad \mathcal{A} : H_0^s(\Omega) \rightarrow H^{-s}(\Omega), \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz,}$$

$\mathcal{A}$  boundedly invertible

example:

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

$$\mathcal{A} = \Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

## 3.1 General Facts

### Theorem 2.

$$e_n^{\text{lin}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \sim e_{n,C}^{\text{non}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \sim n^{-t/d}.$$

### important difference

- if  $\mathcal{A} : H_0^s(\Omega) \rightarrow H^{-s}(\Omega)$  is  $H^{s+t}$ -regular,

$\mathcal{A} : H_0^s(\Omega) \cap H^{s+t}(\Omega) \rightarrow H^{-s+t}(\Omega)$  is isomorphism  $\implies$

optimal basis  $\{g_i, i = 1, \dots, n\}$  independent of  $S$

- easy to implement
- uniform methods sufficient,  $\{V_n\}_{n \in \mathcal{N}}$  uniformly refined,  $\dim V_n \sim n$ ,

$$u \in H^{s+t}(\Omega) \iff \sum_{n=1}^{\infty} [n^{t/d} E_n(u)]^2 \frac{1}{n} < \infty, \quad \text{where} \quad E_n(u) := \inf_{g \in V_n} \|u - g\|_{H^s},$$

- $\mathcal{A}$  not regular  $\implies S(g_i)$  good basis, depends on  $S!$ 
  - precomputation necessary
  - too expansive in practice

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Question: Can we find a "good" basis  $\mathcal{B} \in \mathcal{B}_C$  such that

$$\inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, H^{-s+t}(\Omega), H^s(\Omega)) \sim n^{-t/d} ?$$

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What about a wavelet basis?

## 3.2 Wavelets

Multiresolution Analysis  $\{V_j\}_{j \geq 0}$

$$V_0 \subset V_1 \subset V_2 \subset \dots \quad \overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\Omega)$$

$$V_j = \overline{\text{span}\{\varphi_{j,k}, k \in I_j\}}$$

$$V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\Omega) = \bigoplus_{j=0}^{\infty} W_j$$

$$W_j = \overline{\text{span}\{\psi_{j,k}, k \in \mathcal{J}_j\}}$$

$$\lambda = (j, k), \quad |\lambda| = j, \quad \mathcal{J} = \bigcup_{j=0}^{\infty} (\{j\} \times \mathcal{J}_j)$$

$$(L) \quad \text{diam} (\text{supp} \psi_\lambda) \sim 2^{-|\lambda|}, \quad \lambda \in \mathcal{J}$$

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$$(NE) \quad \|f\|_{B_q^s(L_p(\Omega))} \sim \left( \sum_{|\lambda|=j_0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \left( \sum_{\lambda \in \mathcal{J}, |\lambda|=j} |\langle f, \tilde{\psi}_\lambda \rangle|^p \right)^{q/p} \right)^{1/q},$$

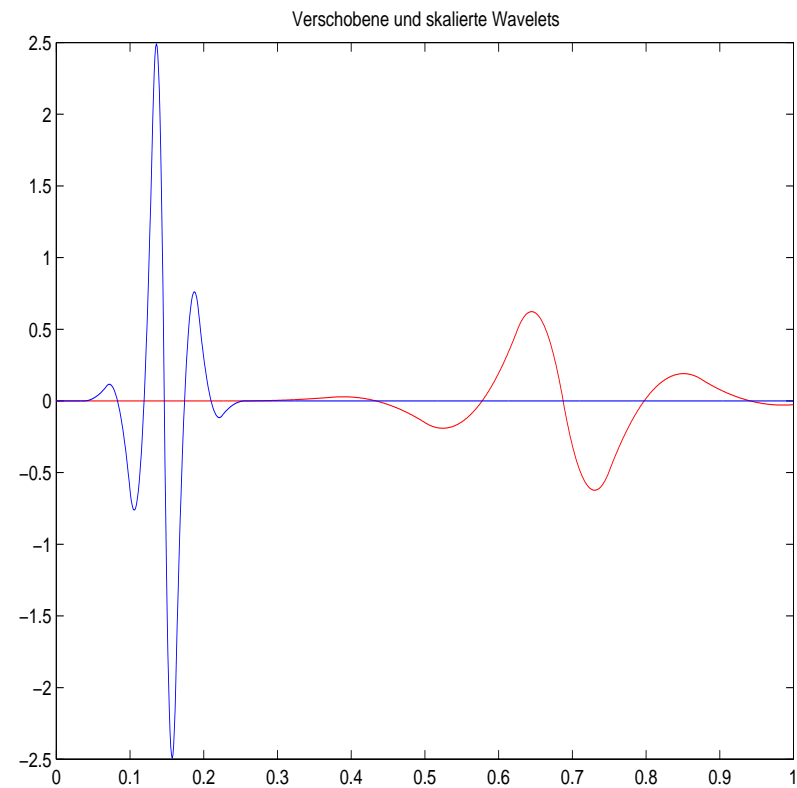
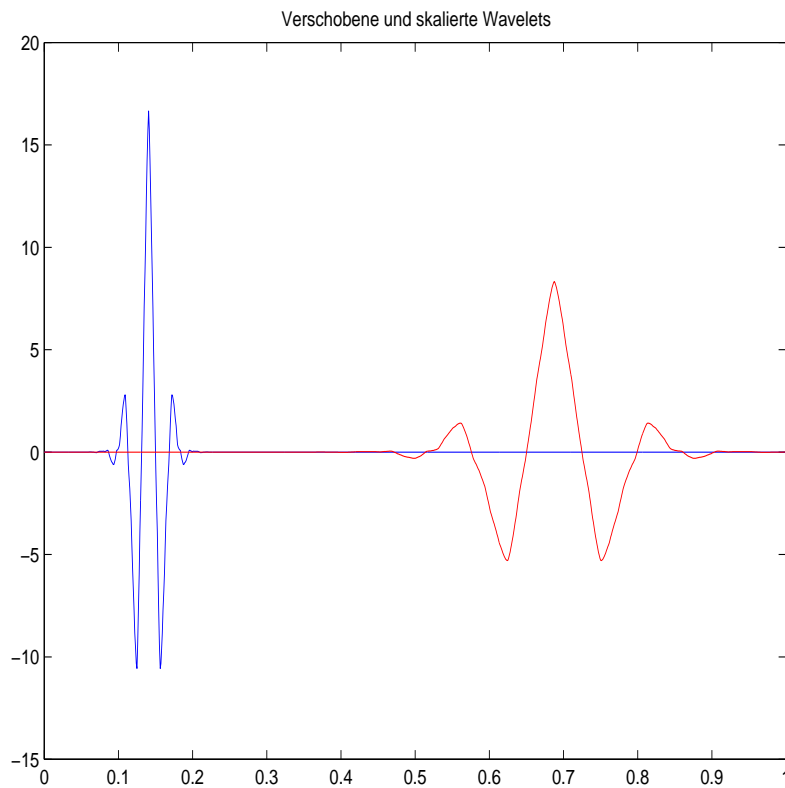
where  $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \mathcal{J}\}$  satisfies  $\langle \psi_\lambda, \tilde{\psi}_\nu \rangle = \delta_{\lambda,\nu}$ ,  $\lambda, \nu \in \mathcal{J}$

$$f \in L_p(\Omega), \quad f \in B_q^s(L_p(\Omega)) \iff |f|_{B_q^s(L_p(\Omega))} < \infty,$$

$$|f|_{B_q^s(L_p(\Omega))} := \left( \int_0^\infty [t^{-s} \omega_r(f, t)_{L_p(\Omega)}]^q dt/t \right)^{1/q}, \quad 0 < q < \infty, \quad r > s$$

$\Omega = \mathbb{R} :$ 

$$\psi_{j+1,k}(x) = 2^{j/2}\psi(2^j x - k), \quad \varphi_{j,k}(x) = \varphi(2^j x - k)$$



## 3.3 The Poisson Equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^d \quad \text{Lipschitz} & (*) \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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Can wavelet algorithms, e.g. best  $n$ -term wavelet approximation, do better?

**Theorem 3.** For the problem  $(*)$ , best  $n$ -term wavelet approximation produces the worst case error estimate:

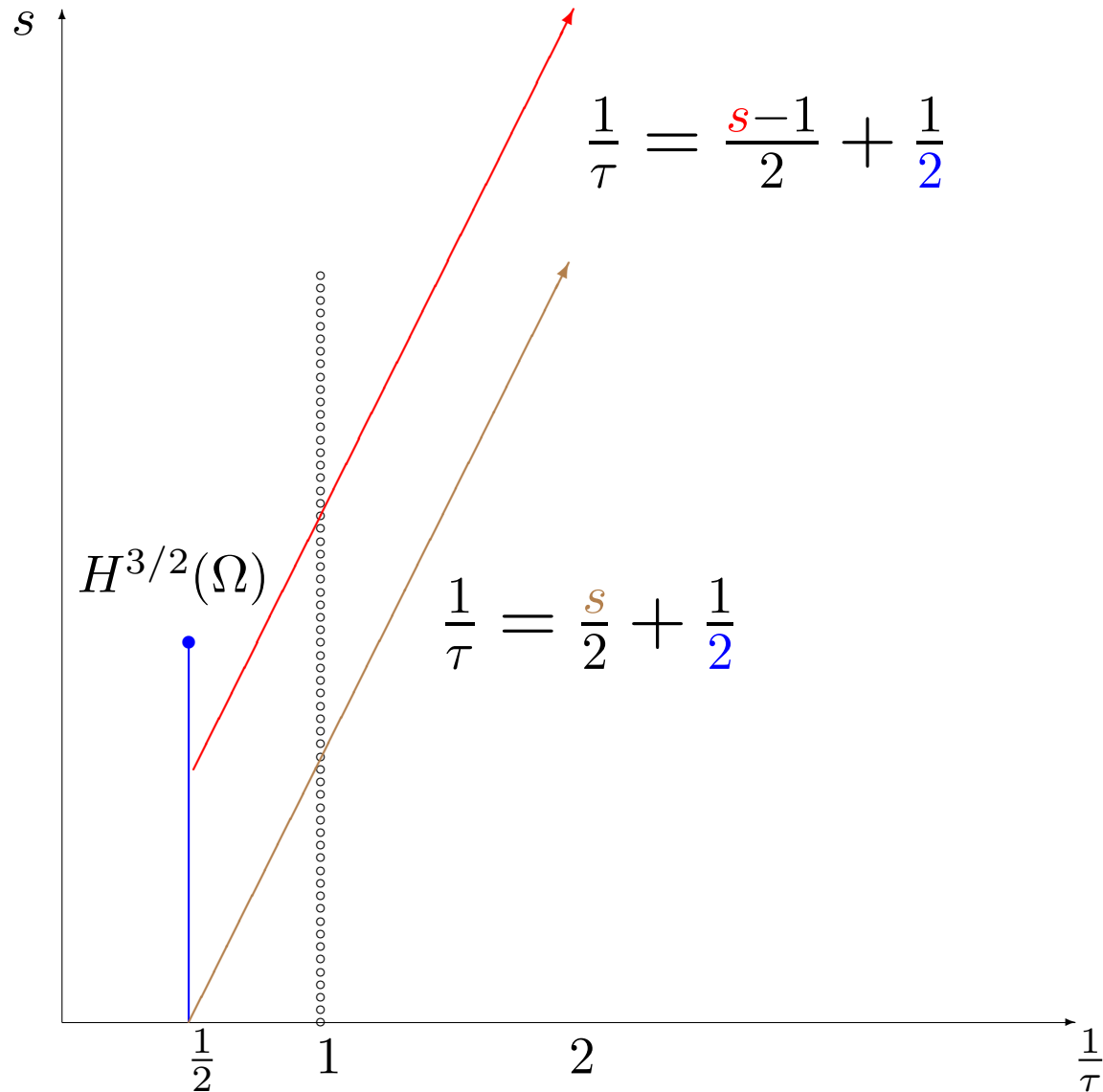
$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq C n^{-((t+1)/3 - \varrho)/d} \quad \text{for all } \varrho > 0,$$

provided that  $\frac{1}{2} < t \leq \frac{3d}{2(d-1)} - 1$ .

- uniform methods:  $\lesssim n^{-1/2d}$
- $n$ -term wavelet:  $\lesssim n^{-((t+1)/3)/d}$
- optimal:  $\lesssim n^{-t/d}$

**Proof:** It is known that

$$\|u - S_n(f)\|_{H^s} \leq C |u|_{B_\tau^\alpha(L_\tau(\Omega))} n^{(\alpha-s)/d}, \quad \frac{1}{\tau} = \frac{(\alpha - s)}{d} + \frac{1}{2}$$

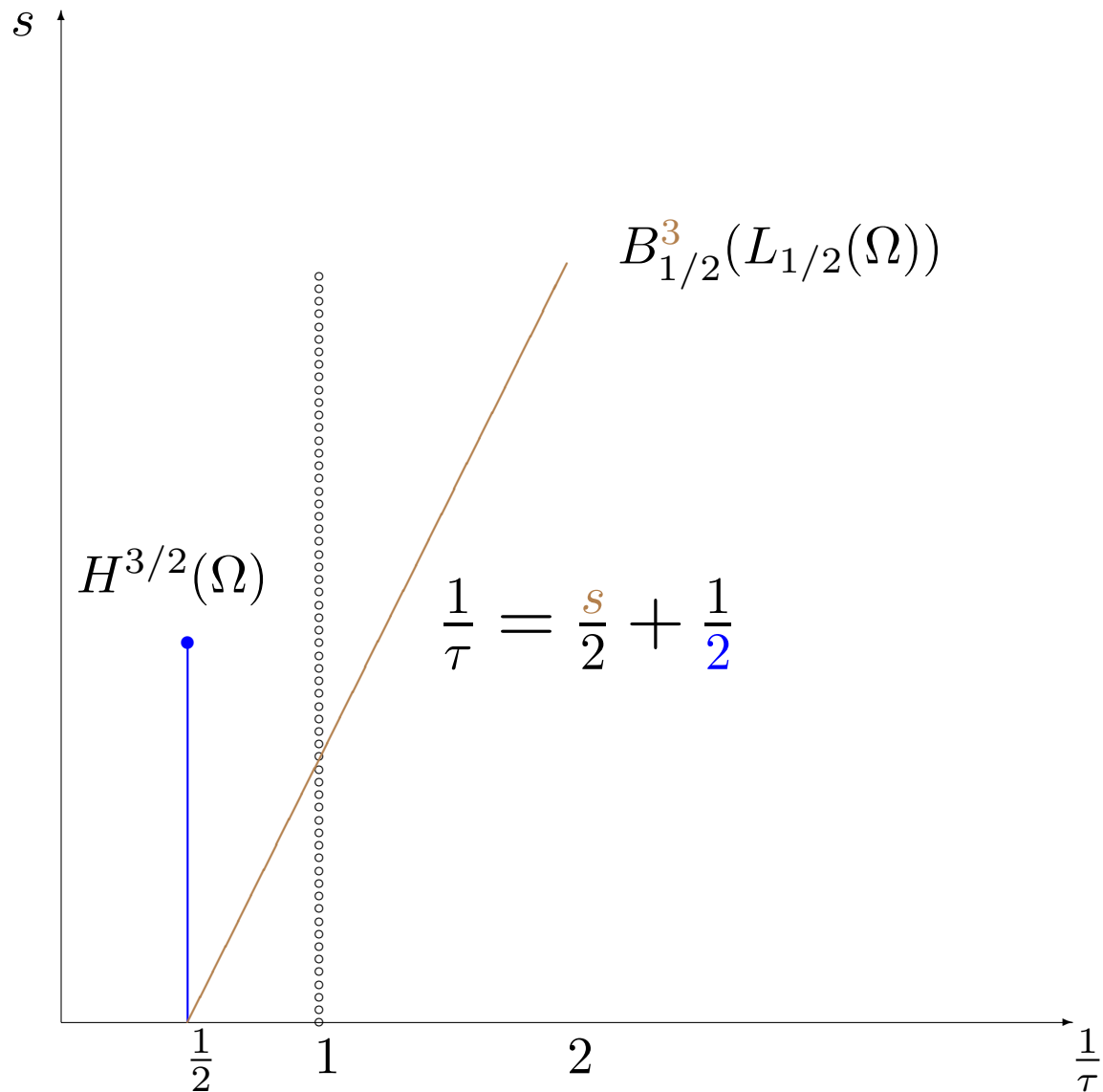
The DeVore–Triebel Diagram,  $d = 2$ 


- estimate in  $B_\tau^s(L_\tau(\Omega))$ ,  $\frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}$  : [D./DeVore]  $\implies$

$$\|u\|_{B_\tau^{t+1-\epsilon}(L_\tau(\Omega))} \leq C \|f\|_{H^{t-1}(\Omega)}, \quad t+1 \leq \frac{3d}{2(d-1)}$$

- estimate in  $B_2^s(L_2(\Omega)) = H^s(\Omega)$  : [Jerison/Kenig]  $\implies$

$$u \in H^{3/2}(\Omega), \quad \|u\|_{H^{3/2}(\Omega)} \leq C \|f\|_{H^{1/2}(\Omega)} \leq C \|f\|_{H^{t-1}(\Omega)}.$$

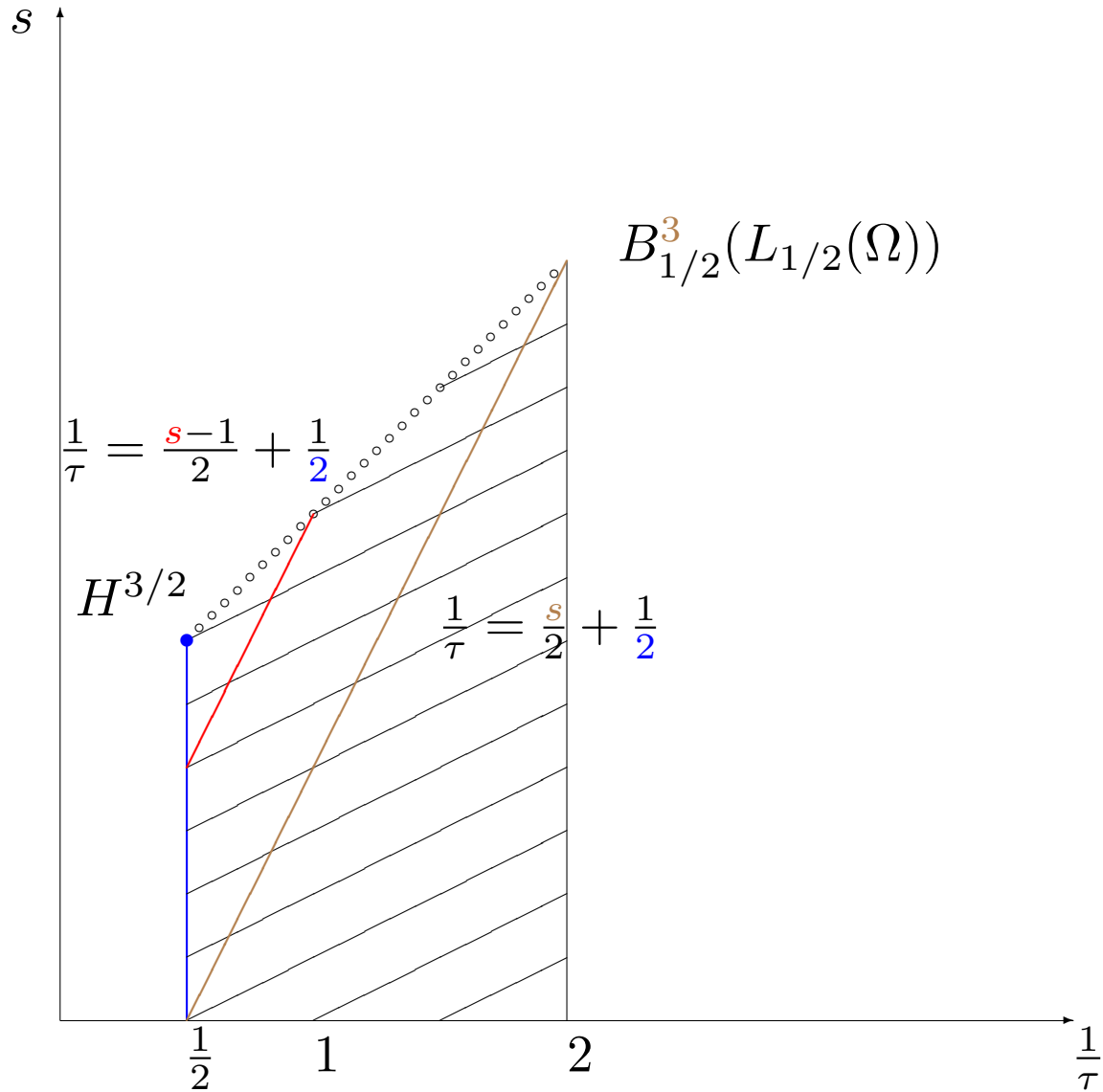
The DeVore–Triebel Diagram,  $d = 2$ 

- interpolation between  $B_\tau^{t+1-\epsilon}(L_\tau(\Omega))$  and  $H^{3/2}(\Omega) \implies$

$$u \in B_\tau^{s^*-\varrho}(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s^* - \varrho - 1}{d} + \frac{1}{2}, \quad s^* = \frac{t+1}{3} + 1,$$

$$\|u\|_{B_\tau^{s^*-\varrho}(L_{\tau^*}(\Omega))} \leq C \|f\|_{H^{t-1}(\Omega)}$$

# The DeVore–Triebel Diagram, $d = 2$



- estimate in  $B_\tau^s(L_\tau(\Omega))$ ,  $\frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}$  :  $u = \tilde{u} + v$ ,

$$-\Delta \tilde{u} = \tilde{f} = \mathcal{E}(f) \quad \text{in} \quad \tilde{\Omega} \supset \Omega$$

$$\tilde{u} = 0 \quad \text{on} \quad \partial \tilde{\Omega}$$

$$\Delta v = 0$$

$$v = g = \text{Tr}(\tilde{u})$$

- estimation of  $\tilde{u}$ : elliptic regularity  $\implies$

$$\tilde{u} \in H^{t+1}(\tilde{\Omega}), \quad \|\tilde{u}\|_{H^{t+1}(\tilde{\Omega})} \lesssim \|\mathcal{E}\| \|f\|_{H^{t-1}(\Omega)}$$

$$H^{t+1}(\tilde{\Omega}) \hookrightarrow B_\tau^{t+1-\epsilon}(L_\tau(\tilde{\Omega})) \implies$$

$$\|\tilde{u}\|_{B_\tau^{t+1-\epsilon}(L_\tau(\tilde{\Omega}))} \lesssim \|\tilde{u}\|_{H^{t+1}(\tilde{\Omega})} \lesssim \|\mathcal{E}\| \|f\|_{H^{t-1}(\Omega)}$$

- estimation of  $v$ : [Jerison/Kenig], trace theorems  $\implies$

$$\begin{aligned} \|v\|_{H^{\rho+1/2}(\Omega)} &\lesssim \|g\|_{H^\rho(\partial\Omega)} = \|\mathrm{Tr}(\tilde{u})\|_{H^\rho(\partial\Omega)} \\ &\lesssim \|\mathrm{Tr}\|\|\mathcal{E}\|\|f\|_{H^{t-1}(\Omega)}, \quad \rho < 1 \end{aligned}$$

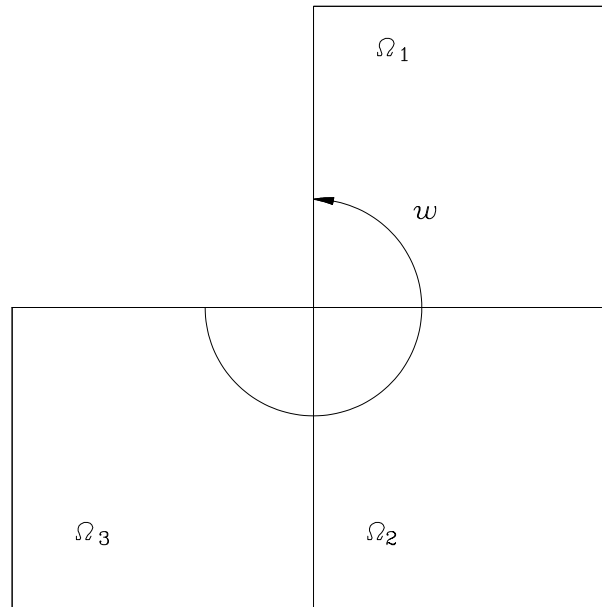
[D./DeVore]:  $v$  harmonic,  $v \in B_p^\beta(L_p(\Omega)) \implies$

$$v \in B_\tau^s(L_\tau(\Omega)), \quad 0 < s < \frac{\beta d}{(d-1)}, \quad \frac{1}{\tau} = \left( \frac{s}{d} + \frac{1}{p} \right), \quad \|v\|_{B_\tau^s(L_\tau(\Omega))} \lesssim \|v\|_{B_p^\beta(L_p(\Omega))}$$

$$B_2^s(L_2(\Omega)) = H^s(\Omega), \quad t+1 \leq \frac{3d}{2(d-1)} \implies$$

$$\|v\|_{B_\tau^{t+1-\epsilon}(L_\tau(\Omega))} \lesssim \|\mathrm{Tr}\|\|\mathcal{E}\|\|f\|_{H^{t-1}(\Omega)}$$

more special domains?



**Theorem 4.** For problem (\*) in a polygonal domain in  $\mathbb{R}^2$ , best  $n$ -term wavelet approximation is almost optimal in the sense that

$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq C n^{-(t-\varrho)/2}, \quad \text{for all } \varrho > 0.$$

**Proof:** [Grisvard]  $\implies f \in H^{t-1}(\Omega) \implies u = u_R + u_S, \quad u_R \in H^{t+1}(\Omega),$

$$u_S = \sum_j \sum_{m < t\omega_j/\pi} c_{j,m} \mathcal{S}_{j,m},$$

$$\mathcal{S}_{j,m}(r_j, \theta_j) = \eta_j(r_j) r_j^{m\pi/\omega_j} \sin(m\pi\theta_j/\omega_j), \quad \eta_j : C^\infty \text{ truncation function}$$

- estimate  $u_R$ :  $H^{t+1} \hookrightarrow B_\tau^{t+1-\varrho}(L_\tau(\Omega)), \quad \varrho > 0 \implies$

$$u_R \in B_\tau^{t+1-\varrho}(L_\tau(\Omega)) \quad \frac{1}{\tau} = \frac{(t-\varrho)}{d} + \frac{1}{2} \quad \text{for all } \varrho > 0$$

- estimate  $u_S$ : [D.]  $\implies$

$$\mathcal{S}_{l,m}(r_l, \theta_l) \in B_\tau^\alpha(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{(\alpha-1)}{d} + \frac{1}{2} \quad \text{for all } \alpha > 0$$

best  $n$ -term wavelet approximation is

- suboptimal for general domains, but superior to uniform schemes
- optimal for polygonal domains

## 4. Numerical Realization: Adaptive Wavelet Schemes

Galerkin scheme

$$\dots S_{j-1} \subset S_j \subset S_{j+1} \subset \dots, \quad \dim S_j = n_j < \infty$$

$$\langle \mathcal{A}(u_j), v \rangle = \langle f, v \rangle \quad \text{for all } v \in S_j$$

$$\Lambda \subset J, \quad S_\Lambda := \text{span}\{\psi_\lambda : \lambda \in \Lambda\}$$

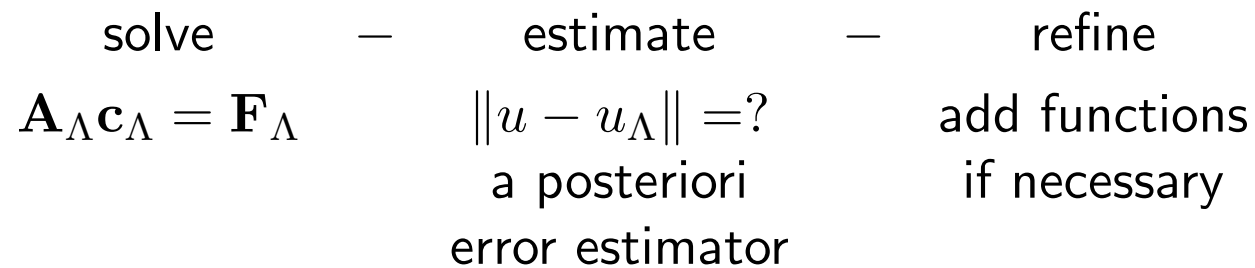
$$\mathbf{A}_\Lambda \mathbf{c}_\Lambda = \mathbf{F}_\Lambda, \quad \mathbf{A}_\Lambda = (\langle \mathcal{A}(\psi_{\lambda'}), \psi_\lambda \rangle)_{\lambda, \lambda' \in \Lambda}$$

$$(\mathbf{F}_\Lambda)_\lambda = \langle f, \psi_\lambda \rangle, \quad \lambda \in \Lambda$$

$$u_\Lambda = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$$

## adaptive scheme: updating strategy

- $S_\Lambda$  depend on  $u$
- self-regulating



- error estimator

$$\begin{aligned} \|u - u_\Lambda\|_{H^s} &\sim \|\mathcal{A}(u - u_\Lambda)\|_{H^{-s}} \\ &\sim \|f - \mathcal{A}(u_\Lambda)\|_{H^{-s}} \\ &\sim \|r_\Lambda\|_{H^{-s}} \\ &\sim \left( \sum_{\lambda \in \mathcal{J} \setminus \Lambda} \underbrace{2^{-2s|\lambda|} |\langle r_\Lambda, \psi_\lambda \rangle|^2}_{\delta_\lambda^2} \right)^{1/2} \end{aligned} \quad (\text{NE})$$

- adaptive refinement strategy

find  $\tilde{\Lambda}$  such that

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\|, \quad \kappa \in (0, 1)$$

catch the bulk of the residual:

$$\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} 2^{-2s|\lambda|} |\langle r_{\Lambda}, \psi_{\lambda} \rangle|^2 \gtrsim \gamma \sum_{\lambda \in J \setminus \Lambda} 2^{-2s|\lambda|} |\langle r_{\Lambda}, \psi_{\lambda} \rangle|^2$$

This adaptive strategy

- is **guaranteed** to converge [D./Dahmen/Hochmuth/Schneider];
- has **optimal** order of convergence, i.e., the same as best  $n$ -term approximation [Cohen/Dahmen/DeVore];
- realizes the optimal order of Theorem 3 and 4 without any precomputation;
- gains efficiency compared to uniform schemes;

- is based on estimates of the **residual**;
- requires knowledge of the right-hand side, e.g., best  $n$ -term approximation of  $f$ .

IBC-model:

- algorithm uses functionals  $L_1(f), L_2(f), \dots$  of the right-hand side
- if  $L_k$  depends on  $L_1(f), \dots, L_{k-1}(f)$ , the algorithm uses adaptive information