Optimal Approximation of Elliptic Problems II: Wavelet Methods

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1. Introduction

- optimal approximation of the solution to

$$\mathcal{A}(u) = f$$

$$\mathcal{A}: \; H \longrightarrow G, \qquad H = H^t \text{ Sobolev space}, \qquad G = H^{-t} \text{ dual space}$$

1. Introduction

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$$\mathcal{A}(u) = f$$

$$\mathcal{A} : \; H \longrightarrow G, \qquad H = H^t \text{ Sobolev space}, \qquad G = H^{-t} \text{ dual space}$$

- elliptic

$$\|\mathcal{A}(u)\|_{H^{-t}} \sim \|u\|_{H^t}, \quad u \in H^t$$

example:

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz} \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

$$\mathcal{A} \; = \; \triangle : \quad H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

Questions:

- How to measure optimality?
- What happens in the special case of elliptic PDEs?
- Do there exist optimal bases/methods?

2. Basic Concepts

$$\mathcal{A}(u) = f, \qquad \mathcal{A}: \quad H \to G, \quad H, G \quad \text{Hilbert spaces}$$

Riesz basis $\mathcal{B} = \{h_i \mid i \in \mathcal{N}\}$:

$$A\Big(\sum_k |\alpha_k|^2\Big)^{1/2} \leq \Big\| \sum_k \alpha_k \, h_k \Big\|_H \leq B\Big(\sum_k |\alpha_k|^2\Big)^{1/2}, \qquad \mathcal{B}_C := \Big\{\mathcal{B} : \; B/A \leq C\Big\}$$

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- nonlinear mappings S_n

$$\mathcal{N}_n := \{S_N : F \to H, \; S_n(f) = u_n = \sum_{k=1}^{n} c_k \, h_{i_k}\}$$

$$e(S_n, F, H) := \sup_{\|f\|_F \leq 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H$$

$$e_{n,C}^{\mathrm{non}}(S, F, H) = \inf_{\mathcal{B} \in \mathcal{B}_C} \inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, F, H), \qquad S := \mathcal{A}^{-1}$$

error of best n–term approximation

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error of best n–term approximation

- linear mapping

$$\mathcal{L}_n := \{S_n : F \to H,\ S_n(f) = \sum_{i=1}^{n} L_i(f) \cdot \tilde{h}_i\}$$

$$e_n^{\text{lin}}(S, F, H) = \inf_{S_n \in \mathcal{L}_n} e(S_n, F, H) = \inf_{S_n \in \mathcal{L}_n} \sup_{\|f\|_F \leq 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H$$

- continuous mapping S_n

$$S_n = \phi_n \circ N_n, \quad N_n : F \to \mathbb{R}^n, \quad \phi_n : \mathbb{R}^n \to H \quad \text{continuous}$$

$$e_n^{\mathrm{cont}}(S, F, H) = \inf_{S_n \in \mathcal{C}_n} e(S_n, F, H)$$

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Theorem 1.

$$e_n^{\text{lin}}(S, F, H) = e_n^{\text{cont}}(S, F, H) \sim e_{n,C}^{\text{non}}(S, F, H)$$

3. Elliptic Problems

$\mathcal{A}(u) = f, \qquad \mathcal{A} \quad : \ H_0^s(\Omega) \to H^{-s}(\Omega), \quad \Omega \subset \mathbb{R}^d$ Lipschitz,

$\mathcal{A}$ boundedly invertible

example:

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

$$\mathcal{A} \ = \ \triangle : \quad H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

3.1 General Facts

Theorem 2.

$$e_n^{\mathrm{lin}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \sim e_{n,C}^{\mathrm{non}}(S, H^{-s+t}(\Omega), H^s(\Omega)) \sim n^{-t/d}.$$

important difference

- if $\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega)$ is H^{s+t}-regular,

  $\mathcal{A} : H_0^s(\Omega) \cap H^{s+t}(\Omega) \to H^{-s+t}(\Omega)$ is isomorphism $\Longrightarrow$

  optimal basis $\{g_i, i = 1, \ldots, n\}$ independent of S

  - easy to implement
  - uniform methods sufficient, $\{V_n\}_{n \in \mathcal{N}}$ uniformly refined, $\dim V_n \sim n$,

$$u \in H^{s+t}(\Omega) \Longleftrightarrow \sum_{n=1}^{\infty} [n^{t/d} E_n(u)]^2 \frac{1}{n} < \infty, \quad \text{where} \quad E_n(u) := \inf_{g \in V_n} \|u - g\|_{H^s},$$

- $\mathcal{A}$ not regular $\Longrightarrow$ $S(g_i)$ good basis, depends on S!
  - precomputation necessary
  - too expansive in practice

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Question: Can we find a "good" basis $\mathcal{B} \in \mathcal{B}_C$ such that

$$\inf_{S_n \in \mathcal{N}_n(\mathcal{B})} e(S_n, H^{-s+t}(\Omega), H^{s}(\Omega)) \sim n^{-t/d} \,?$$

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What about a wavelet basis?

3.2 Wavelets

Multiresolution Analysis $\{V_j\}_{j\geq 0}$

$$V_0 \subset V_1 \subset V_2 \subset \dots \qquad \overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\Omega)$$

$$V_j = \overline{\text{span}\{\varphi_{j,k},\ k \in I_j\}}$$

$$V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\Omega) = \oplus_{j=0}^{\infty} W_j$$

$$W_j = \overline{\text{span}\{\psi_{j,k},\ k \in \mathcal{J}_j\}}$$

$$\lambda = (j,k), \quad |\lambda| = j, \quad \mathcal{J} = \bigcup_{j=0}^{\infty} (\{j\} \times \mathcal{J}_j)$$

(L) $\quad$ diam (suppψ_λ) $\sim 2^{-|\lambda|}, \quad \lambda \in \mathcal{J}$

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(NE)
$$\|f\|_{B_q^s(L_p(\Omega))} \sim \left(\sum_{|\lambda|=j_0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \left(\sum_{\lambda\in\mathcal{J},|\lambda|=j} |\langle f, \tilde{\psi}_\lambda\rangle|^p \right)^{q/p} \right)^{1/q},$$

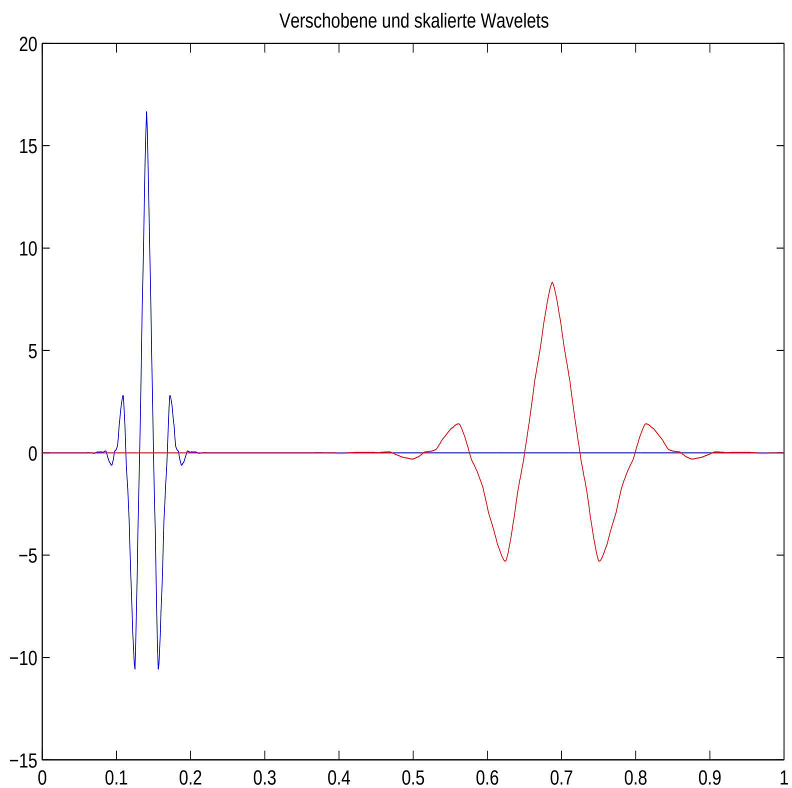
where $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \mathcal{J}\}$ satisfies $\langle \psi_\lambda, \tilde{\psi}_\nu\rangle = \delta_{\lambda,\nu}, \quad \lambda, \nu \in \mathcal{J}$

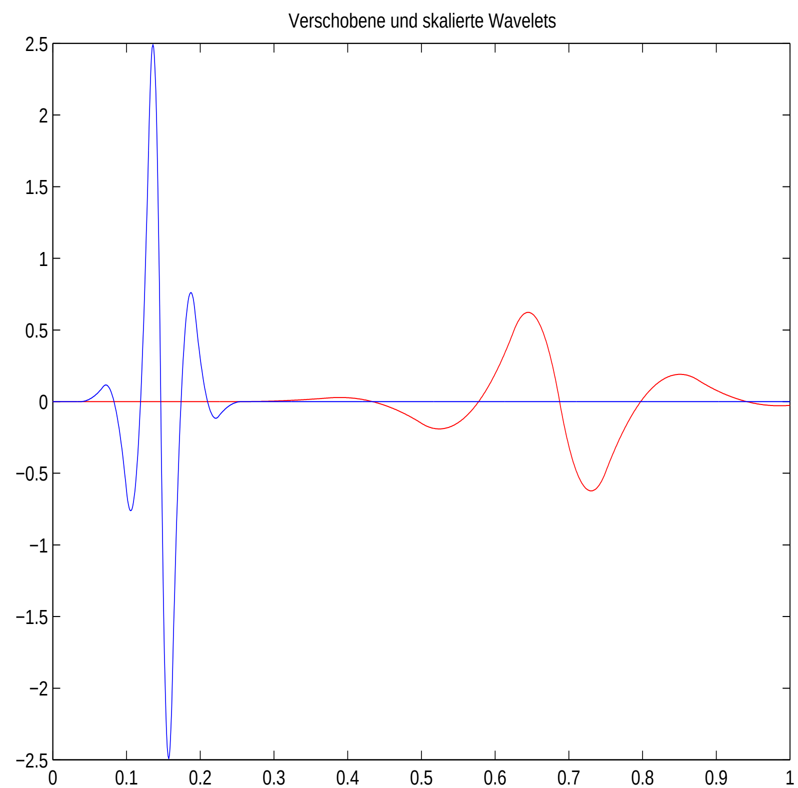
$$f \in L_p(\Omega), \quad f \in B_q^s(L_p(\Omega)) \iff |f|_{B_q^s(L_p(\Omega))} < \infty,$$

$$|f|_{B_q^s(L_p(\Omega))} := \left(\int_0^\infty [t^{-s}\omega_r(f,t)_{L_p(\Omega)}]^q dt/t \right)^{1/q}, \; 0 < q < \infty, \quad r > s$$

$\Omega = I\!R:$

$$\psi_{j+1,k}(x) = 2^{j/2}\psi(2^j x - k), \quad \varphi_{j,k}(x) = \varphi(2^j x - k)$$





3.3 The Poisson Equation

$$
\begin{aligned}
-\triangle u &= f \quad \text{in} \quad \Omega \subset I\!R^d \quad \text{Lipschitz} \qquad (*) \\
u &= 0 \quad \text{on} \quad \partial\Omega
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Can wavelet algorithms, e.g. best n–term wavelet approximation, do better?

Theorem 3. *For the problem* $(*)$, *best* n*-term wavelet approximation produces the worst case error estimate:*

$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq C\, n^{-(\frac{(t+1)}{3}-\varrho)/d} \quad \textit{for all} \quad \varrho > 0,$$

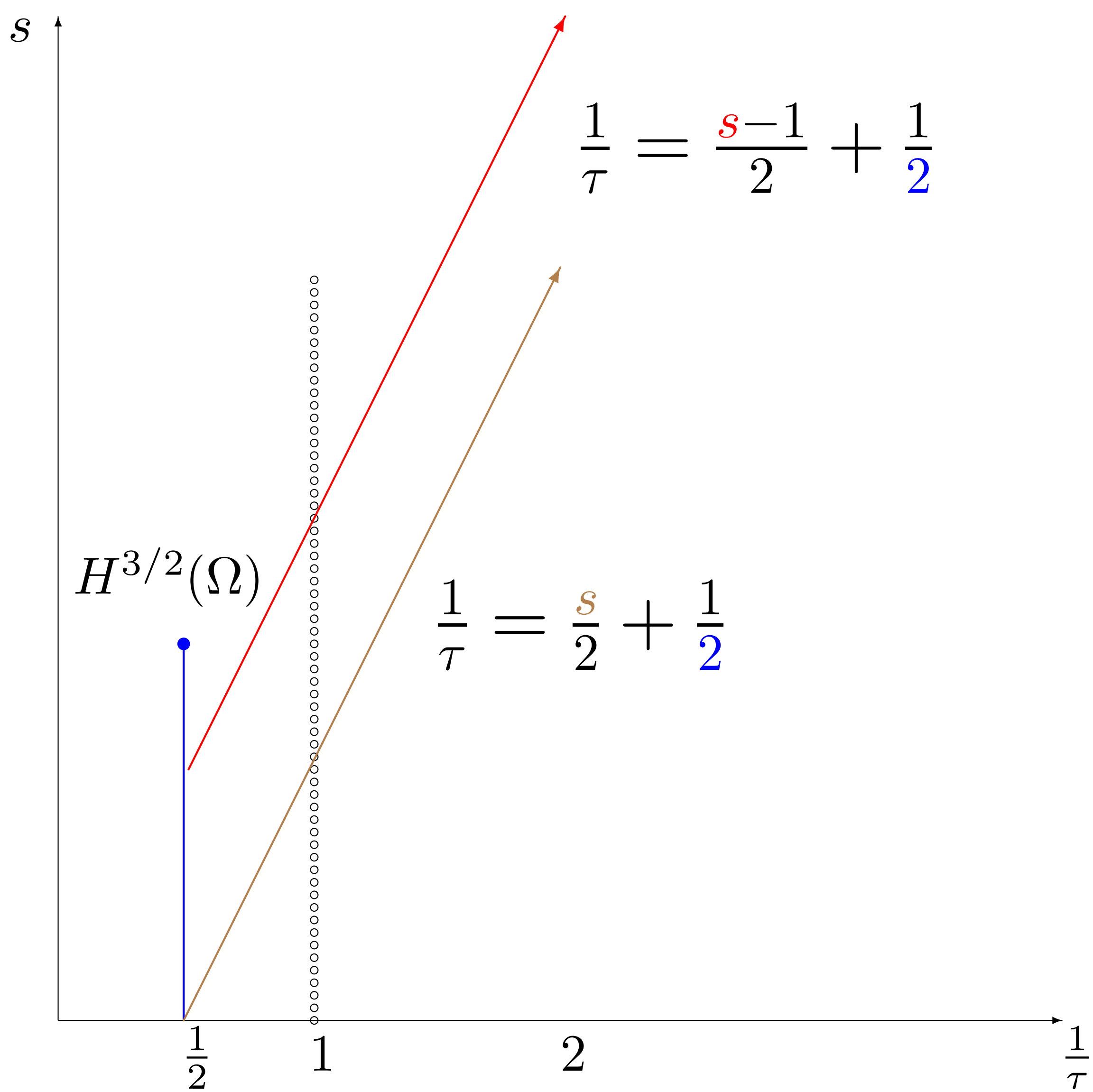
provided that $\frac{1}{2} < t \leq \frac{3d}{2(d-1)} - 1$.

- uniform methods: $\lesssim n^{-1/2d}$
- n–term wavelet: $\lesssim n^{-(\frac{(t+1)}{3})/d}$
- optimal: $\lesssim n^{-t/d}$

Proof: It is known that

$$\|u - S_n(f)\|_{H^s} \leq C|u|_{B^\alpha_\tau(L_\tau(\Omega))} n^{(\alpha-s)/d}, \qquad \frac{1}{\tau} = \frac{(\alpha - s)}{d} + \frac{1}{2}$$

The DeVore–Triebel Diagram, $d = 2$



- estimate in $B^s_\tau(L_\tau(\Omega))$, $\frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}$: [D./DeVore] $\Longrightarrow$

$$\|u\|_{B^{t+1-\epsilon}_\tau(L_\tau(\Omega))} \leq C\|f\|_{H^{t-1}(\Omega)}, \qquad t+1 \leq \frac{3d}{2(d-1)}$$

- estimate in $B^s_2(L_2(\Omega)) = H^s(\Omega)$: [Jerison/Kenig] $\Longrightarrow$

$$u \in H^{3/2}(\Omega), \qquad \|u\|_{H^{3/2}(\Omega)} \leq C\|f\|_{H^{1/2}(\Omega)} \leq C\|f\|_{H^{t-1}(\Omega)}.$$

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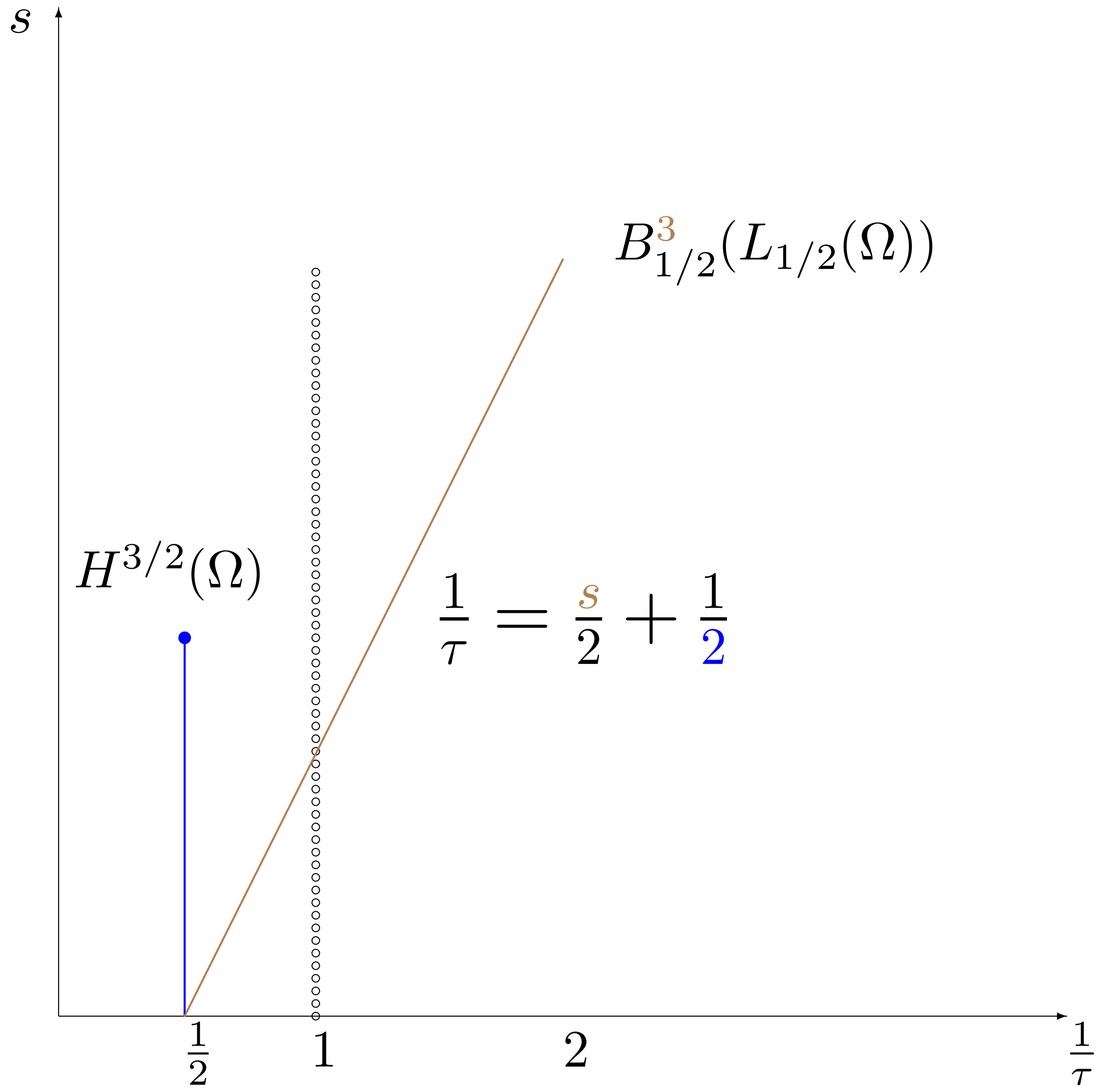

- interpolation between $B_\tau^{t+1-\epsilon}(L_\tau(\Omega))$ and $H^{3/2}(\Omega) \Longrightarrow$

$$u \in B_\tau^{s^*-\varrho}(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s^*-\varrho-1}{d} + \frac{1}{2}, \quad s^* = \frac{t+1}{3} + 1,$$

$$\|u\|_{B_\tau^{s^*-\varrho}(L_{\tau*}(\Omega))} \leq C\|f\|_{H^{t-1}(\Omega)}$$

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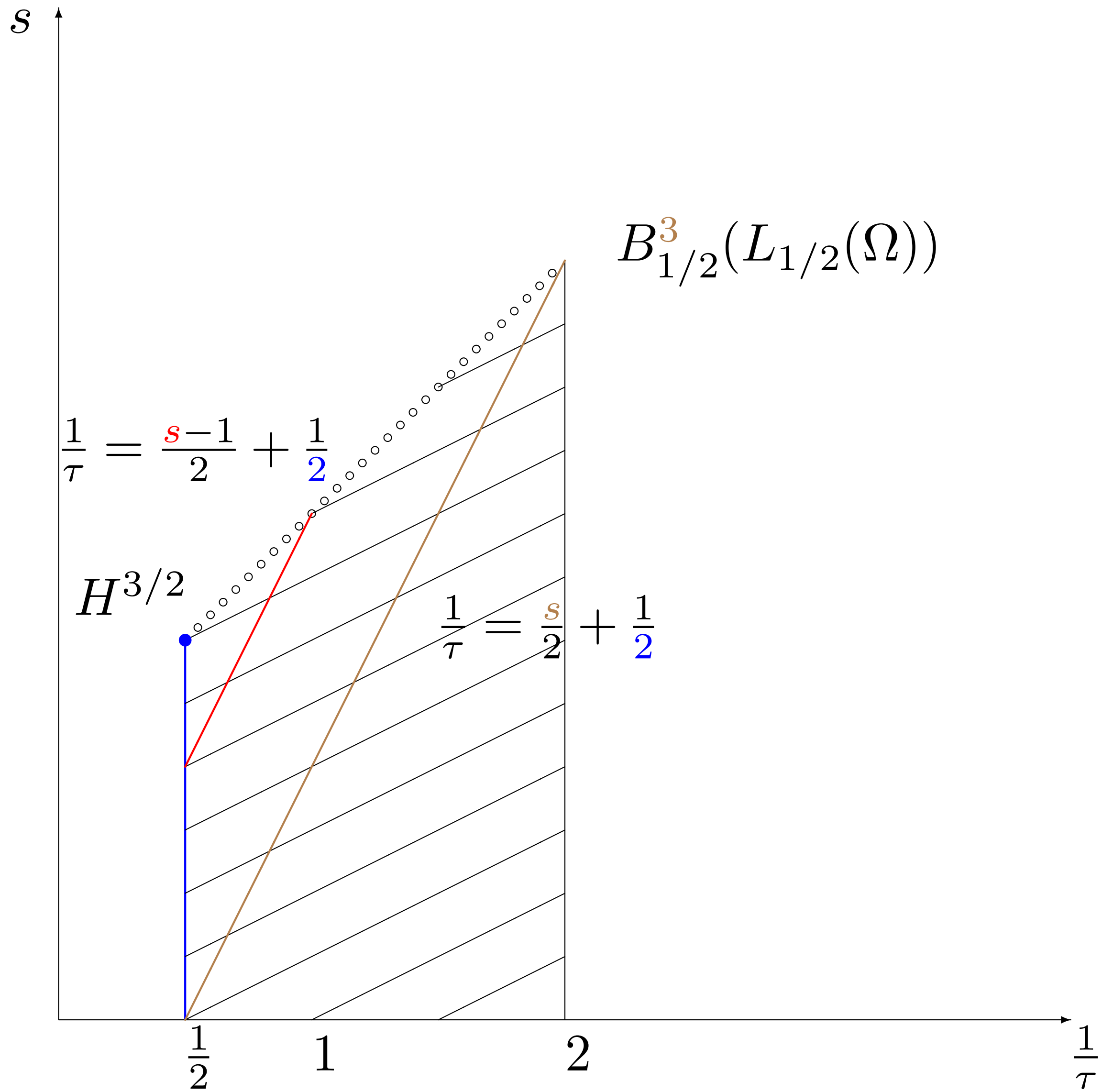


- estimate in $B^{s}_{\tau}(L_\tau(\Omega))$, $\quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}$: $\quad u = \tilde{u} + v$,

$$\begin{aligned} -\triangle \tilde{u} &= \tilde{f} = \mathcal{E}(f) \quad \text{in} \quad \widetilde{\Omega} \supset \Omega \\ \tilde{u} &= 0 \quad \text{on} \quad \partial\widetilde{\Omega} \end{aligned}$$

$$\begin{aligned} \triangle v &= 0 \\ v &= g = \mathrm{Tr}(\tilde{u}) \end{aligned}$$

- estimation of $\tilde{u}$: elliptic regularity $\Longrightarrow$

$$\tilde{u} \in H^{t+1}(\widetilde{\Omega}), \quad \|\tilde{u}\|_{H^{t+1}(\widetilde{\Omega})} \lesssim \|\mathcal{E}\| \|f\|_{H^{t-1}(\Omega)}$$

$$H^{t+1}(\widetilde{\Omega}) \hookrightarrow B^{t+1-\epsilon}_{\tau}(L_\tau(\widetilde{\Omega})) \Longrightarrow$$

$$\|\tilde{u}\|_{B^{t+1-\epsilon}_{\tau}(L_\tau(\widetilde{\Omega}))} \lesssim \|\tilde{u}\|_{H^{t+1}(\widetilde{\Omega})} \lesssim \|\mathcal{E}\| \|f\|_{H^{t-1}(\Omega)}$$

- estimation of v: [Jerison/Kenig], trace theorems $\Longrightarrow$

$$\begin{aligned}\|v\|_{H^{\rho+1/2}(\Omega)} &\lesssim \|g\|_{H^{\rho}(\partial\Omega)} = \|\mathrm{Tr}(\tilde{u})\|_{H^{\rho}(\partial\Omega)} \\ &\lesssim \|\mathrm{Tr}\|\|\mathcal{E}\|\|f\|_{H^{t-1}(\Omega)}, \qquad \rho < 1\end{aligned}$$

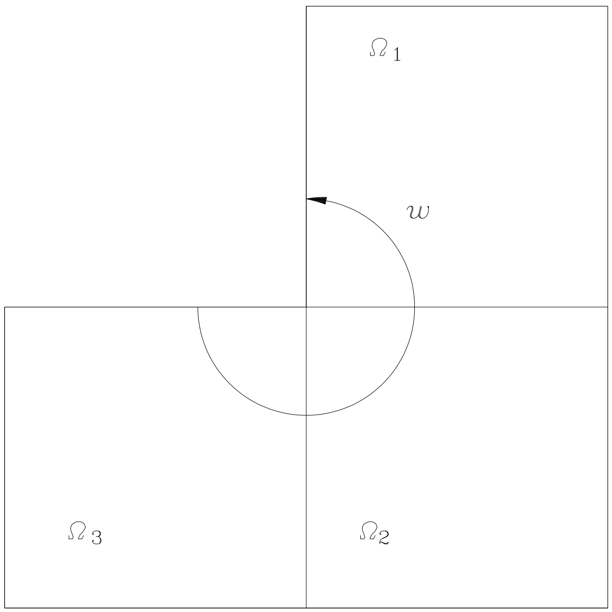
[D./DeVore]: v harmonic, $v \in B_p^{\beta}(L_p(\Omega)) \Longrightarrow$

$$v \in B_\tau^s(L_\tau(\Omega)), \quad 0 < s < \frac{\beta d}{(d-1)}, \quad \frac{1}{\tau} = \left(\frac{s}{d} + \frac{1}{p}\right), \quad \|v\|_{B_\tau^s(L_\tau(\Omega))} \lesssim \|v\|_{B_p^{\beta}(L_p(\Omega))}$$

$$B_2^s(L_2(\Omega)) = H^s(\Omega), \qquad t+1 \leq \frac{3d}{2(d-1)} \Longrightarrow$$

$$\|v\|_{B_\tau^{t+1-\epsilon}(L_\tau(\Omega))} \lesssim \|\mathrm{Tr}\|\|\mathcal{E}\|\|f\|_{H^{t-1}(\Omega)}$$

more special domains?



Theorem 4. *For problem (∗) in a polygonal domain in* $\mathbb{R}^2$*, best* n*-term wavelet approximation is almost optimal in the sense that*

$$e(S_n, H^{t-1}(\Omega), H^1(\Omega)) \leq Cn^{-(t-\varrho)/2}, \qquad \text{for all} \quad \varrho > 0.$$

Proof: [Grisvard] $\Longrightarrow f \in H^{t-1}(\Omega) \Longrightarrow u = u_R + u_S, \qquad u_R \in H^{t+1}(\Omega),$

$$u_S = \sum_j \sum_{m<t\omega_j/\pi} c_{j,m}\mathcal{S}_{j,m},$$

$\mathcal{S}_{j,m}(r_j,\theta_j) = \eta_j(r_j) r_j^{m\pi/\omega_j} \sin(m\pi\theta_j/\omega_j), \quad \eta_j : \quad C^\infty$ truncation function

- estimate u_R: $\quad H^{t+1} \hookrightarrow B_\tau^{t+1-\varrho}(L_\tau(\Omega)), \quad \varrho > 0 \Longrightarrow$

$$u_R \in B_\tau^{t+1-\varrho}(L_\tau(\Omega)) \qquad \frac{1}{\tau} = \frac{(t-\varrho)}{d} + \frac{1}{2} \qquad \text{for all} \quad \varrho > 0$$

- estimate u_S: [D.] $\Longrightarrow$

$$\mathcal{S}_{l,m}(r_l,\theta_l) \in B_\tau^{\alpha}(L_\tau(\Omega)), \qquad \frac{1}{\tau} = \frac{(\alpha-1)}{d} + \frac{1}{2} \quad \text{for all} \quad \alpha > 0$$

best n–term wavelet approximation is

- suboptimal for general domains, but superior to uniform schemes
- optimal for polygonal domains

4. Numerical Realization: Adaptive Wavelet Schemes

Galerkin scheme

$$\ldots S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots, \qquad \mathrm{dim} S_j = n_j < \infty$$

$$\langle \mathcal{A}(u_j), v \rangle = \langle f, v \rangle \quad \text{for all} \quad v \in S_j$$

$$\Lambda \subset J, \qquad S_\Lambda := \mathrm{span}\{\psi_\lambda : \lambda \in \Lambda\}$$

$$\mathbf{A}_\Lambda \mathbf{c}_\Lambda = \mathbf{F}_\Lambda, \qquad \mathbf{A}_\Lambda = (\langle \mathcal{A}(\psi_{\lambda'}), \psi_\lambda \rangle)_{\lambda, \lambda' \in \Lambda}$$

$$(\mathbf{F}_\Lambda)_\lambda = \langle f, \psi_\lambda \rangle, \; \lambda \in \Lambda$$

$$u_\Lambda = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$$

adaptive scheme: updating strategy

- S_Λ depend on u
- self–regulating

solve	–	estimate	–	refine
$\mathbf{A}_\Lambda \mathbf{c}_\Lambda = \mathbf{F}_\Lambda$		$\|u - u_\Lambda\| = ?$ a posteriori error estimator		add functions if necessary

- error estimator

$$\begin{aligned} \|u - u_\Lambda\|_{H^s} &\sim \|\mathcal{A}(u - u_\Lambda)\|_{H^{-s}} \\ &\sim \|f - \mathcal{A}(u_\Lambda)\|_{H^{-s}} \\ &\sim \|r_\Lambda\|_{H^{-s}} \\ &\sim \left(\sum_{\lambda \in \mathcal{J} \setminus \Lambda} \underbrace{2^{-2s|\lambda|} |\langle r_\Lambda, \psi_\lambda \rangle|^2}_{\delta_\lambda^2} \right)^{1/2} \qquad \text{(NE)} \end{aligned}$$

- adaptive refinement strategy

find $\tilde{\Lambda}$ such that

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\|, \qquad \kappa \in (0, 1)$$

catch the bulk of the residual:

$$\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} 2^{-2s|\lambda|} |\langle r_\Lambda, \psi_\lambda \rangle|^2 \gtrsim \gamma \sum_{\lambda \in J \setminus \Lambda} 2^{-2s|\lambda|} |\langle r_\Lambda, \psi_\lambda \rangle|^2$$

This adaptive strategy

- is guaranteed to converge [D./Dahmen/Hochmuth/Schneider];
- has optimal order of convergence, i.e., the same as best n–term approximation [Cohen/Dahmen/DeVore];
- realizes the optimal order of Theorem 3 and 4 without any precomputation;
- gains efficiency compared to uniform schemes;

- is based on estimates of the residual;
- requires knowledge of the right–hand side, e.g., best n–term approximation of f.

IBC–model:

- algorithm uses functionals $L_1(f), L_2(f), \dots$ of the right–hand side
- if L_k depends on $L_1(f), \dots, L_{k-1}(f)$, the algorithm uses adaptive information