Weighted Coorbit Spaces and Banach Frames on Homogeneous Spaces

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1. Motivation
2. Coorbit Spaces
3. Banach Frames
4. Nonlinear Approximation
5. Applications

1. Motivation

- Classical problem: analyze/decompose/approximate... a given signal/function

  Basic step: decomposition into suitable building blocks

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

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- common thread: square integrable group representations

  $\mathcal{H}$: Hilbert space, $\mathcal{G}$: Lie group, $U : \mathcal{G} \longrightarrow \mathcal{L}(\mathcal{H})$

  U is square-integrable if $\exists\ \psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi \rangle_{\mathcal{H}}|^2 \, d\nu(g) < \infty$$

Such a function ψ is called admissible (analyzing wavelet).

Examples:

- Weyl–Heisenberg group $\mathcal{G}_{WH}^{red}$, $\mathcal{H} = L_2(\mathbb{R})$

$$\mathcal{G}_{WH}^{red} \cong \mathbb{R}^2 \times S^1 \qquad (p, q, \phi) \circ (p', q', \phi') = (p + p', q + q', \phi + \phi' + p'q),$$

$$U(p, q, \phi)f(x) = \exp(i(\lambda\phi + q(x - \lambda p)))f(x - \lambda p)$$

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- affine group, $\mathcal{H} = L_2(\mathbb{R})$

$$\mathcal{A} := \{(a, b) \in \mathbb{R}^2,\ a \neq 0\}, \qquad (a, b) \circ (a', b') = (aa', ab' + b)$$

$$U(a, b) f(x) = |a|^{-1/2} f\left(\frac{x - b}{a}\right)$$

- integral transform

$$V_\psi : \mathcal{H} \longrightarrow L_2(\mathcal{G}) \qquad V_\psi f(h) := \langle f, U(h^{-1})\psi \rangle_{\mathcal{H}}$$

$\mathcal{G}_{WH}^{red} \rightsquigarrow$ Gabor transform, affine group $\rightsquigarrow$ wavelet transform

$$(V_\psi(f) * V_\psi(\psi))(h) = V_\psi(f)(h)$$

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- smoothness spaces, coorbit spaces

$$\mathcal{C}o(Y) := \{ f \mid V_\psi(f) \in Y \}, \qquad \|f\|_{\mathcal{C}o(Y)} = \|V_\psi(f)\|_Y$$

$\mathcal{G}_{WH}^{red} : \mathcal{C}o(L_p(\mathcal{G}))$ modulation spaces M_{pp}^0

affine group : Besov spaces $B_p^{1/p-1/2}(L_p(\mathbb{R}))$

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weighted coorbit spaces

$\mathcal{G}^{red}_{WH} : \mathcal{C}o(L_{p,w}(\mathcal{G}))$, $w(q,p) = (1+|p|)^{2s}$ modulation spaces M^s_{pp}

- Discretization necessary!

$$f = \sum_i c_i U(h_i^{-1})\psi = \sum_i c_i \psi_i$$

optimal: (stable) basis! Example: affine group, $a = 2^{-j},\ b = 2^{-j}k \rightsquigarrow$

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k} 2^{j/2}\psi(2^j x - k) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(x)$$

advantages:

- non–redundant representation, efficient decomposition and reconstruction algorithms

- characterization of function spaces

$$f \in H^s \Longleftrightarrow \sum_{j\in\mathbb{Z}} 2^{2js} \sum_{k\in\mathbb{Z}} |c_{j,k}|^2 < \infty,$$

$\Longrightarrow$ preconditioning strategies

- vanishing moments,

$$\int_{I\!R} x^{\alpha}\psi(x)dx = 0, \qquad \alpha < r$$

$\Longrightarrow$ compression strategies

- locality,

$$\text{supp}\psi_{j,k} \subset Q_{j,k}, \quad |Q_{j,k}| \le C2^{-j}$$

disadvantages:

- lack of flexibility

- problem of adaptation to domains and manifolds not satisfactorily solved

- way out: frames: a system $\{h_m\}_{m\in\mathbb{Z}}$ is called a frame, if

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{m\in\mathbb{Z}} |\langle f, h_m\rangle|^2 \leq B\|f\|_{\mathcal{H}}^2 \qquad f = \sum_{m\in\mathbb{Z}} c_m h_m$$

$$\mathcal{S}(f) := \sum_{m\in\mathbb{Z}} \langle f, h_m\rangle h_m \quad \text{frame operator } \mathcal{S}$$

Theorem 1. *Let $\{h_m\}_{m\in\mathbb{Z}}$ be a frame. Then*

i) $\mathcal{S}$ is invertible and $B^{-1}I \leq \mathcal{S}^{-1} \leq A^{-1}I$;

ii) $\{h^m\}_{m\in\mathbb{Z}}$, $h^m := \mathcal{S}^{-1}h_m$ is also a frame, the dual frame;

iii) every $f \in \mathcal{H}$ has an expansion

$$f = \sum_{m\in\mathbb{Z}} \langle f, \mathcal{S}^{-1}h_m\rangle h_m = \sum_{m\in\mathbb{Z}} \langle f, h_m\rangle \mathcal{S}^{-1}h_m.$$

disadvantage:

- redundancy

advantage:

- gain of flexibility!

- foundations of modern frame theory: Feichtinger/Gröchenig theory, since 1986

Banach frames

$$f = \sum_i c_i U(h_i^{-1})\psi = \sum_i c_i \psi_i$$

$$\frac{1}{B}\|f\|_{\mathcal{C}o(L_p)} \leq \|(\langle f, \psi_i \rangle)\|_{\ell_p} \leq \frac{1}{A}\|f\|_{\mathcal{C}o(L_p)}$$

Attention: no automatic reconstruction!

Questions:

- How can Gabor/wavelet transforms on manifolds and associated coorbit spaces be defined?

- Is it possible to obtain Banach frames for these coorbit spaces?

- Convergence order of best N–term approximation?

- Applications, e.g., to the spheres?

Strategy: Combine the following building blocks:

- representations involving quotient groups [B. Torresani, J.–P. Antoine...]
- Feichtinger–Gröchenig theory
- nonlinear approximation by frames [K. Gröchenig/S. Samarah],[R. DeVore/V. Temlyakov...]
- local Fourier analysis on spheres [B. Torresani]

- (2002): program works for $\mathcal{C}o(L_p)$!
- Question: what about weighted spaces?

2. Coorbit Spaces

$\mathcal{N}$ manifold, $\mathcal{H} = L_2(\mathcal{N})$

- First step: find group $\mathcal{G}$ with representation in $L_2(\mathcal{N})$

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- Way out: quotients $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$, $\mathcal{G}_{\mathcal{F}}$ closed subgroup

  $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\mathcal{F}}$, $\sigma : \mathcal{G}/\mathcal{G}_{\mathcal{F}} \to \mathcal{G}$ Borel section

  U is strictly square integrable mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$, if $\exists \psi \in L_2(\mathcal{N})$ such that

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi \rangle_{L_2(\mathcal{N})}$$

  is an isometry.

- Problem: no group structure, no convolution, no reproducing kernel....

$$\begin{aligned} R(h,l) &:= \langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= \langle \psi, U(\sigma(h)\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \end{aligned}$$

$$C \geq \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(h,l)| d\mu(l) \qquad (*)$$

Proposition 1. *Let U be strictly square integrable* $\Longrightarrow$

$$\langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = V_\psi f(h), \quad f \in L_2(\mathcal{N})$$

- weighted coorbit spaces

weight function $w(g \circ \tilde{g}) \leq w(g)\, w(\tilde{g})$

$$L_{p,w}(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) := \{f \ : \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |f(h)|^p w(\sigma(h))^p \, d\mu(h) < \infty\} \qquad (1 \leq p < \infty)$$

$$L_{\infty,w}(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) := \{f \ : \ \operatorname{ess} \sup_{h \in \mathcal{G}/\mathcal{G}_{\mathcal{F}}} |f(h)| w(\sigma(h)) < \infty\}$$

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$$H_{1,w} := \{f \in L_2(\mathcal{N}) : \ V_\psi f \in L_{1,w}(\mathcal{G}/\mathcal{G}_{\mathcal{F}})\},$$

$$\|f\|_{H_{1,w}} := \|V_\psi f\|_{L_{1,w}(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}$$

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$H_{1,w} \hookrightarrow L_2(\mathcal{N}) \hookrightarrow H'_{1,w}$ $\qquad H'_{1,w} :=$ dual space

$$\tilde{R}(h,l) := R(h,l)\frac{w(\sigma(l))}{w(\sigma(h))}$$

$$C \geq \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |\tilde{R}(h,l)|d\mu(l) \quad \text{and} \quad C \geq \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |\tilde{R}(h,l)|d\mu(h) \qquad (**)$$

$$(**) \Longrightarrow U(\sigma(h)^{-1})\psi \in H_{1,w} \Longrightarrow \quad V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle, \qquad f \in H'_{1,w}!$$

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coorbit spaces:

$$M_{p,w} := \{ f \in H'_{1,w} : \ V_\psi f \in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \}, \quad \|f\|_{M_{p,w}} := \|V_\psi f\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}$$

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$$\mathcal{M}_{p,w} := \{F \in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \ : \ \langle F, R(h,\cdot)\rangle = F\}$$

Proposition 2. *(Correspondence Principle) Let U be a strictly square integrable representation of $\mathcal{G} \bmod (\mathcal{G}_{\mathcal{F}}, \sigma) \Longrightarrow M_{p,w}$ and $\mathcal{M}_{p,w}$ are isometrically isomorphic, i.e., $\forall f \in M_{p,w}$*

$$\langle V_\psi(f), R(h,\cdot)\rangle = V_\psi(f)$$

Moreover: $M_{\infty,w} = H'_{1,w}$.

3. Banach Frames

$e \in \mathcal{U} \subset \mathcal{G}, \quad X = (x_i)_{i \in \mathcal{I}}$ is called $\mathcal{U}$-dense if $\displaystyle\bigcup_{i \in \mathcal{I}} \mathcal{U} x_i = \mathcal{G}$

$\mathcal{I}_\sigma := \{ i \in \mathcal{I} : \ \sigma(\mathcal{G}/\mathcal{G}_{\mathcal{F}}) \cap \mathcal{U} x_i \neq \emptyset \}, \quad x_i = \sigma(h_i)$

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$$\mathrm{osc}^l_\mathcal{U}(l,h) := \sup_{u \in \mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi \rangle|$$

$$\mathrm{osc}^r_\mathcal{U}(l,h) := \sup_{u \in \mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(\sigma(l)\sigma(h)^{-1}u)\psi \rangle|$$

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Decomposition:

Theorem 2. *Let U be a strictly square integrable representation of $\mathcal{G} \bmod (\mathcal{G}_\mathcal{F}, \sigma)$ in $L_2(\mathcal{N})$. Choose $\mathcal{U}$ so small that*

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \operatorname{osc}^l_{\mathcal{U},w}(l,h)\,d\mu(l) < 1 \text{ and } \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \operatorname{osc}^l_{\mathcal{U},w}(l,h)\,d\mu(h) < 1 \qquad (C)\,.$$

Technical conditions $\Longrightarrow$ any $f \in M_{p,w}$, $1 \leq p \leq \infty$, has an expansion

$$f = \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi,$$

where

$$\frac{1}{B}\|f\|_{M_{p,w}} \leq \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \leq \frac{1}{A}\|f\|_{M_{p,w}}.$$

Idea of Proof:

Atomic decomposition:

$$T_\varphi F(h) := \sum_{i\in\mathcal{I}_\sigma} \langle F, \varphi_i \circ \sigma \rangle \, R(h_i, h), \quad (\varphi_i)_{i\in\mathcal{I}} \quad \text{partition of unity}$$

$(C) \Longrightarrow T_\varphi$ invertible on $\mathcal{M}_{p,w}$,

$$F(h) = T_\varphi T_\varphi^{-1} F(h) = \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma \rangle R(h_i, h)$$

correspondence principle $\Longrightarrow$

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i \, U(\sigma(h_i)^{-1})\psi \ , \qquad c_i = c_i(f) := \langle T_\varphi^{-1} V_\psi(f), \varphi_i \circ \sigma \rangle.$$

frame bounds:

$$\|f\|_{M_{p,w}} \leq B\,\|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} = B\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}}$$

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direct check:

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_{1,w}} = \|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\|_{L_{1,\frac{1}{w}}(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leq C_1\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{1,\frac{1}{w}}}$$

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_{\infty,w}} = \|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{G}_\mathcal{F})} \leq C_2\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{\infty,\frac{1}{w}}}$$

frame bounds:

$$\|f\|_{M_{p,w}} \leq B \, \|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma \rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} = B\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}}$$

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complex interpolation $\Longrightarrow$

$$(L_{1,w}, L_{\infty,w})_{[\theta]} = L_{p,w} \quad \text{and} \quad (\ell_{1,w}, \ell_{\infty,w})_{[\theta]} = \ell_{p,w}, \qquad \frac{1}{p} = 1-\theta$$

Riesz–Thorin Interpolation Theorem $\Longrightarrow$

$$\|f\|_{M_{p,w}} \leq B \, \|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma \rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}}, \qquad 1 \leq p \leq \infty$$

Reconstruction:

Theorem 3. *Impose the same assumptions as in Theorem 2 with* $\mathrm{osc}^r_{\mathcal{U},w}(l,h)$ *instead of* $\mathrm{osc}^l_{\mathcal{U},w}(l,h)$. *Then the set*

$$\{\psi_i := U(\sigma(h_i)^{-1})\psi : i \in \mathcal{I}_\sigma\}$$

is a Banach frame for $M_{p,w}$. *This means that*

(i) $f \in M_{p,w}$ *if and only if* $(\langle f, \psi_i \rangle)_{i\in\mathcal{I}_\sigma} \in \ell_{p,\frac{1}{w}}$;

(ii) there exist two constants $0 < A' \le B' < \infty$ *such that*

$$A' \, \|f\|_{M_{p,w}} \le \|(\langle f, \psi_i \rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \le B' \, \|f\|_{M_{p,w}};$$

(iii) there exists a bounded, linear reconstruction operator S *from* $\ell_{p,\frac{1}{w}}$ *to* $M_{p,w}$ *such that* $S\left((\langle f, \psi_i \rangle)_{i\in\mathcal{I}_\sigma}\right) = f$.

4. Nonlinear Approximation

Quality of approximation schemes based on Banach frames? Especially, best N–term approximation

$$\Sigma_n := \{S \in M_p : \; S = \sum_{i \in J} a_i \psi_i, \; J \subseteq \mathcal{I}_\sigma, \; \mathrm{card} J \leq n\}$$

$$E_n(f)_{M_p} := \inf_{S \in \Sigma_n} \|f - S\|_{M_p} \; .$$

- wavelet transform: [DeVore/Jawerth/Popov] $\rightsquigarrow$ Besov spaces
- Gabor transform: [Gröchenig/Samarah] $\rightsquigarrow$ modulation spaces

Theorem 4. *Let* $\{\psi_i : \; i \in \mathcal{I}_\sigma\}$ *be a Banach frame for* M_p, $1 \leq p \leq \infty$. *If* $1 \leq p < q$, $\alpha := 1/p - 1/q$ *and* $f \in M_p$, *then*

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(n^\alpha E_n(f)_{M_q} \right)^p \right)^{1/p} \leq c \|f\|_{M_p}, \qquad \textit{where} \quad c < \infty.$$

5. Applications

5.1 Gabor Frames on the Spheres

Aims:

- construction of a windowed Fourier transform on spheres [B. Torresani]
- modulations spaces
- associated Banach frames

$$\mathcal{G} := E(n) = SO(n) \ltimes \mathbb{R}^n$$

$$(R,p) \circ (\tilde{R},\tilde{p}) = (R\tilde{R}, R\tilde{p} + p)$$

$$(R,p)^{-1} = (R^{-1}, -R^{-1}p)$$

$$(U(R,p))f(s) := e^{i<s,p>} f(R^{-1}s) \ , \quad s \in S^{n-1}$$

not square integrable!

$S^1: \quad \mathcal{G}_{\mathcal{F}} \cong \{(0,0,p_2) \in \mathcal{G}\}, \quad \Pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\mathcal{F}}$

flat section $\sigma(\theta, p_1) = (\theta, p_1, 0)$

Lemma 1. [B. Torresani] $\operatorname{supp} \psi \subset [-\pi/2, \pi/2]$ *and*

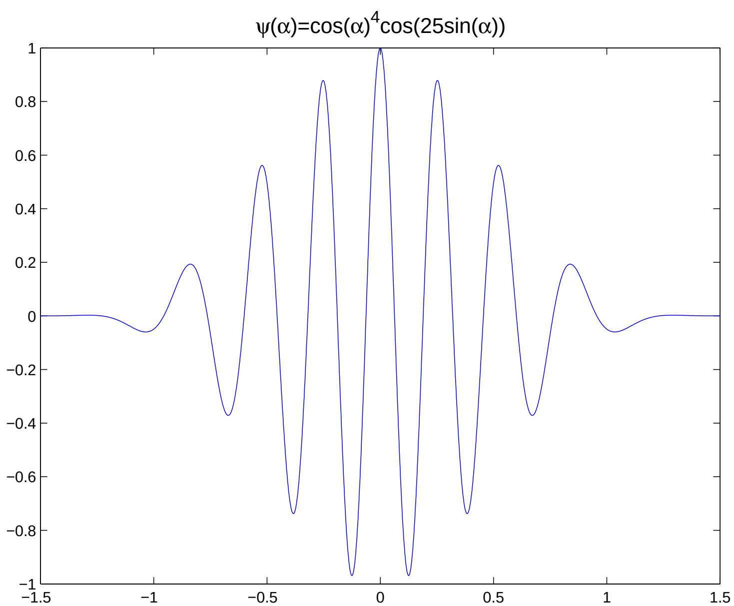
$$2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\alpha)|^2}{\cos \alpha} d\alpha = 1 \implies$$

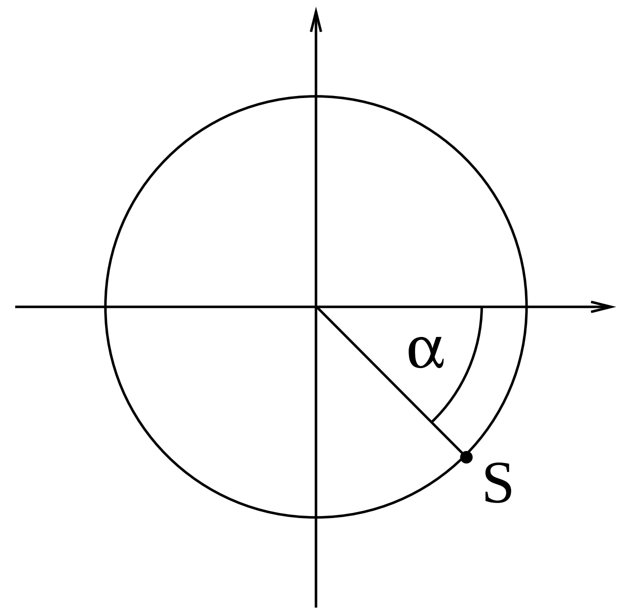
$$L_2(S^1) \ni g \mapsto V_\psi g \in L_2(\mathcal{G}/\mathcal{G}_F)$$

is an isometry.

Example:

$$\psi(\alpha) = \cos^4(\alpha) \cdot \chi_{[-\pi/2,\pi/2]}(\alpha)$$





- so far: Banach frames for generalized modulation spaces M_p established!
- $M_{p,w}$ is under construction

5.2 Mixed Smoothness Spaces

Aim:

- mixed forms of Besov and modulation spaces
- affine Weyl–Heisenberg group

$$\mathcal{G}_{aWH} = \mathcal{G}_{WH}^{red} \ltimes I\!R_+$$

$$(p, q, a, \phi) \circ (p', q', a', \phi') = p + ap', q + a^{-1}q', aa', \phi + \phi' + qap')$$

$$U(p, q, a, 0)f(x) = a^{-1/2} e^{iq(x-p)} f\left(\frac{x-p}{a}\right)$$

representations of $\mathcal{G}_{aWH}$ not square integrable [Kalisa/Torresani, 1993]!

- way out: use quotients!

$$\mathcal{G}_{\mathcal{F}} := \{(0,0,a,\phi) \in \mathcal{G}_{aWH}\}, \qquad \sigma(p,q) = (p,q,\beta(p,q),0\}$$

square integrable

$\mathcal{G}_{WH}^{red}$: modulation spaces

affine group : Besov spaces

$(\mathcal{G}_{aWH}/\mathcal{G}_{\mathcal{F}}, \sigma)$ something in between!

Perspectives:

- numerical schemes, higher–dimensional spheres
- wavelet frames, generalized Besov spaces on spheres
- more general manifolds
- mixed smoothness spaces

Summary

- Banach frames for coorbit spaces
- integrability modulo subgroups
- generalized reproducing kernels, generalized coorbit spaces
- (weighted) $\mathcal{U}$–oscillation $\Longrightarrow$ Banach frames
- applications to spheres