Besov Regularity for Elliptic Boundary Value Problems

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1. Motivation
2. Adaptivity: Theoretical Analysis
3. Besov Regularity

1. Motivation

- Numerical treatment of elliptic boundary value problems

$$
\begin{array}{rclll}
L(u) & = & F & \text{in} & \Omega \subset \mathbb{R}^d \quad \text{Lipschitz} \\
B(u) & = & 0 & \text{on} & \partial\Omega
\end{array}
$$

- weak formulation

$$a(u,v) = \langle F, v \rangle \qquad \text{for all} \quad v \in H_B^m(\Omega) \subset H^m(\Omega)$$

- example:

$$
\begin{array}{rclll}
-\triangle u & = & F & \text{in} & \Omega \subset \mathbb{R}^d \quad \text{Lipschitz} \\
u & = & 0 & \text{on} & \partial\Omega
\end{array}
$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad H_B^m(\Omega) = H_0^1(\Omega)$$

- Galerkin approach

$$\dots S_{j-1} \subset S_j \subset S_{j+1} \subset \dots,$$

$$S_j = \text{span}\{\zeta_1, \dots, \zeta_{n_j}\}, \qquad \text{dim} S_j = n_j < \infty$$

$$a(u_j, v) = \langle F, v \rangle \quad \text{for all} \quad v \in S_j$$

$$\mathbf{A}_j \mathbf{c}_j = \mathbf{F}_j, \qquad \mathbf{A}_j = (a(\zeta_{l'}, \zeta_l))_{l,l'=1,\dots,n_j}$$

$$\mathbf{F}_j = (\langle F, \zeta_l \rangle)_{l=1}^{n_j}$$

$$u_j = \sum_{l=1}^{n_j} (\mathbf{c}_j)_l \zeta_l$$

finite elements, wavelets...

- adaptive methods: updating strategy

  - S_j depends on u

  - self–regulating

Question : Do we gain efficiency?

2. Adaptivity: Theoretical Analysis

- wavelets

multiresolution analysis $\{V_j\}_{j\geq 0}$

$$V_0 \subset V_1 \subset V_2 \subset \ldots \qquad \overline{\bigcup_{j=0}^{\infty} V_j} = L_2(\Omega)$$

$$V_j = \overline{\text{span}\{\varphi_{j,k},\ k \in I_j\}}$$

$$V_{j+1} = V_j \oplus W_{j+1} \quad V_0 = W_0 \quad L_2(\Omega) = \oplus_{j=0}^{\infty} W_j$$

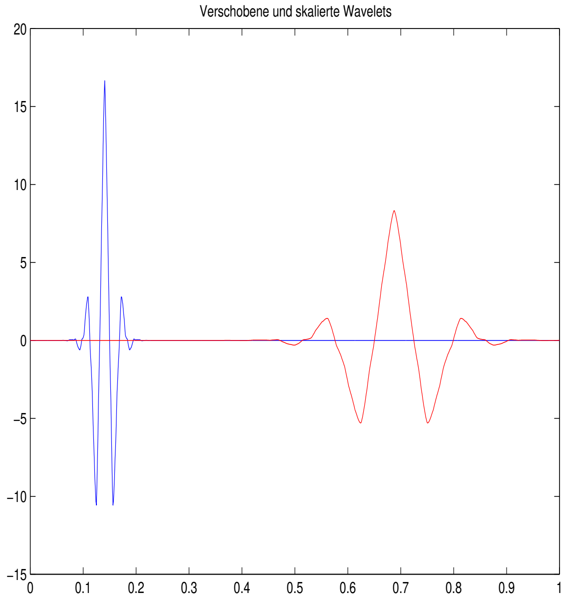
$$W_j = \overline{\text{span}\{\psi_{j,k},\ k \in J_j\}},$$

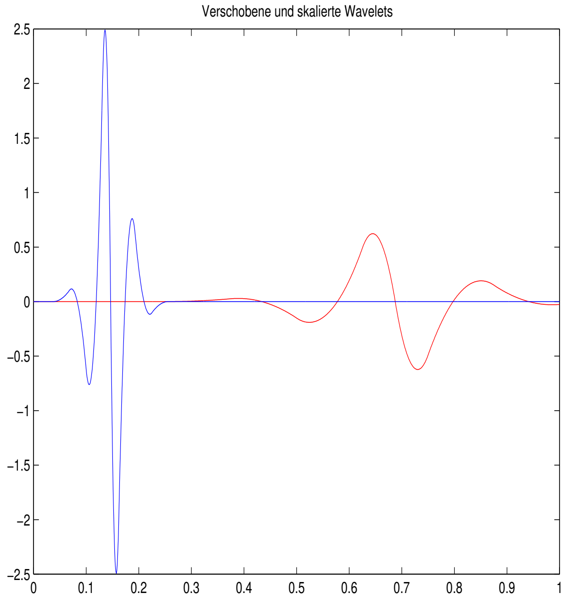
(L) $\quad \text{supp}\psi_{j,k} \subset Q_{j,k}, \quad |Q_{j,k}| \lesssim 2^{-jd}$

(CP) $\quad \displaystyle\int_\Omega x^\gamma \psi_{j,k}(x)dx = 0, \qquad |\gamma| \leq N$

$\Omega = I\!R$:

$\psi_{j+1,k}(x) = 2^{j/2}\psi(2^j x - k), \quad \varphi_{j,k}(x) = \varphi(2^j x - k)$





- nonadaptive schemes

$$E_j(f) = \inf_{g \in V_j} \|f - g\|_{L_p(\Omega)} \lesssim 2^{-sj} |f|_{W^s(L_p(\Omega))}$$

$$E_j(f) = \mathcal{O}(n_j^{-s/d}) \Longleftarrow f \in W^s(L_p(\Omega))$$

- adaptive schemes

  "ideal" algorithm: best n–term approximation

$$\mathcal{M}_n := \{g = \sum_{(j,k)\in\Lambda} c_{j,k}\psi_{j,k} \mid |\Lambda| = n\}$$

$$\sigma_n(f)_{L_p(\Omega)} := \inf_{g \in \mathcal{M}_n} \|f - g\|_{L_p(\Omega)} \sim \|f - g_n\|_{L_p(\Omega)}$$

$$g_n = \sum_{(j,k)\in\Lambda_n} d_{j,k}\psi_{j,k}, \quad \Lambda_n \mathrel{\widehat{=}} n \text{ biggest wavelet coefficients}$$

$$\sigma_n(f)_{L_p(\Omega)} = \mathcal{O}(n^{-s/d}) \Longleftarrow f \in B^s_\tau(L_\tau(\Omega)) \quad \frac{1}{\tau} = \left(\frac{s}{d} + \frac{1}{p}\right)$$

Question: $u \in B^s_\tau(L_\tau(\Omega)),\ 0 < s < s^*$?

3 model problems

Dirichlet problem

$$
\begin{aligned}
\triangle v &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz,} \\
v &= h \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Neumann problem

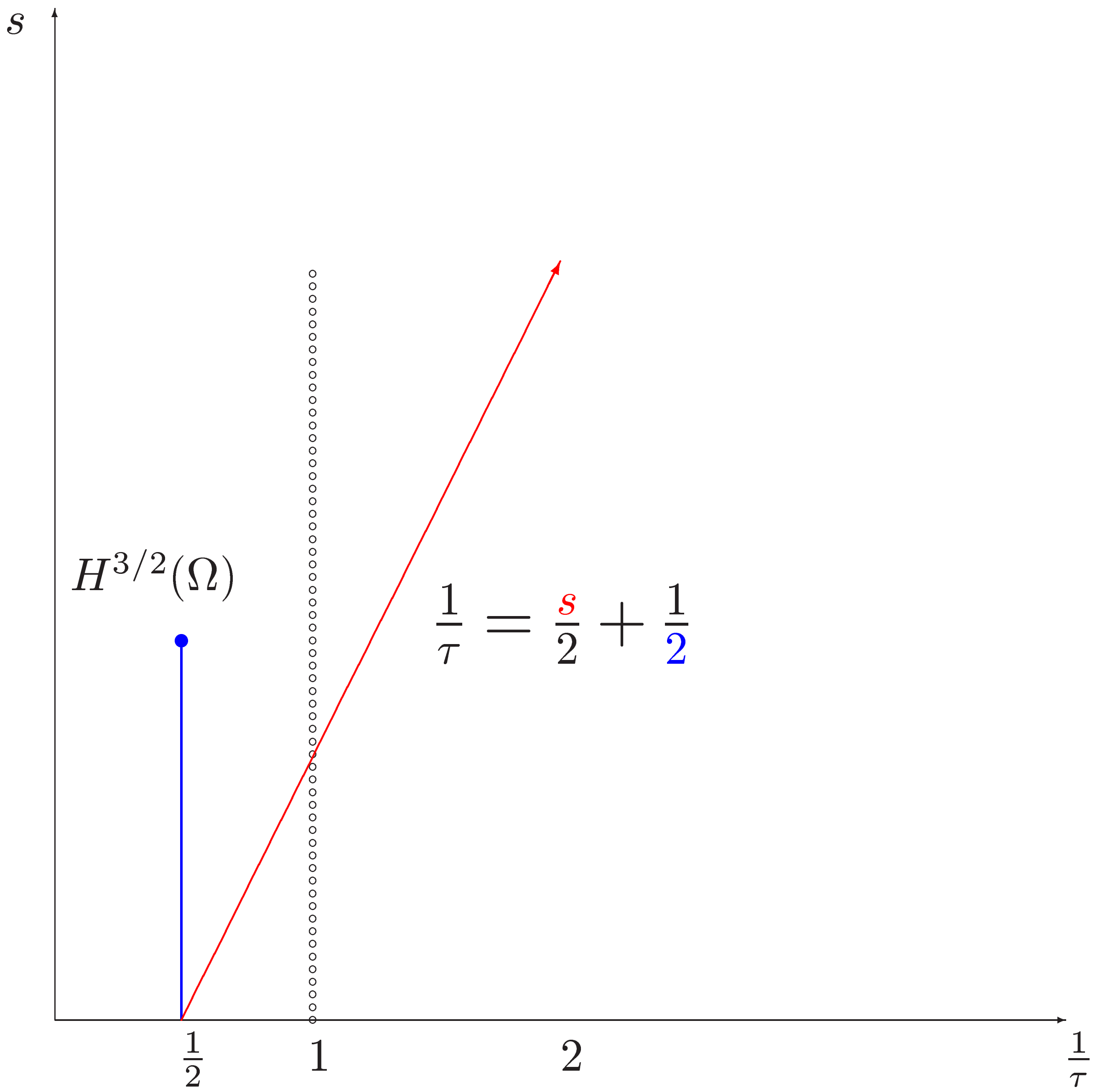
$$
\begin{aligned}
\triangle v &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz,} \\
\frac{\partial v}{\partial n} &= g \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

inhomogeneous Neumann problem

$$
\begin{aligned}
-\triangle w &= F \quad \text{in} \quad \Omega \subset \mathbb{R}^d \quad \text{Lipschitz,} \\
w &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

$F \in L_2(\Omega) \Longrightarrow w \in H^{3/2}(\Omega)$!

The DeVore Diagram, $d = 2$



3. Besov Regularity

Theorem 1. [DeVore/D.] *$\mathbb{R}^d \supset \Omega$ bounded, Lipschitz, $1 < p_0 < p < p_0'$, $h \in B_p^\alpha(L_p(\partial\Omega)), 0 < \alpha < 1$,*

$$\left.\begin{aligned} \triangle v &= 0, \\ v|_{\partial\Omega} &= h \end{aligned}\right\} \Longrightarrow v \in B_\tau^s(L_\tau(\Omega)), \ s < \frac{d(\alpha + 1/p)}{(d-1)}$$

$1/\tau = (s/d + 1/p)$

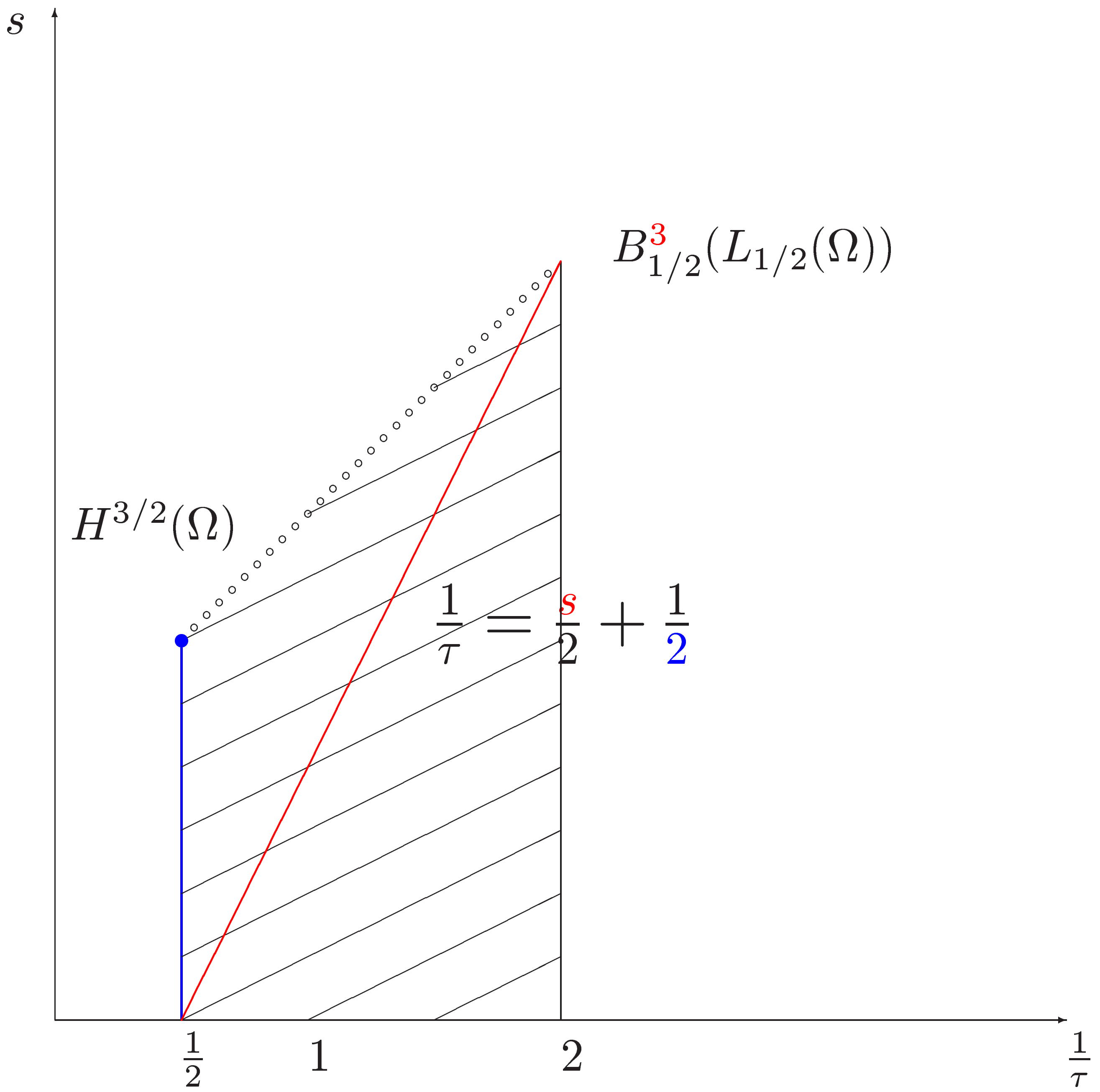
Proof:

Theorem 2. [Jerison/Kenig] *same conditions $\Longrightarrow$ $v \in B_p^{\alpha+1/p}(L_p(\Omega))$*

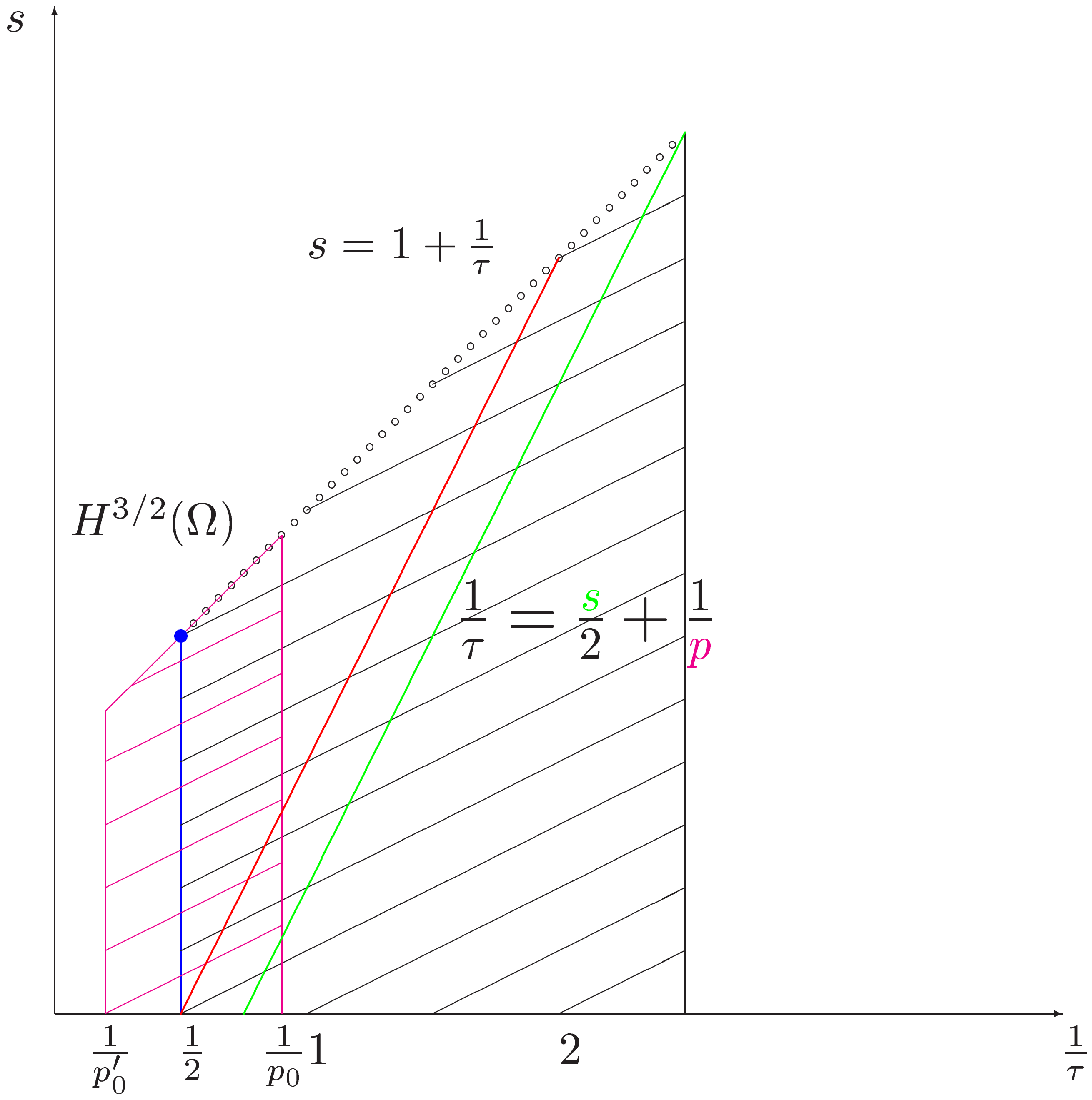
Theorem 3. [DeVore/D.] *$\mathbb{R}^d \supset \Omega$ bounded, Lipschitz, $1 < p < \infty$, v harmonic, $v \in B_p^\beta(L_p(\Omega)) \Longrightarrow$*

$$v \in B_\tau^s(L_\tau(\Omega)), \ 0 < s < \frac{\beta d}{(d-1)}, \ \frac{1}{\tau} = \left(\frac{s}{d} + \frac{1}{p}\right)$$

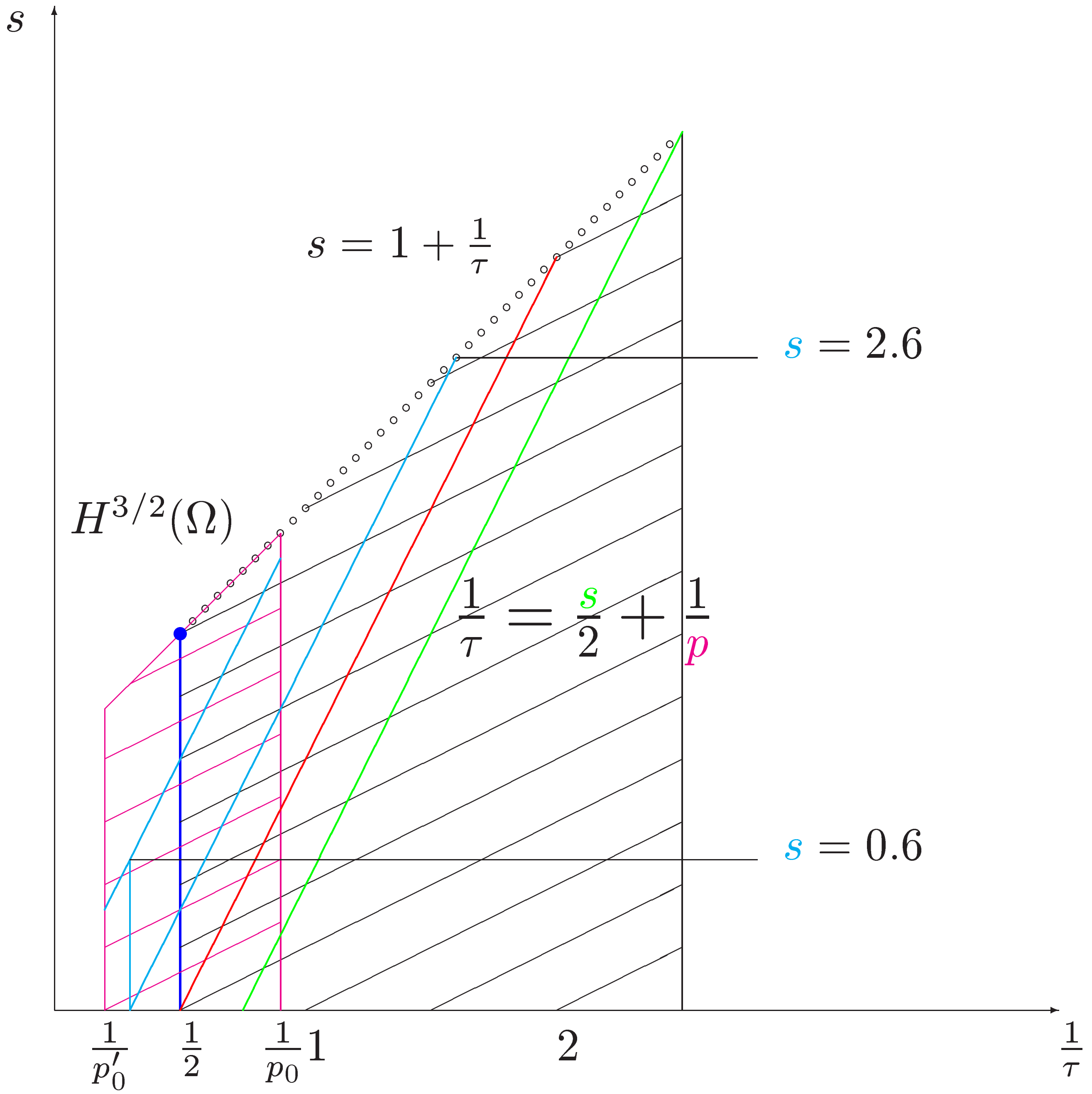
The DeVore Diagram



Regularity spaces for the Dirichlet problem,
$h \in B^1_2(L_2(\partial\Omega)),\ v \in B^{3/2}_2(L_2(\Omega)),\ \Omega \subset \mathbb{R}^2.$



Regularity spaces for the Dirichlet problem, $p_0 < p < p_0'$



Regularity spaces for the Poisson equation,
$F \in B_p^{\mu-2}(L_p(\Omega))$, $\mu - 2 = 2/p$

further results:

- general Dirichlet problems
- variable coefficients
- special domains
- noncontinuous coefficients
- Stokes problem...

Theorem 4. $\mathbb{R}^d \supset \Omega$ *bounded, Lipschitz,* $1 < p_0 < p < p_0'$, $g \in B_p^{\alpha-1}(L_p(\partial\Omega)), 0 < \alpha < 1,$

$$\left.\begin{aligned} \triangle v &= 0, \\ \frac{\partial v}{\partial n}|_{\partial\Omega} &= g \end{aligned}\right\} \Longrightarrow v \in B_\tau^s(L_\tau(\Omega)),\ s < \frac{d(\alpha + 1/p)}{(d-1)}$$

$$1/\tau = (s/d + 1/p)$$

Proof:
D. Zanger (2000): systematic study of the Neumann problem, idea: use Theorem 2!

$$\Gamma_a(Q) := \{x \in \Omega \mid |X - Q| < (1+a)\mathrm{dist}(X, \partial\Omega)\}$$

$$M(u)(Q) := \sup\{|u(P)| \mid P \in \Gamma_1(Q)\}$$

$$\mathcal{S}h(X) := -\frac{1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{h(Q)}{|X-Q|^{d-2}} dQ$$

$$\mathcal{K}^* h(Q) := \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle P - Q, n(P)\rangle}{|P-Q|^d} h(Q) dQ$$

$$\mathcal{T} := \frac{1}{2} I - \mathcal{K}^*$$

$$L_p(\partial\Omega)_{1^\perp} := \{h \in L_p(\partial\Omega) \mid \int_{\partial\Omega} h(Q)dQ = 0\}$$

Theorem 5. [Dahlberg/Kenig/Verchota...]
$1 < p < 2+\epsilon :\ \mathcal{T} :\ L_p(\partial\Omega)_{1^\perp} \longrightarrow L_p(\partial\Omega)_{1^\perp}$, *and*

$$v(X) = \frac{1}{\omega_d(d-2)} \int_{\partial\Omega} |X-Q|^{2-d} (\frac{1}{2}I - \mathcal{K}^*)^{-1}(g)(Q)dQ$$
$$= (\mathcal{S}\mathcal{T}^{-1}g)(X)$$

is harmonic and satisfies

- $\|M(\nabla v)\|_{L_p(\partial\Omega)},\ \|M(v)\|_{L_p(\partial\Omega)} \leq C$
- $\dfrac{\partial v}{\partial n} = g$ *nontangentially a.e.*

inverse Calderón operator Υ:

$$\Upsilon :\ L_p(\partial\Omega)_{1^\perp} \longrightarrow W^1(L_p(\partial\Omega))$$
$$h \longrightarrow \Upsilon(h) := (\mathcal{S} \circ \mathcal{T}^{-1})(h)|_{\partial\Omega}$$

Theorem 6. [Zanger] $p_0 < p < p_0',\ 0 < \alpha < 1, \Longrightarrow$

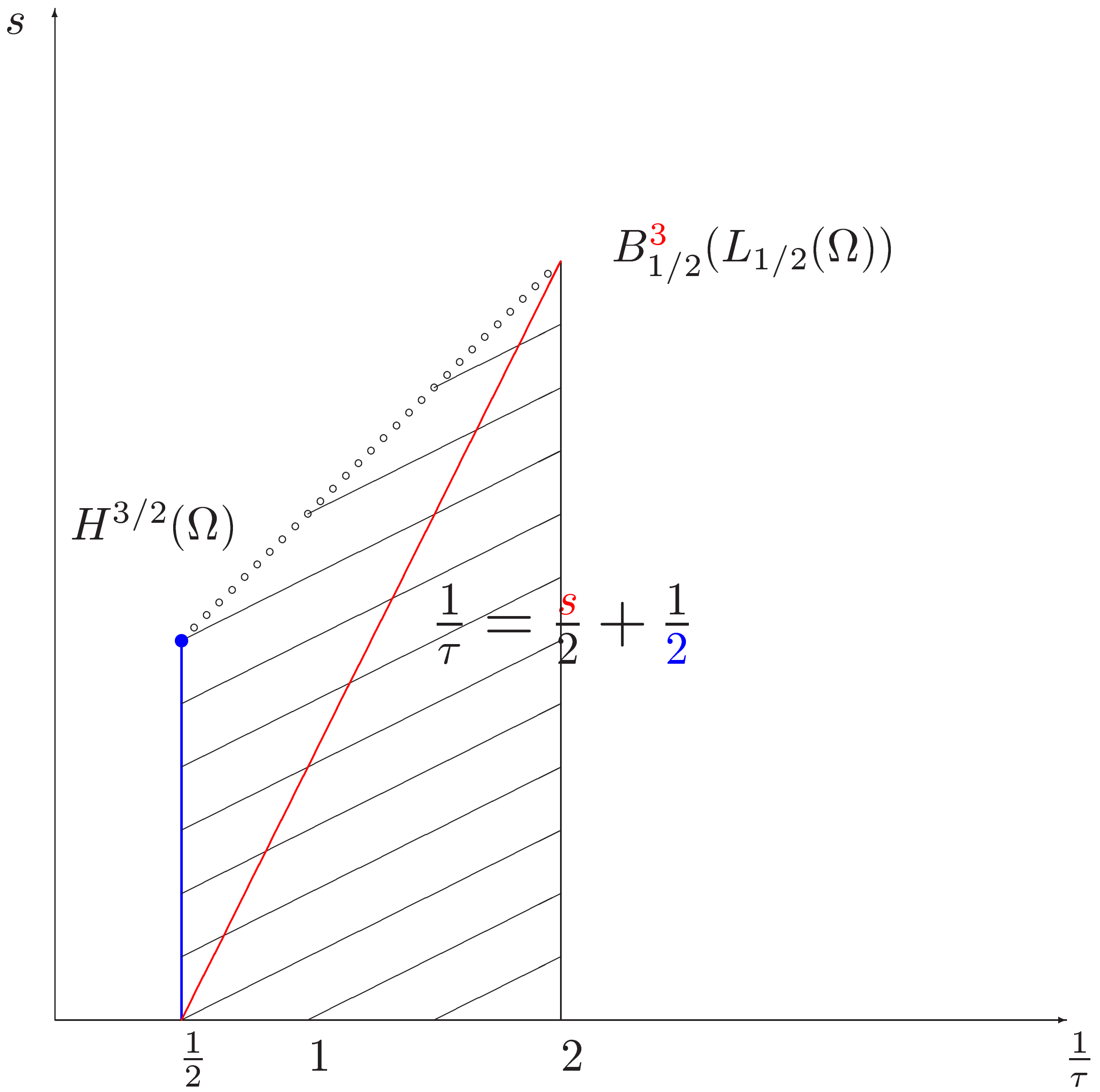
$$\|\Upsilon h\|_{B_p^\alpha(L_p(\partial\Omega))} \leq C\|h\|_{B_p^{\alpha-1}(L_p(\partial\Omega))}$$

Theorem 6 + Theorem 2 $\Longrightarrow v \in B_p^{\alpha+1/p}(L_p(\Omega))$,
Theorem 3 $\Longrightarrow$

$$v \in B_\tau^s(L_\tau(\Omega)), \quad s < \frac{d(\alpha+1/p)}{(d-1)}, \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}$$

$\square$

The DeVore Diagram



Regularity spaces for the Dirichlet problem,
$g \in L_2(\partial\Omega),\ v \in B_2^{3/2}(L_2(\Omega)),\ \Omega \subset \mathbb{R}^2.$

$$(B^{2-\mu}_{p'}(L_{p'}(\Omega)))^*_{1\perp} = \left\{ f \in (B^{2-\mu}_{p'}(L_{p'}(\Omega)))^* \; | F(1) = 0 \right\}$$

Theorem 7. *$\mathbb{R}^d \supset \Omega$ Lipschitz, $p_0 < p < p_0'$ $\quad F \in (B^{2-\mu}_{p'}(L_{p'}(\Omega)))^*_{1\perp}$, $\mu > 1 + 1/p, 1/p + 1/p' = 1$,*

$$\left.\begin{aligned} \triangle w &= F, \\ \frac{\partial w}{\partial n}|_{\partial\Omega} &= 0 \end{aligned}\right\} \quad \Longrightarrow \quad w \in B^s_\tau(L_\tau(\Omega)),$$

$$0 < s < \min\left\{ \frac{d}{d-1}(1 + \frac{1}{p}), \mu \right\} \quad 1/\tau = s/d + 1/p$$

Proof:

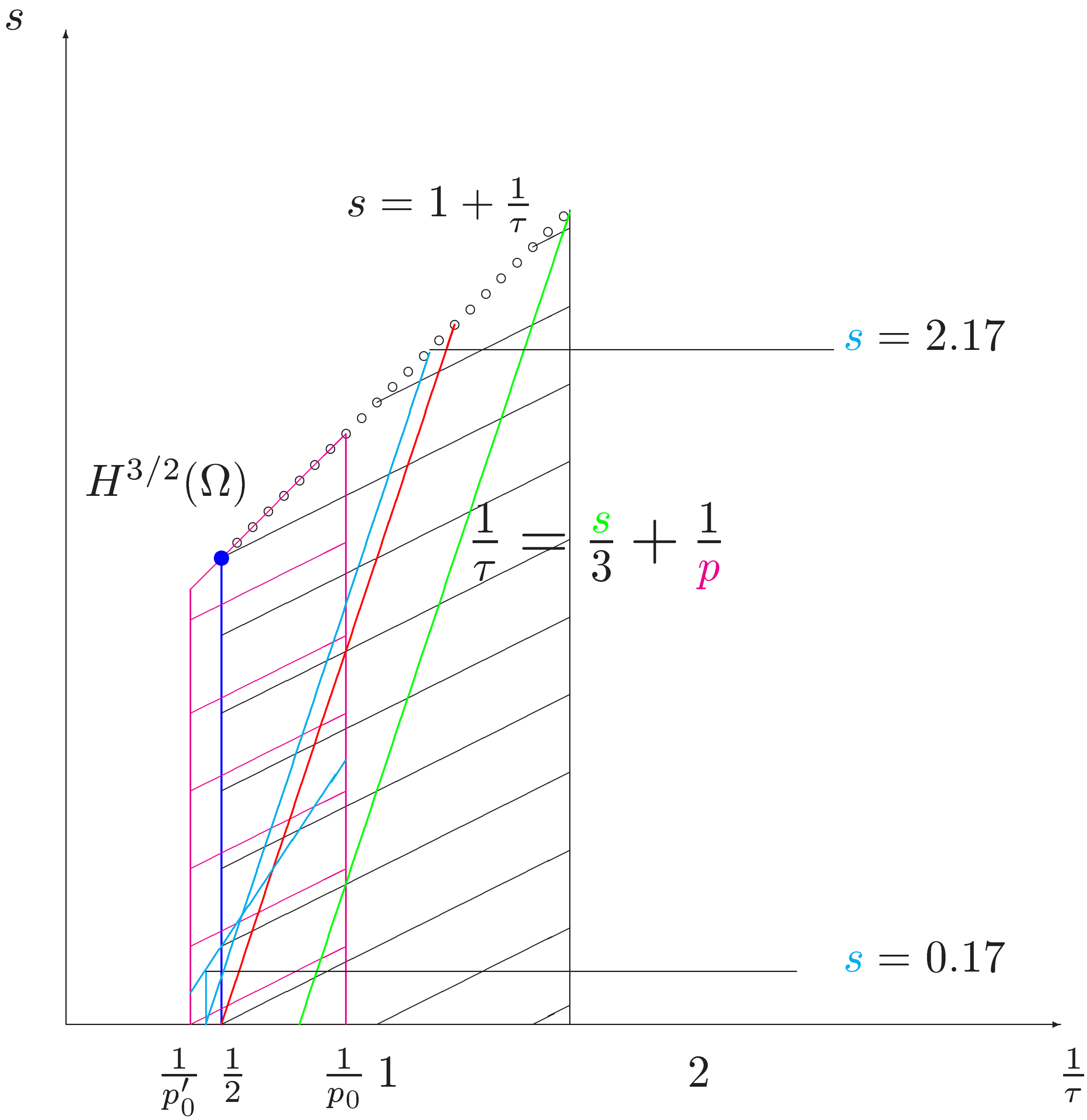
$$\begin{aligned} N(x) :&= C_d |x|^{2-d}, \quad R_\Omega \ \text{restriction operator} \\ \tilde{w} &= N * (R^*_\Omega F), \quad R^*_\Omega F \in B^{\mu-2}_p(L_p(\mathbb{R}^d)), \Longrightarrow \\ \triangle \tilde{w} &= \triangle N * (R^*_\Omega F) = R^*_\Omega F \end{aligned}$$

$$\begin{aligned} \triangle v &= 0 & \text{in} \quad \Omega \\ \frac{\partial v}{\partial n} &= \frac{\partial \tilde{w}}{\partial n} =: g & \text{on} \quad \partial\Omega \end{aligned}$$

$$w = \tilde{w} - v$$

$$
\begin{array}{rcl}
\text{elliptic regularity} & \Longrightarrow & \tilde{w} \in B^{\mu}_{p}(L_p(\mathbb{R}^d)), \\
 & \Longrightarrow & \tilde{w} \in B^{s}_{\tau}(L_\tau(\Omega)), \\
\text{Zanger} & \Longrightarrow & g \in B^{\beta-1}_{p}(L_p(\partial\Omega)),\ \beta < 1, \\
\text{Theorem 4} & \Longrightarrow & v \in B^{(d/(d-1))(\beta+1/p)}_{\tau}(L_\tau(\Omega))
\end{array}
$$

$\square$



Regularity spaces for the inhomogenous Neumann problem, $F \in (B_{p'}^{2-\mu}(L_{p'}(\Omega)))^*_{1^\perp}$, $\mu - 2 = 2/(3p) - 1/2$

Practical realization

convergent algorithms

W. Dahmen, R. Hochmuth, R. Schneider, and S. D., Stable Multiscale Bases and Local Error Estimation for Elliptic Problems, *Appl. Numer. Math.* **23(1)** (1997), 21–48.

asymptotically optimal algorithms

A. Cohen, W. Dahmen, and R. DeVore, Adaptive Wavelet Methods for Elliptic Operator Equations — Convergence Rates, *Math. Comp.* **70** (2001), 27–75.

Implementation (data structures etc.):

A. Barinka, T. Barsch, P. Charton, A. Cohen, W. Dahmen, K. Urban, and S. D., Adaptive Wavelet Schemes for Elliptic Problems — Implementation and Numerical Experiments, IGPM–Preprint No. 173, RWTH Aachen, 1999, to appear in: *SIAM J. Scientific Comput.*

Summary

- adaptive numerical schemes for elliptic PDE's
- theoretical analysis
- Besov regularity
  - $u \in B^s_\tau(L_\tau(\Omega)), \quad 1/\tau = s/d + 1/p,\ 0 < s < s^*$?
  - model problems, new regularity theorems, s^* sufficiently large