. . .

Wavelets: History and Theory

Stephan Dahlke
University of Bremen
FB 3, ZeTeM
P.O. Box 330440
28359 Bremen
http://www.math.uni-bremen.de/∼dahlke

1. Motivation
2. History
3. Multiresolution Analysis
4. Construction of Wavelets
5. Properties of Wavelets

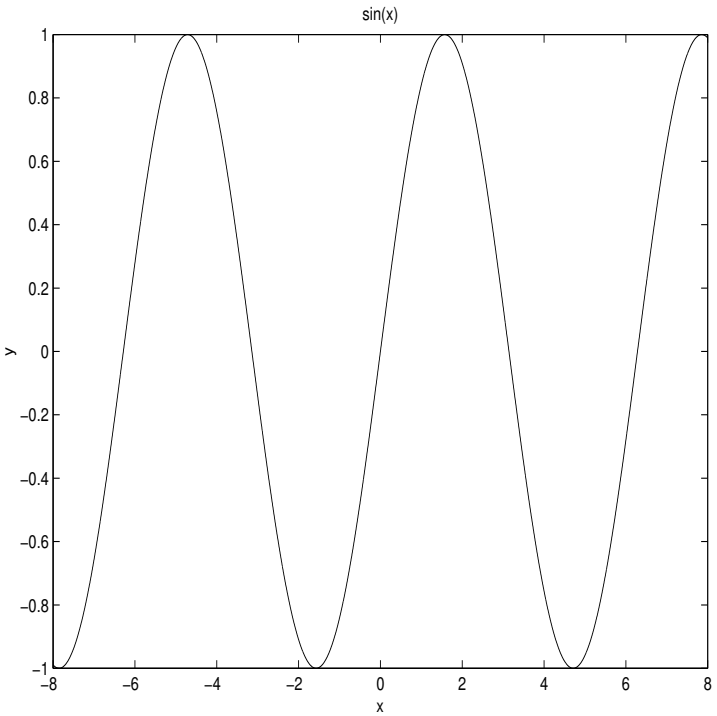
1 Motivation

representation/analysis/decomposition of signals
signal $\hat{=}$ function $y = f(x),\ x \in \mathbb{R}$
decomposition into suitable building blocks, "basis functions"
"classical" method: Fourier expansion,

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \quad c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

building blocks of the form



Successful in many cases!
Problem: basis functions not local!

Idea: construct basis functions by dilating and translating (one or several) "mother functions".

Wavelets: specific bases of the form

$$\psi_{i,j,k}(x) := 2^{jd/2}\psi_i(2^j x - k), \qquad i \in I,\ j \in \mathbb{Z},\ k \in \mathbb{Z},$$

supp $\psi_i := \{x \in \mathbb{R} \mid \psi(x) \neq 0\} \subset Q = [x_0, x_1] \Longrightarrow$

supp $\psi_{i,j,k} \subset 2^{-j}(Q + k)$

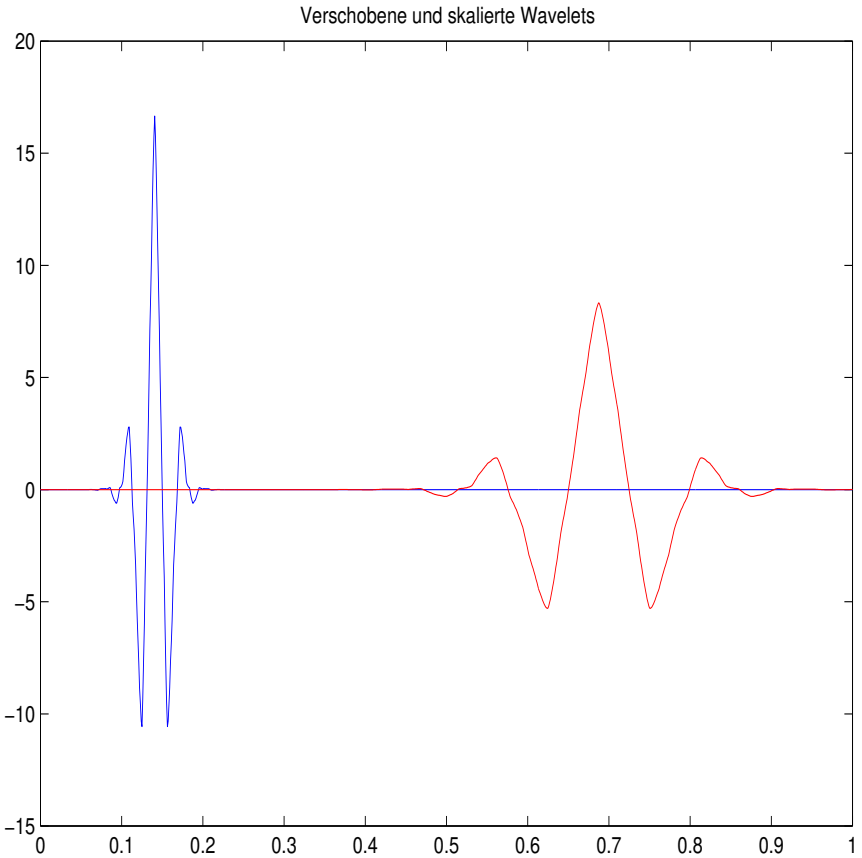
$\{\psi_i\}_{i \in I}$ mother wavelets

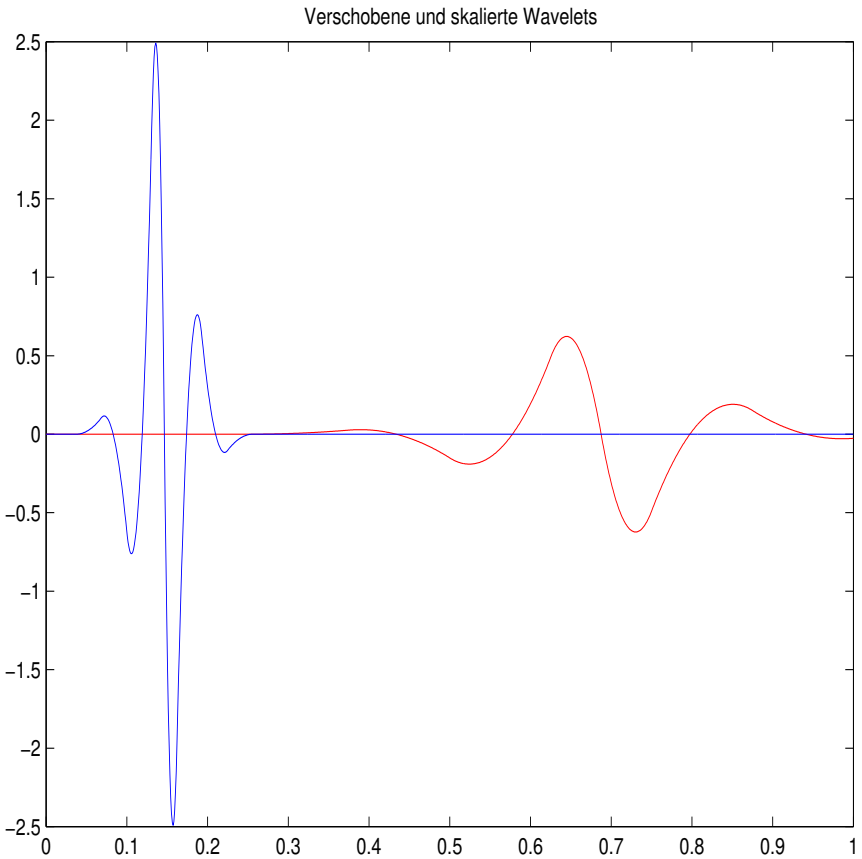
$$f(x) = \sum_{i \in I} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{i,j,k}\psi_{i,j,k}(x)$$

Wavelet representation is local !

$c_{i,j,k} \hat{=}$ behaviour of f on $2^{-j}(Q + k) \Longrightarrow$

higher scales j correspond to the resolution of finer details!


Verschobene und skalierte Wavelets
20
15
10
5
0
−5
−10
−15
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Verschobene und skalierte Wavelets
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

2 History

- time–frequency analysis, Gabor 1946
  windowed Fourier transform

$$C_{m,n} = \int_{I\!R} f(x) g(x - nx_0) e^{-im\omega_0 x} dx$$

- geophysics, Morlet 1982
  Idea: use dilations!

$$\begin{aligned} C_{j,k} &= \int_{I\!R} f(x) 2^{j/2} \overline{\psi(2^j x - k)} dx, \\ \hat{\psi}(\xi) &= \xi^2 e^{-\xi^2/2} \end{aligned}$$

The Fourier transform of f is the function

$$\hat{f}(\xi) \quad = \quad \int_{I\!R} f(x) e^{-ix\xi} dx$$

Questions: Morlet's approach well–defined, invertibility etc.

- mathematical physics, Marseille team, Grossmann/Paul 1984

$$U_{a,b} \quad : \quad \psi \longmapsto |a|^{-1/2} \psi\left(\frac{\cdot - b}{a}\right) \quad \text{coherent states}$$

$$(Wf)_{a,b} \quad :=: \quad |a|^{-1/2} \int_{I\!R} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx$$

$$= \quad \langle f, U_{a,b}(\psi) \rangle_{L_2}$$

$$L_2(I\!R) \quad = \quad \left\{ f \mid \int_{I\!R} |f(x)|^2 dx < \infty \right\}$$

$$\langle f, g \rangle_{L_2} \quad = \quad \int_{I\!R} f(x) \overline{g(x)} dx$$

$$\|f\|_{L_2} \quad = \quad \langle f, f \rangle_{L_2}^{1/2} = \left(\int_{I\!R} |f(x)|^2 dx \right)^{1/2}$$

$a = 2^{-j},\ b = 2^{-j}k \rightsquigarrow$ Morlet's approach!

"mathematical microscope"

If the admissibility condition

$$\int_{I\!R} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty$$

is satisfied, then W is an isometry from $L_2(I\!R)$ into $L_2(I\!R\backslash\{0\} \times I\!R, a^{-2}dadb)$. W is called continuous wavelet transform, ψ admissible wavelet, mother wavelet

$$f = c_\psi^{-1} \int_{I\!R} \int_{I\!R\backslash\{0\}} \langle f, U_{a,b}\psi \rangle U_{a,b}(\psi) \frac{dadb}{a^2}$$

$\{e_n\}_{n\in\mathbb{Z}}$ is called frame, if

$$A\|f\|_{L_2}^2 \le \sum_{n\in\mathbb{Z}} |\langle f, e_n\rangle_{L_2}|^2 \le B\|f\|_{L_2}^2$$

$$\left\{ |a_0|^{-m/2}\psi\left(\frac{\cdot - nb_0a_0^m}{a_0^m}\right)\right\}_{m,n\in\mathbb{Z}}$$ wavelet frame

- operator theory, Y. Meyer 1985

$$\begin{aligned}(Tf)(x) &= \int_{\mathbb{R}^d} K(x-y)f(y)dy\\ (\tau_a f)(x) &= f(ax)\\ \tau_{a^{-1}} T\tau_a &= T\\ K(ax) &= a^{-d}K(x)\end{aligned}$$

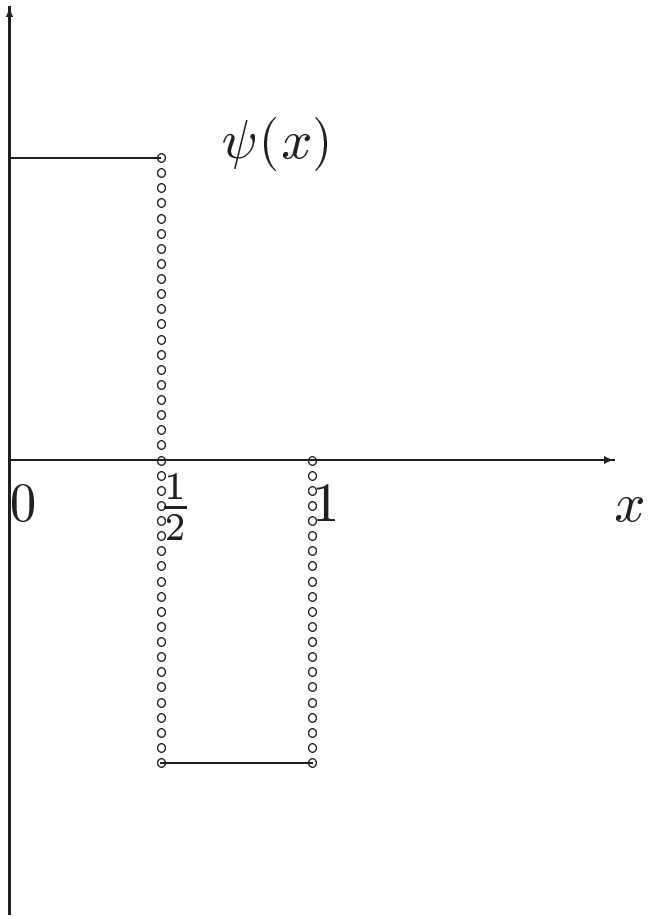
Question: does there exist $\psi \in L_2(\mathbb{R})$, ψ sufficiently smooth and localized, such that

$$\{2^{j/2}\psi(2^j\cdot -k)\}_{j,k\in\mathbb{Z}}$$

is an orthonormal basis of $L_2(\mathbb{R})$?

Example : Haar basis, A. Haar 1910

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{else} \end{cases}$$



ψ is localized, but not very smooth! Meyer's conjecture: Heisenberg uncertainty principle $\Longrightarrow$
ψ does not exist!
But : Meyer developed ψ localized, $\psi \in C^\infty$!
Question: higher dimensions etc.

- signal analysis, filter design, S. Mallat 1986

$$
\begin{aligned}
P_j(f) &= \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}, \quad \varphi_{j,k}(x) := 2^{j/2}\varphi(2^j x - k) \\
Q_j(f) &= \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} \\
P_j(f) &\longrightarrow 0, \quad j \longrightarrow -\infty \\
P_j(f) &\longrightarrow f, \quad j \longrightarrow \infty \\
Q_j &= P_j - P_{j-1}
\end{aligned}
$$

- P.–G. Lemarié 1988
  wavelets with exponential decay

- I. Daubechies 1988
  $\psi \in C^N$, compact support

- approximation theory, spline theory, R.-Q. Jia/ C.A. Micchelli 1991
  general construction method in $L_2(\mathbb{R}^d)$

- once again theoretical physics, J.P. Antoine 1992
  continuous Wavelet transform in $\mathbb{R}^d$

- A. Cohen/ I. Daubechies/ J.–C. Feauveau 1992
  biorthogonal wavelets

- A. Cohen/ I. Daubechies 1993
  wavelets for general scalings

- C. de Boor, R. DeVore, A. Ron 1993
  generalized construction principle

- $\sim$1995 wavelets on domains

- J.P. Antoine 1996
  continuous wavelet transform on spheres

Definition. *i)* $\{\psi_i\}_{i\in I}$, $\psi_i \in L_2(\mathbb{R}^d)$, *is called a family of (orthonormal mother) wavelets* $\Longleftrightarrow$

$$\psi_{i,j,k}(x) := 2^{dj/2}\psi_i(2^j x - k), \quad i \in I,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d, \tag{1}$$

constitute ONB of $L_2(\mathbb{R}^d)$,

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}. \tag{2}$$

The system

$$\Psi = \{\psi_{i,j,k},\ i \in I,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\} \tag{3}$$

is called wavelet basis.
ii) A system Ψ *of the form* (1),(3) *is called pre-wavelet basis* $\Longleftrightarrow$

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle = 0 \qquad \text{für } j \neq j', \tag{4}$$

ℓ_2*–stable for* $j = j'$, *i.e.,*

$$\begin{aligned} C_1 \sum_{i\in I} \|\{c_{i,j,k}\}_{k\in\mathbb{Z}^d}\|^2_{\ell_2} &\leq \|\sum_{i\in I,k\in\mathbb{Z}^d} c_{i,j,k}\psi_{i,j,k}\|^2_{L_2(\mathbb{R}^d)} \\ &\leq C_2 \sum_{i\in I} \|\{c_{i,j,k}\}_{k\in\mathbb{Z}^d}\|^2_{\ell_2} \end{aligned} \tag{5}$$

$$\|\{c_k\}_{k\in\mathbb{Z}^d}\|_{\ell_2} := (\sum_{k\in\mathbb{Z}^d} |c_k|^2)^{1/2} \tag{6}$$

iii) Two systems $\Psi, \tilde{\Psi}$ *of the form* (1),(3) *are called biorthogonal wavelet basis* $\Longleftrightarrow$

$$\langle \psi_{i,j,k}, \tilde{\psi}_{i',j',k'} \rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}. \tag{7}$$

Literature:

- Yves Meyer: *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics **37**, Cambridge 1992.

- Charles K. Chui: *An Introduction to Wavelets*, Academic Press, Boston 1992.

- Ingrid Daubechies: *Ten Lectures on Wavelets*, CBMS–NSF Regional Conference Series in Applied Mathematics, **61**, SIAM, Philadelphia 1992.

- Alfred Louis, Peter Maass, and Andreas Rieder, *Wavelets: Theorie und Anwendungen*, Teubner, Stuttgart, 1994.

- Gilbert Walter, *Wavelets and Other Orthogonal Systems with Applications*, CRC Press 1994.

- Albert Cohen and Robert Ryan, *Wavelets and Multiscale Signal Processing*, Chapman & Hall, 1995.

- Martin Vetterli and Jelena Kovačević, *Wavelets and Subband Coding*, Prentice Hall, 1995.

- Mathias Holschneider, *Wavelets: An Analysis Tool*, Oxford University Press, Oxford 1995

- Jean–Pierre Kahane and Pierre–Gilles Lemarié–Rieusset, *Fourier Series and Wavelets*, Gordon and Breach Science Publishers, Luxembourg, 1995.

- Mladen Wickerhauser, *Adaptive Wavelet Analysis*, Vieweg, 1996.

- Wolfgang Dahmen: *Wavelets and Multiscale Methods for Operator Equations*, Acta Numerica **6**, Cambridge University Press, Cambridge 1997.

- P. Wojtaszczyk: *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge 1997.

- Stephane Mallat, *A Wavelet Tour of Signal Processing*, Academic Press 1998.

- C. Blatter, *Wavelets – Eine Einführung*, Advanced Lectures in Mathematics, Vieweg, 1998.

Applications:

- signal/image processing, compression, denoising..
- numerical treatment of operator equations (preconditioning, matrix compression, adaptivity)
- geophysics
- meteorology
- astrophysics
- tomography, inverse problem...

2 Multiresolution Analysis

2.1 Definition and Examples

S. Mallat, motivation: signal analysis, mathematical model for a "scaling transform" T_r, mapping, which assigns to "signal" f with "finite energy", $f \in L_2(\mathbb{R})$, an "approximation" $T_r(f)$ with "resolution" r

reasonable requirements for T_r:

- $r = q^j,\ q = 2$
- T_{2^j} linear, $T_{2^j}^2(f) = T_{2^j}(f) \Longrightarrow$
  $T_{2^j} \longleftrightarrow$ projector onto $V_j \subset L_2(\mathbb{R})$
- causality principle, coarse resolutions determined by finer resolutions $\Longrightarrow V_j \subset V_{j+1}$
- no resolution is preferred $\Longrightarrow$
  $g(x) \in V_j \Longleftrightarrow g(2x) \in V_{j+1}$
- resolution 2^j requires 2^j data per unit length $\Longrightarrow \exists$ isomorphism $\mathcal{I} : V_0 \longrightarrow \ell_2(\mathbb{Z})$
- translation invariance, approximation of translated signals $\hat{=}$ translated approximation of the original signals, $\Longrightarrow$

$$T_0 f(x) = g(x) \Longleftrightarrow T_0(f(x-k)) = g(x-k)$$

$$\mathcal{I}(g(x)) = \{\lambda_l\}_{l\in\mathbb{Z}} \Longleftrightarrow \mathcal{I}(g(x-k)) = \{\lambda_{l-k}\}_{l\in\mathbb{Z}}$$

- $j \longrightarrow \infty \Longrightarrow T_{2^j} f \longrightarrow f$
  $j \longrightarrow -\infty \Longrightarrow T_{2^j} f \longrightarrow 0 \Longrightarrow$

$$\Longrightarrow \overline{\bigcup_{-\infty}^{\infty} V_j} = L_2(\mathbb{R}), \qquad \bigcap_{-\infty}^{\infty} V_j = \{0\}$$

Definition 1. *A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L_2(\mathbb{R}^d)$ is called a multiresolution analysis (M.R.A.) of $L_2(\mathbb{R}^d)$ if*

$$\subset V_{j-1} \subset V_j \subset V_{j+1} \subset \dots \tag{8}$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L_2(\mathbb{R}^d) \tag{9}$$

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\} \tag{10}$$

$$f(\cdot) \in V_j \Longleftrightarrow f(2\cdot) \in V_{j+1} \tag{11}$$

$$f(\cdot) \in V_0 \Longleftrightarrow f(\cdot - k) \in V_0 \qquad \forall k \in \mathbb{Z}^d \tag{12}$$

$$\exists\, \varphi \in V_0 \text{ s.t. } V_0 := \overline{\text{span}\{\varphi(\cdot - k),\, k \in \mathbb{Z}^d\}} \tag{13}$$

$$C_3||\lambda||_{\ell_2} \leq ||\sum_{k\in\mathbb{Z}^d} \lambda_k\, \varphi(\cdot - k)||_{L_2} \leq C_4||\lambda||_{\ell_2} \tag{14}$$

φ: generator of the multiresolution analysis

Aim : find a family $\{\psi_i\}_{i\in I}$ such that

$$\begin{aligned}
W_0 &= \overline{\operatorname{span}\{\psi_i(\cdot-k) \mid k\in\mathbb{Z}^d, i\in I\}}\\
V_1 &= V_0\oplus W_0,\ V_0\perp W_0,\quad \langle\psi_i(\cdot),\psi_{i'}(\cdot-k)\rangle=\delta_{i,i'}\delta_{0,k},\\
W_j &:= \{f(\cdot)\in L_2(\mathbb{R}^d)|f(2^{-j}\cdot)\in W_0\},\\
W_j &= \overline{\operatorname{span}\{2^{jd/2}\psi_i(2^j\cdot-k) \mid k\in\mathbb{Z}^d, i\in I\}}
\end{aligned}$$

(11), $f(x)\in V_j\Longleftrightarrow f(2x)\in V_{j+1}\Longrightarrow$

$$\begin{aligned}
V_{j+1} &= V_j\oplus W_j, \qquad\qquad V_j\perp W_j\\
V_{j+1} &= V_{j-1}\oplus W_{j-1}\oplus W_j\\
&\vdots \qquad\quad \vdots \qquad\quad \vdots\\
V_{j+1} &= V_{-J}\oplus W_{-J}\oplus W_{-J+1}\oplus\cdots\oplus W_j,
\end{aligned}$$

(9), (10), $\overline{\bigcup_{j=-\infty}^{\infty}V_j}=L_2(\mathbb{R}^d)$, $\bigcap_{j=-\infty}^{\infty}V_j=\{0\},\Rightarrow$
$L_2(\mathbb{R}^d)=\bigoplus_{j=-\infty}^{\infty}W_j\quad\Longrightarrow$

$$\psi_{i,j,k}(x)=2^{dj/2}\psi_i(2^jx-k),\quad i\in I, j\in\mathbb{Z}, k\in\mathbb{Z}^d,$$

is wavelet basis! Wavelets describe the detail information between scale j and scale $j+1$!

(11),(13) $\Longrightarrow$

$$V_j = \overline{\text{span}\{\varphi(2^j \cdot -k) \mid k \in \mathbb{Z}^d\}} \Longrightarrow$$

$$\psi_i(x) = \sum_{l \in \mathbb{Z}^d} b_l^i \varphi(2x - l)$$

Task: determine the coefficients b_l^i !

(8), (11), (13), (14) $\Longrightarrow$

$\varphi \in V_0 \subset V_1 = \overline{\text{span}\{\varphi(2 \cdot -k) \mid k \in \mathbb{Z}^d\}} \Longrightarrow$

φ is a refinable function,

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} a_k \varphi(2x - k), \quad \mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d),$$

$$a(z) = \sum_{k \in \mathbb{Z}^d} a_k z^k, \quad z = (z_1, \ldots, z_d), \ |z_i| = 1$$

$\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d}$ is called the mask of φ

$a(z)$ is called the symbol of φ

Fourier trafo, $z = e^{-i\xi} = (e^{-i\xi_1}, \ldots, e^{-i\xi_d}) \Longrightarrow$

$$\hat{\varphi}(\xi) = 2^{-d} a(e^{-i\xi/2}) \hat{\varphi}(\xi/2)$$

Algorithm to construct M.R.A.'s

- given $\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$;
- find φ such that

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} a_k \varphi(2x - k);$$

- define

$$V_0 = \overline{\text{span}\{\varphi(\cdot - k),\ k \in \mathbb{Z}^d\}};$$

- define

$$V_j := \{f \mid f(x) = g(2^j x),\ g \in V_0\}.$$

Example 1.

$$V_0 = \{f \in L_2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\pi, \pi]\}$$

$$f \in V_j \Longleftrightarrow f(x) = g(2^j x), \quad g \in V_0.$$

$\{V_j\}_{j \in \mathbb{Z}}$ is M.R.A.!

$\widehat{g(2^j \cdot)}(\xi) = 2^{-j} \hat{g}(2^{-j}\xi) \Longrightarrow$

$f \in V_j \Longleftrightarrow \operatorname{supp}\ \hat{f} \subset [-2^j\pi, 2^j\pi] \Longrightarrow$

$V_j \subset V_{j+1}$ etc.

Example 2. M.R.A. with cardinal B–splines

Definition 2.

$$N_m(x) := \int_0^1 N_{m-1}(x-t)dt$$

where

$$N_1(x) := \chi_{[0,1)}(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \textit{else} \end{cases}$$

is called cardinal B–spline of order m.

Theorem 1.

N_m *is a piecewise polynomial of degree* $m-1$;

$N_m \geq 0$;

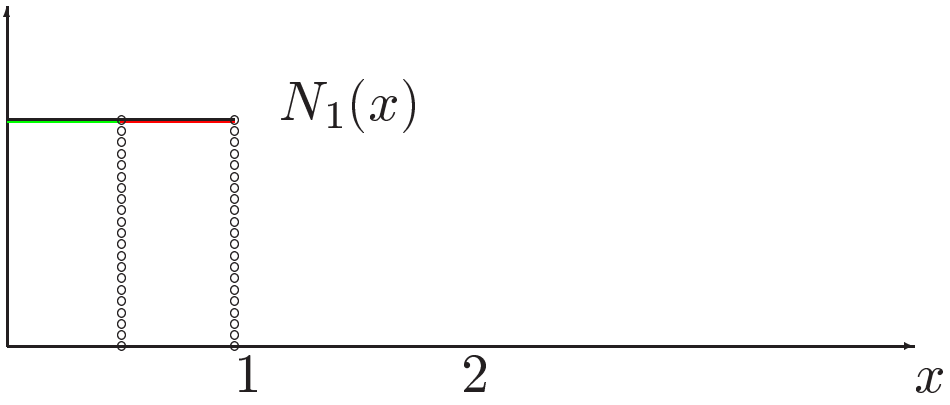
supp $N_m \subset [0, m]$;

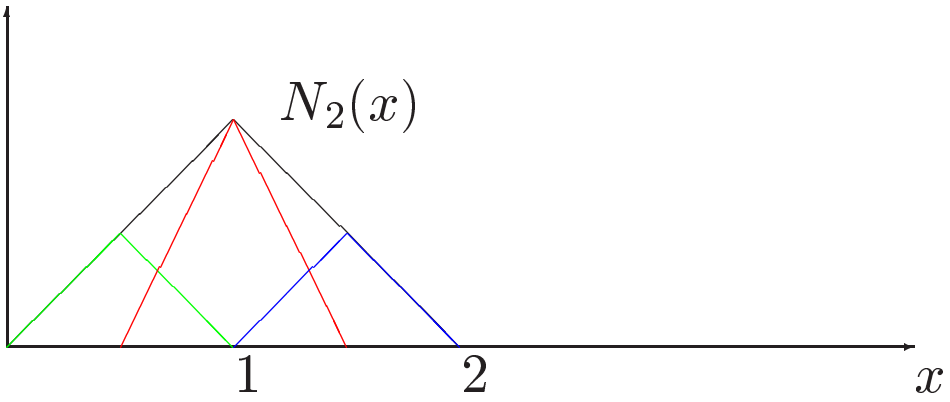
$N_m \in C^{m-2}$, $m \geq 2$;

$$N_m(x) = 2^{1-m} \sum_{k=0}^{m} \binom{m}{k} N_m(2x-k); \qquad (15)$$

$$a(z) = 2^{1-m}(1+z)^m. \qquad (16)$$

(15),(16) $\Longrightarrow$ N_m is a refinable function! N_m is the generator of a M.R.A.





2.2 The Spaces $\mathcal{L}_2$ and $\mathcal{E}_2$, Orthogonality

$$
\begin{aligned}
f^0 &:= \sum_{k\in\mathbb{Z}^d} |f(\cdot - k)| \\
|f|_2 &:= \|f^0\|_{L_2([0,1)^d)} = \left(\int_{[0,1)^d} |f^0|^2 dx\right)^{1/2} \\
\mathcal{L}_2(\mathbb{R}^d) &:= \{f \mid |f|_2 < \infty\} \\
f \in \mathcal{E}_2(\mathbb{R}^d) &\iff \|f(\cdot + k)\|_{L_2([0,1)^d)} \le C_5 q^{|k|} \\
&\qquad \forall\ k \in \mathbb{Z}^d,\ 0 < q < 1 \\
\mathcal{E}_2 &\subset \mathcal{L}_2 \subset L_2
\end{aligned}
$$

Theorem 2. *If $\varphi \in \mathcal{L}_2(\mathbb{R}^d)$ is refinable, $\mathbf{a} \in \ell_1(\mathbb{Z}^d), \sum_{k\in\mathbb{Z}} |a_k| < \infty$, translates ℓ_2–stable $\Longrightarrow$ φ is the generator of a M.R.A.*

Theorem 3. *$\varphi \in \mathcal{E}_2$ generator of a M.R.A. $\Longrightarrow$*

$$
\hat{\phi}(\xi) := \frac{\hat{\varphi}(\xi)}{(\sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(\xi + 2\pi k)|^2)^{1/2}}
$$

is generator of the same M.R.A., $\phi \in \mathcal{E}_2$,

$$
\langle \phi(\cdot), \phi(\cdot - k)\rangle = \int_{\mathbb{R}^d} \phi(x)\overline{\phi(x-k)}dx = \delta_{0,k}.
$$

3. Construction of Wavelets

3.1 Orthonormal Wavelets in $L_2(\mathbb{R})$

Theorem 4. *$\phi \in \mathcal{L}_2(\mathbb{R})$ $(\mathcal{E}_2(\mathbb{R}))$ generator of a M.R.A. $\{V_j\}_{j\in\mathbb{Z}}$, $\mathbf{a} \in \ell_1(\mathbb{Z})$, $\langle \phi(\cdot), \phi(\cdot - k)\rangle = \delta_{0,k}$.*

$$\begin{aligned}
\psi(x) &:= \sum_{k\in\mathbb{Z}} (-1)^k \overline{a_{1-k}}\phi(2x-k) =: \sum_{k\in\mathbb{Z}} b_k \phi(2x-k) \\
\hat{\psi}(\xi) &= \frac{1}{2} b(z)\hat{\phi}(\xi/2) = -\frac{1}{2} z\overline{a(-z)}\hat{\phi}(\xi/2) \qquad (17) \\
b(z) &= \sum_{k\in\mathbb{Z}} b_k z^k \quad \Longrightarrow
\end{aligned}$$

i.) $\langle \psi(\cdot), \psi(\cdot - k)\rangle = \delta_{0,k}, W_0 = \overline{\text{span}\{\psi(\cdot - k) \mid k \in \mathbb{Z}\}}$, *satisfies* $V_1 = V_0 \oplus W_0$, *and* $W_0 \perp V_0$;

ii.) $\{2^{j/2}\psi(2^j \cdot - k)\}_{j,k\in\mathbb{Z}}$ *is ONB of* $L_2(\mathbb{R})$;

iii.) $\psi \in \mathcal{L}_2(\mathbb{R}) \quad (\mathcal{E}_2(\mathbb{R}))$.

Sketch of Proof:

i), (17) $\Longrightarrow$ ii), iii)

i) we have to show :

$$\mathcal{A}(z) \quad = \quad \frac{1}{2}\begin{pmatrix} a(z) & a(-z) \\ b(z) & b(-z) \end{pmatrix} \tag{18}$$

is unitary, i.e., $\mathcal{A}(z)\overline{\mathcal{A}(z)^T} = E$

$$\langle \phi(\cdot), \phi(\cdot - k)\rangle = \delta_{0,k} \Rightarrow |a(z)|^2 + |a(-z)|^2 = 4 \tag{19}$$

(CQF condition) (17), (19) $\Longrightarrow$ (18) satisfied

$$\begin{aligned}
&\mathcal{A}(z)\overline{\mathcal{A}(z)^T} \\
&= \frac{1}{4}\begin{pmatrix} a(z) & a(-z) \\ -z\overline{a(-z)} & z\overline{a(z)} \end{pmatrix}\begin{pmatrix} \overline{a(z)} & -z^{-1}a(-z) \\ \overline{a(-z)} & z^{-1}a(z) \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} |a(z)|^2 + |a(-z)|^2 & 0 \\ 0 & |a(z)|^2 + |a(-z)|^2 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = E
\end{aligned}$$

a.) $\langle \psi(\cdot), \phi(\cdot - k)\rangle = 0 \quad \forall\ k \in \mathbb{Z}$;

b.) $\langle \psi(\cdot), \psi(\cdot - k)\rangle = \delta_{0,k}, \quad \forall\ k \in \mathbb{Z}$;

c.) $V_1 = V_0 \oplus W_0$.

only a.)

$$\langle \psi(\cdot), \phi(\cdot - k) \rangle = 0 \Longleftrightarrow \langle \hat{\psi}(\cdot), \widehat{\phi(\cdot - k)} \rangle = 0$$

$$\begin{aligned}
&\langle \hat{\psi}(\cdot), \widehat{\phi(\cdot - k)} \rangle \\
&= \int_{I\!R} \hat{\psi}(\xi) \overline{\hat{\phi}(\xi)} e^{ik\xi} \, d\xi \\
&= \int_{I\!R} \frac{1}{2} b(z) \frac{1}{2} \overline{a(z)} e^{ik\xi} \left| \hat{\phi}\left(\frac{\xi}{2}\right) \right|^2 \, d\xi, \quad z = e^{-i\xi/2}, \\
&= \frac{1}{4} \sum_{l \in \mathbb{Z}} \int_{[0,4\pi)+4\pi l} e^{ik\xi} b(z) \overline{a(z)} \left| \hat{\phi}\left(\frac{\xi}{2}\right) \right|^2 \, d\xi \\
&= \frac{1}{4} \int_{[0,4\pi)} e^{ik\xi} b(z) \overline{a(z)} \underbrace{\sum_{l \in \mathbb{Z}} \left| \hat{\phi}\left(\frac{\xi + 4\pi l}{2}\right) \right|^2}_{=1} \, d\xi \\
&= \frac{1}{4} \left[\int_{[0,2\pi)} e^{ik\xi} b(z) \overline{a(z)} \, d\xi + \int_{(2\pi,4\pi)} e^{ik\xi} b(z) \overline{a(z)} \, d\xi \right] \\
&= \frac{1}{4} \int_{[0,2\pi)} e^{ik\xi} \left[b(z) \overline{a(z)} + b(-z) \overline{a(-z)} \right] \, d\xi \\
&= 0
\end{aligned}$$

Algorithm to construct a wavelet basis:

- find refinable $\varphi \in \mathcal{E}_2(\mathbb{R}^d)$, $\mathbf{a} \in \ell_1(\mathbb{Z})$, ℓ_2–stable translates,

- compute

$$\hat{\phi}(\xi) = \frac{\hat{\varphi}(\xi)}{\left(\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi k)|^2\right)^{1/2}};$$

- determine the corresponding symbol $\tilde{a}(z) = 2\hat{\phi}(\xi)/\hat{\phi}(\xi/2)$;

- calculate $b(z)$ and $\hat{\psi}(\xi) = \frac{1}{2} b(z)\hat{\phi}(\xi/2)$.

Example 3. B–spline wavelets für N_2. $N_2 \in \mathcal{E}_2$, $N_2 \rightsquigarrow \tilde{N}_2(x) := N_2(x+1)$

$$\begin{aligned}
\hat{\varphi}(\xi) &:= \frac{\hat{\tilde{N}}_2(\xi)}{(\sum_{k\in\mathbb{Z}} |\hat{\tilde{N}}_2(\xi + 2\pi k)|^2)^{1/2}} \\
&= \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 \frac{1}{(1 - \frac{2}{3}\sin^2(\xi/2))^{1/2}} \\
\tilde{a}(z) &= 2\cos^2(\xi/4)\frac{(1 - \frac{2}{3}\sin^2(\xi/4))^{1/2}}{(1 - \frac{2}{3}\sin^2(\xi/2))^{1/2}} \\
\hat{\psi}(\xi) &= \frac{1}{2} b(z)\hat{\varphi}(\xi/2)
\end{aligned}$$

$$= -e^{-i\xi/2}\sin^2\left(\frac{\xi}{4}\right)\left(\frac{\sin\left(\frac{\xi}{4}\right)}{\frac{\xi}{4}}\right)^2\left(\frac{1-\frac{2}{3}\cos^2\left(\frac{\xi}{4}\right)}{1-\frac{2}{3}\sin^2\left(\frac{\xi}{4}\right)}\right)^{1/2}$$
$$\left(1-\frac{2}{3}\sin^2\left(\frac{\xi}{2}\right)\right)^{-1/2}$$

3.2 Wavelets with Compact Support

Problem: supp φ compact $\not\Longrightarrow$ supp ϕ compact

Theorem 5. $\exists\ \psi^N,\ \phi^N \in C^{\beta N}(\mathbb{R})$, *supp* $\psi^N = [-(N-1), N]$, *supp* $\phi^N = [0, 2N-1]$, *and* $\langle \phi^N(\cdot), \phi^N(\cdot - k)\rangle = \delta_{0,k}$.

3.3 Pre–Wavelets

- Disadvantage: not completely orthonormal (Parseval etc.)
- Advantage: weaker conditions, compact support can be preserved, no orthonormalization procedure
- Task: finde $b(z)$ such that

$$\mathcal{A}(z) = \frac{1}{2}\begin{pmatrix} a(z) & a(-z) \\ b(z) & b(-z) \end{pmatrix}$$

is nonsingular

Example 4. B–spline pre–wavelets for N_2

$$\begin{aligned} \psi(x) &:= \sum_{k\in\mathbb{Z}} b_k N_2(2x-k) \\ b(z) &= 2\left(\frac{1-z}{2}\right)^2 \sum_{l=0}^{2} N_4(l+1)(-z)^l \end{aligned}$$

3.4 Biorthogonal Wavelets

2 systems $\Psi,\ \tilde{\Psi}$

$$\langle \psi_{i,j,k}, \tilde{\psi}_{i',j',k'} \rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}$$

- flexibility
- compact support can be preserved
- computations similar to the orthonormal case

$$f = \sum_{i,j,k} \langle f, \tilde{\psi}_{i,j,k} \rangle \psi_{i,j,k} = \sum_{i',j',k'} \langle f, \psi_{i',j',k'} \rangle \tilde{\psi}_{i',j',k'}$$

2 M.R.A.'s $\{V_j\}_{j\in\mathbb{Z}},\ \{\tilde{V}_j\}_{j\in\mathbb{Z}}$

$$V_0 \perp \tilde{W}_0, \quad \tilde{V}_0 \perp W_0, \quad V_0 \oplus W_0 = V_1, \quad \tilde{V}_0 \oplus \tilde{W}_0 = \tilde{V}_0$$

$$\langle \varphi(\cdot), \tilde{\varphi}(\cdot - k)\rangle = \delta_{0,k}, \qquad k \in \mathbb{Z}^d \tag{20}$$

$$\varphi(x) = \sum_{k\in\mathbb{Z}^d} a_k \varphi(2x-k), \qquad \tilde{\varphi}(x) = \sum_{k\in\mathbb{Z}^d} d_k \tilde{\varphi}(2x-k)$$

$d = 1$:

Theorem 6. *$\varphi,\ \tilde{\varphi} \in \mathcal{L}_2$, refinable, $\mathbf{a}, \mathbf{d} \in \ell_1(\mathbb{Z})$, generators of M.R.A.'s, ℓ_2–stable , (20) satisfied $\Longrightarrow$*

$$\psi(x) := \sum_{k\in\mathbb{Z}} (-1)^k \overline{d_{1-k}} \varphi(2x-k)$$

$$\hat{\psi}(x) := \sum_{k\in\mathbb{Z}} (-1)^k \overline{a_{1-k}} \tilde{\varphi}(2x-k)$$

d.h.

$$\hat{\psi}(\xi) = -\frac{z}{2}\overline{d(-z)}\hat{\varphi}(\xi/2)$$

$$\hat{\tilde{\psi}}(\xi) = -\frac{z}{2}\overline{a(-z)}\hat{\tilde{\varphi}}(\xi/2)$$

give rise to a biorthogonal wavelet basis.

3.5 Wavelets in $L_2(\mathbb{R}^d)$

We need $2^d - 1$ mother wavelets!

one way: tensor products

Theorem 7. *ϕ, ψ corresponding to Theorem 4, $e \in \{0,1\}^d$, $e \neq (0,\ldots,0) \Longrightarrow$*

$$\psi^e(x_1,\ldots,x_d) = \psi^{e_1}(x_1)\psi^{e_2}(x_2)\cdots\psi^{e_d}(x_d)$$

$$\psi^{e_i}(x_i) = \begin{cases} \varphi(x_i), & e_i = 0 \\ \psi(x_i), & e_i = 1 \end{cases}$$

is a family of orthonormal (mother) wavelets in $L_2(\mathbb{R}^d)$.

Example: $d = 2 \implies$

$$\{\phi(x)\psi(y),\ \psi(x)\phi(y),\ \psi(x)\psi(y)\}$$

is family of wavelets!

$$V_0 = \overline{\mathsf{span}\{\phi(x-k)\phi(y-l) \mid k,l \in \mathbb{Z}\}}$$

in general: find $b^e(z),\ e \in \{0,1\}^d, e \neq 0$, such that

$$\mathcal{A}(z) = 2^{-d} \begin{pmatrix} a(z) & a((-1)^{e_1}z) & \dots & a((-1)^{e_{2^d-1}}z) \\ b^{e_1}(z) & b^{e_1}((-1)^{e_1}z) & \dots & b^{e_1}((-1)^{e_{2^d-1}}z) \\ \vdots & & & \vdots \\ b^{e_{2^d-1}}(z) & b^{e_{2^d-1}}((-1)^{e_1}z) & \dots & b^{e_{2^d-1}}((-1)^{e_{2^d-1}}z) \end{pmatrix}$$

$b((-1)^\epsilon z) := b((-1)^{\epsilon_1}z_1, \dots, (-1)^{\epsilon_d}z_d)$

is unitary or nonsingular. This is highly nontrivial!

more general: scaling matrices

$$\psi_{i,j,k}(x) := |\det M|^{j/2}\psi_i(M^j x - k),$$

M integer, expanding, i.e., $|\lambda| > 1,\ \lambda$ EV of M

4. Properties of Wavelets and M.R.A.'s?

4.1 Reproduction of Polynomials

$\varphi \in C_0^r(\mathbb{R}^d) := \{g \mid g \in C^r(\mathbb{R}^d) \text{ and } \operatorname{supp} g \text{ compact}\} \Longrightarrow$

$$x^\alpha = \sum_{k \in \mathbb{Z}^d} c_k^\alpha \varphi(x-k), \quad |\alpha| \leq r$$

4.2 Oscillation

Reproduction of polynomials $\longrightarrow$ vanishing moments $\longrightarrow$ oscillations

Theorem 8. *Let $\varphi \in C_0^r(\mathbb{R}^d)$, ψ_i pre–wavelet, supp ψ_i compact$\Longrightarrow$*

$$\int_{\mathbb{R}} x^\alpha \psi(x) dx \quad = \quad 0 \qquad \textit{for all} \quad 0 \leq |\alpha| \leq r.$$

Proof:

$$\begin{aligned}
\int_{\mathbb{R}^d} x^\alpha \overline{\psi_i(x)} dx &= \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} c_k^\alpha \varphi(x-k) \overline{\psi_i(x)} dx \\
&= \sum_{k \in \mathbb{Z}^d} c_k^\alpha \underbrace{\langle \varphi(\cdot - k), \psi_i(\cdot) \rangle}_{=0!} \\
&= 0
\end{aligned}$$

4.3 Approximation

$f \in L_2(\mathbb{R}^d)$, $v = \partial^\gamma f \in L_2(\mathbb{R}^d)$ is called weak derivative of f, if

$$\int_{\mathbb{R}^d} g(x)v(x)dx \quad = \quad (-1)^{|\gamma|} \int_{\mathbb{R}^d} \partial^\gamma g(x) f(x) dx$$
$$\forall\ g \in C_0^\infty(\mathbb{R}^d),$$

the space

$$H^r := \{f \mid f \text{ possesses weak derivative } \partial^\gamma f,\ |\gamma| \le r\}$$

is called the Sobolev space, $|f|_{H^r} := (\sum_{|\gamma|\le r} \|\partial^\gamma f\|_{L_2}^2)^{1/2}$

Theorem 9. *φ generator of a M.R.A., $\varphi \in C_0^r(\mathbb{R}^d)$, $f \in H^r \Longrightarrow$ Jackson inequality*

$$\inf_{g\in V_j} \|f - g\|_{L_2(\mathbb{R}^d)} \quad \le \quad C_6 2^{-jr} |f|_{H^r}.$$

4.4 Regularity

- How can we construct a refinable function for $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}^d}$?
- How can we estimate its regularity?

f "smooth" $\Longleftrightarrow$ $\hat{f}$ "rapidly decaying"

Idea: find representation of $\hat{\varphi}$, show : $\hat{\varphi}$ rapidly decaying

$$\begin{aligned}
\hat{\varphi}(\xi) &= 2^{-d} a(e^{-i\xi/2})\hat{\varphi}(\xi/2) \\
\hat{\varphi}(\xi) &= 2^{-d} a(e^{-i\xi/2}) 2^{-d} a(e^{-i\xi/4})\hat{\varphi}(\xi/4) \\
\vdots &= \vdots \quad \vdots \\
\hat{\varphi}(\xi) &= \prod_{j=1}^{\infty} \frac{a(e^{-i2^{-j}\xi})}{2^d}\hat{\varphi}(0).
\end{aligned}$$

Theorem 10. *(d=1)* $\sum_{k\in\mathbb{Z}} |a_k||k|^{\epsilon} < \infty$,

$$\begin{aligned}
a(z) &= (1+z)^M Q(z) \quad \sup_{|z|=1} |Q(z)| \le 2^{-m} \Longrightarrow \\
|\hat{\varphi}(\xi)| &\le C_8(1+|\xi|)^{-1-m-\varepsilon}, \quad \textit{i.e.} \quad \varphi \in C^m(\mathbb{R}).
\end{aligned}$$

$(1+z)^M$ "nice" part, $Q(z)$ "ugly" part

(17) $\Longrightarrow$ corresponding wavelet has the same regularity!

Theorem 11. *Let φ be the generator of a M.R.A., $\varphi \in C^r(\mathbb{R}^d)$, supp φ compact* $\Longrightarrow$ *Bernstein inequality*

$$|f|_{H^r} \le C_9 2^{jr} \|f\|_{L_2(\mathbb{R}^d)}, \quad f \in V_j$$

4.5 Wavelets and Function Spaces

$$f = \sum_{i \in I, j \in \mathbb{Z}, k \in \mathbb{Z}^d} c_{i,j,k} \psi_{i,j,k}$$

regularity of $f \longleftrightarrow$ decay of $c_{i,j,k}$!

Theorem 12. *φ generator of a M.R.A., supp φ compact, $\varphi \in C^r(\mathbb{R}^d)$. Let Ψ denote the corresponding pre–wavelet system $\Longrightarrow$*

$$f \in H^s \iff \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{i \in I, k \in \mathbb{Z}^d} |c_{i,j,k}|^2 < \infty, \quad s < r.$$