Wavelets: Applications in Image Processing and Numerical Analysis

Stephan Dahlke
University of Bremen
FB 3, ZeTeM
P.O. Box 330440
28359 Bremen

1. Applications in Image Processing
2. Applications in Numerical Analysis

1. Applications of Wavelets in Signal/Image Processing

first of all: signal analysis, $f \in L_2(\mathbb{R})$
Daubechies wavelets $\phi = \phi^N,\ \psi = \psi^N$
$T_{2^j} \longleftrightarrow$ orthogonal projector P_j onto V_j,

$$\begin{aligned} P_j(f)(x) &= \sum_{k\in\mathbb{Z}} \langle f(\cdot), 2^{j/2}\phi(2^j\cdot -k)\rangle 2^{j/2}\phi(2^j x - k) \\ &=: \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \phi_{j,k} \end{aligned}$$

'measuring device' $\rightsquigarrow \{\lambda_k^j\}_{k\in\mathbb{Z}}, \qquad \lambda_k^j = \langle f, \phi_{j,k}\rangle$

5.1 Decomposition of Signals

causality principle $\Longrightarrow P_{j-1}$ determined by P_j

$$\begin{aligned} \lambda_l^{j-1} &= \langle f, \phi_{j-1,l}\rangle \\ &= \langle f, 2^{(j-1)/2}\phi(2^{j-1}\cdot -l)\rangle \\ &= \langle f, 2^{(j-1)/2}\sum_{k\in\mathbb{Z}} a_k\phi(2^j\cdot -2l-k)\rangle \\ &= 2^{-1/2}\sum_{k\in\mathbb{Z}} \overline{a}_{k-2l}\langle f, 2^{j/2}\phi(2^j\cdot -k)\rangle \end{aligned}$$

$$= \quad 2^{-1/2} \sum_{k \in \mathbb{Z}} \overline{a}_{k-2l} \lambda_k^j$$

consequently:

$$\begin{aligned} P_{j-1}(f) &= \sum_{l \in \mathbb{Z}} \lambda_l^{j-1} \phi_{j-1,l} \\ &= \sum_{l \in \mathbb{Z}} 2^{-1/2} \Big(\sum_{k \in \mathbb{Z}} \overline{a}_{k-2l} \lambda_k^j \Big) \phi_{j-1,l}. \end{aligned}$$

a induces discrete filter H

$$\lambda^{j-1} = H\lambda^j, \qquad \lambda_l^{j-1} = \sum_{k \in \mathbb{Z}} 2^{-1/2} \overline{a}_{k-2l} \lambda_k^j. \tag{1}$$

$H \triangleq$ low–pass filter

W_j describes the detail information added to V_j!

$$\begin{aligned} Q_j(f) &:= \sum_{l \in \mathbb{Z}} \langle f, \psi_{j,l} \rangle \psi_{j,l} = \sum_{l \in \mathbb{Z}} c_l^j \psi_{j,l} \\ &= \sum_{l \in \mathbb{Z}} \langle f, \sum_{k \in \mathbb{Z}} 2^{j/2} b_k \phi(2^{j+1} \cdot -2l - k) \rangle \psi_{j,l} \\ &= \sum_{l \in \mathbb{Z}} 2^{-1/2} \Big(\sum_{k \in \mathbb{Z}} \overline{b}_{k-2l} \langle f, \phi_{j+1,k} \rangle \Big) \psi_{j,l} \\ &= \sum_{l \in \mathbb{Z}} 2^{-1/2} \Big(\sum_{k \in \mathbb{Z}} \overline{b}_{k-2l} \lambda_k^{j+1} \Big) \psi_{j,l} \end{aligned}$$

b also induces discrete filter D

$D \hat{=}$ high–pass filter

$$c^j = D\lambda^{j+1}, \qquad c^j_l = 2^{-1/2} \sum_{k\in\mathbb{Z}} \bar{b}_{k-2l}\lambda^{j+1}_l \qquad (2)$$

iteration $\Longrightarrow$ fast wavelet transform, pyramid algorithm:

$$\begin{tikzcd}
\lambda^j \arrow[r, "H"] \arrow[dr, "D"] & \lambda^{j-1} \arrow[r, "H"] \arrow[dr, "D"] & \lambda^{j-2} \arrow[r, "H"] \arrow[dr, "D"] & \lambda^{j-3} \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & {} \\
& c^{j-1} & c^{j-2} & c^{j-3} & {}
\end{tikzcd}$$

filter $(\mathcal{F}\lambda)_l := \sum_{k\in\mathbb{Z}} h_{l-k}\lambda_k = \{h * \lambda\}_l$

$$\begin{aligned}
(\mathcal{H}\lambda)_l &= 2^{-1/2} \sum_{k\in\mathbb{Z}} \bar{a}_{k-l}\lambda_k = \{2^{-1/2}\tilde{\mathbf{a}} * \lambda\}_l, \tilde{a}_l = \bar{a}_{-l} \\
(\mathcal{D}\lambda)_l &= 2^{-1/2} \sum_{k\in\mathbb{Z}} \bar{b}_{k-l}\lambda_k = \{2^{-1/2}\tilde{\mathbf{b}} * \lambda\}_l, \tilde{b}_l = \bar{b}_{-l}
\end{aligned}$$

interpretation of (1), (2)

- compute $2^{-1/2}\tilde{\mathbf{a}} * \lambda^j,\ 2^{-1/2}\tilde{\mathbf{b}} * \lambda^j$

- neglect odd indices downsampling

5.2 Reconstruction

$V_j = V_{j-1} \oplus W_{j-1} \Longrightarrow$

$$\begin{aligned}
\phi_{j,l} &= \sum_{k\in\mathbb{Z}} \langle \phi_{j,l}, \phi_{j-1,k}\rangle \phi_{j-1,k} + \sum_{n\in\mathbb{Z}} \langle \phi_{j,l}, \psi_{j-1,n}\rangle \psi_{j-1,n} \\
&= 2^{-1/2} \sum_{k\in\mathbb{Z}} \overline{a}_{l-2k}\phi_{j-1,k} + 2^{-1/2} \sum_{n\in\mathbb{Z}} \overline{b}_{l-2n}\psi_{j-1,n} \Longrightarrow
\end{aligned}$$

$$\begin{aligned}
P_j(f) &= \sum_{l\in\mathbb{Z}} \langle f, \phi_{j,l}\rangle \phi_{j,l} = \sum_{l\in\mathbb{Z}} \lambda_l^j \phi_{j,l} \\
&= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} (\langle f, \sum_{k\in\mathbb{Z}} \overline{a}_{l-2k}\phi_{j-1,k} + \sum_{n\in\mathbb{Z}} \overline{b}_{l-2n}\psi_{j-1,n}\rangle)\phi_{j,l} \\
&= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} a_{l-2k}\langle f, \phi_{j-1,k}\rangle + \sum_{n\in\mathbb{Z}} b_{l-2n}\langle f, \psi_{j-1,n}\rangle)\phi_{j,l} \\
&= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} a_{l-2k}\lambda_k^{j-1} + \sum_{n\in\mathbb{Z}} b_{l-2n}c_n^{j-1})\phi_{j,l}
\end{aligned}$$

$$\lambda_l^j = 2^{-1/2} \sum_{k\in\mathbb{Z}} a_{l-2k}\lambda_k^{j-1} + 2^{-1/2} \sum_{n\in\mathbb{Z}} b_{l-2n}c_n^{j-1} \qquad (3)$$

$$
\begin{aligned}
(\overline{\mathcal{H}}\lambda)_l &= 2^{-1/2}\sum_{k\in\mathbb{Z}} a_{l-k}\lambda_k = \{2^{-1/2}\mathbf{a}*\lambda\}_l, \\
(\overline{\mathcal{D}}\lambda)_l &= 2^{-1/2}\sum_{k\in\mathbb{Z}} b_{l-k}\lambda_k = \{2^{-1/2}\mathbf{b}*\lambda\}_l
\end{aligned}
$$

interpretation of (3)

- insert zeros at odd indices upsampling
- compute $2^{-1/2}\mathbf{a}*\lambda^{j-1},\ 2^{-1/2}\mathbf{b}*c^{j-1}$
- add the two terms

$$
\lambda^j \xrightarrow{\text{decomposition}} \binom{\lambda^{j-1}}{c^{j-1}} \xrightarrow[\text{denoising}]{\text{compression}} \binom{\tilde{\lambda}^{j-1}}{\tilde{c}^{j-1}} \xrightarrow{\text{reconstruction}} \tilde{\lambda}^j
$$

5.3 Compression

linear methods:
approximate f by the spaces $\{V_j\}_{j\in\mathbb{Z}}$, i.e.:

$$d_k^n = \begin{cases} c_k^n & n \le j \\ 0 & \text{else} \end{cases}$$

Jackson inequality:

$$\inf_{g\in V_j} \|f-g\|_{L_2(\mathbb{R}^d)} \le C_6 2^{-jr} |f|_{H^r} \quad \Longrightarrow$$

linear methods perform well for smooth signals f!

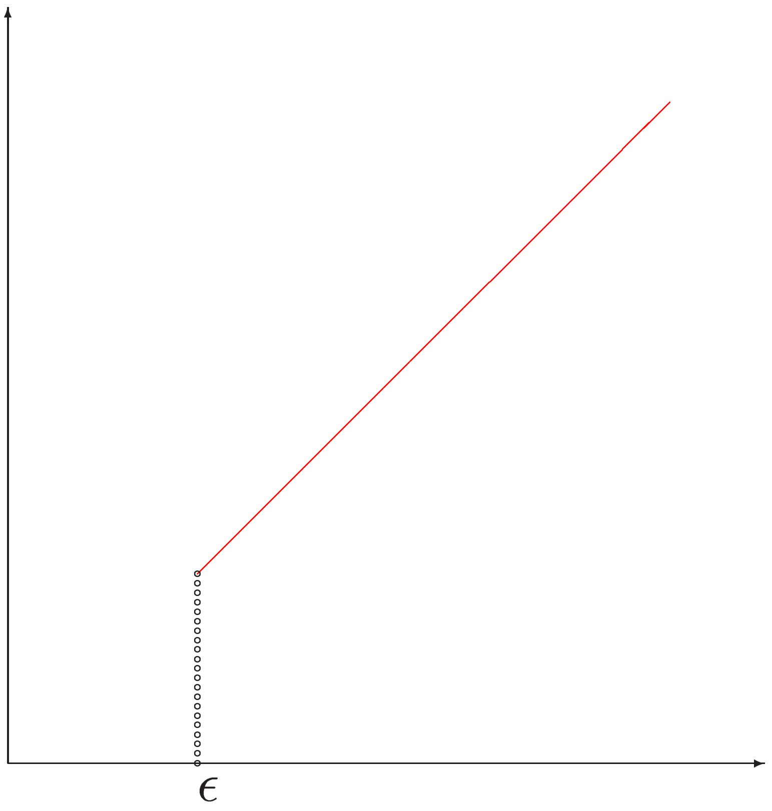
nonlinear methods:
idea: f locally "smooth" $\Longrightarrow$

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \text{rest} \\ c_k^j &= \langle f, \psi_{j,k} \rangle \\ &= \underbrace{\langle f(x_0), \psi_{j,k}\rangle}_{=0} + \underbrace{\langle f'(x_0)(x-x_0), \psi_{j,k}\rangle}_{=0} \\ &\qquad + \frac{1}{2} \underbrace{\langle f''(x_0)(x-x_0)^2, \psi_{j,k}\rangle}_{=0} + \langle \text{rest}, \psi_{j,k}\rangle \end{aligned}$$

hard thresholding

$$d_k^j = \left\{ \begin{array}{cc} c_k^j & |c_k^j| > \epsilon \\ 0 & \text{else} \end{array} \right. \Longrightarrow$$

$$f^\epsilon = \sum_{j,k} d_k^j \psi_{j,k} = \sum_{|c_k^j| \geq \epsilon} c_k^j \psi_{j,k}$$



5.4 Denoising

$$\tilde{f} = \sum_{j,k} \tilde{c}_k^j \psi_{j,k}$$

$$\tilde{c}_k^j = c_k^j + \eta_k^j, \qquad \eta_k^j \text{ independent } N(0,\sigma)\text{–distributed}$$

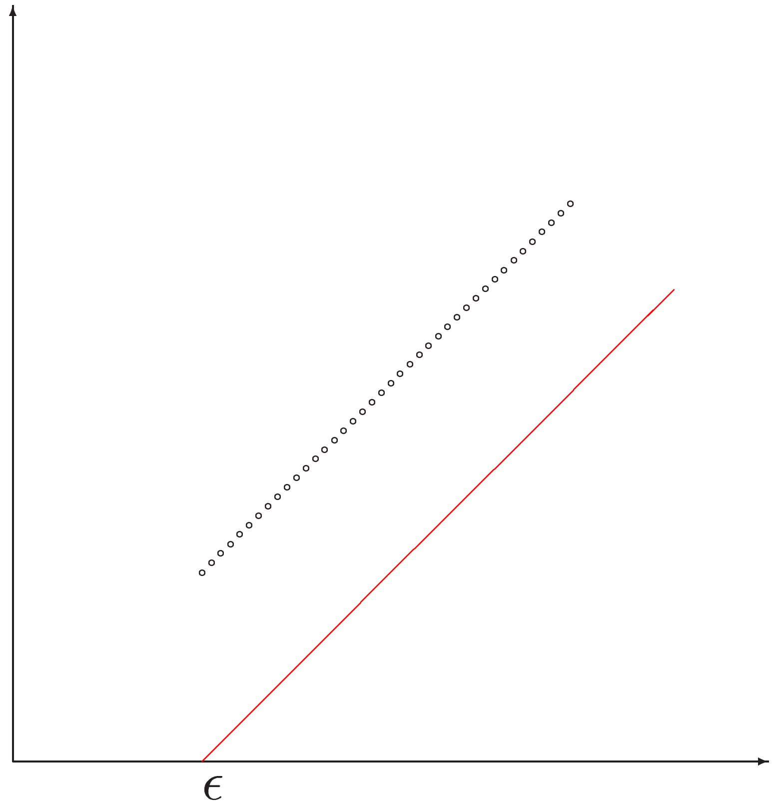
soft thresholding, shrinkage

$$s_\epsilon(t) = \begin{cases} t-\epsilon, & t > \epsilon \\ 0, & |t| \le \epsilon \\ t+\epsilon, & t < -\epsilon \end{cases} \implies$$

$$\tilde{f}_\epsilon = \sum_{j,k} s_\epsilon(c_k^j + \eta_k^j)\psi_{j,k}$$

ϵ can be determined such that

$$E(\|f - \tilde{f}_\epsilon\|) = \min!$$


ϵ

5.5 The Multivariate Case, Image Analysis

tensor product approach $(d=2)$

$$\{\phi(x)\psi(y), \psi(x)\phi(y), \psi(x)\psi(y)\}$$

is a system of mother wavelets,

$$V_0 = \overline{\mathsf{span}\{\phi(x-k)\phi(y-l) \mid k,l \in \mathbb{Z}\}}$$

$$\begin{aligned}
\langle f, \phi_{j-1,k}\phi_{j-1,l}\rangle &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{a}_{n-2k}\overline{a}_{t-2l}\langle f, \phi_{j,n}\phi_{j,t}\rangle \\
\langle f, \phi_{j,k}\psi_{j,l}\rangle &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{a}_{n-2k}\overline{b}_{t-2l}\langle f, \phi_{j+1,n}\phi_{j+1,t}\rangle \\
\langle f, \psi_{j,k}\phi_{j,l}\rangle &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{b}_{n-2k}\overline{a}_{t-2l}\langle f, \phi_{j+1,n}\phi_{j+1,t}\rangle \\
\langle f, \psi_{j,k}\psi_{j,l}\rangle &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{b}_{n-2k}\overline{b}_{t-2l}\langle f, \phi_{j+1,n}\phi_{j+1,t}\rangle
\end{aligned}$$

$$
\begin{aligned}
(H\lambda)_{k,l} &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{a}_{n-2k}\overline{a}_{t-2l}\lambda_{n,t}\\
(D^1\lambda)_{k,l} &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{a}_{n-2k}\overline{b}_{t-2l}\lambda_{n,t}\\
(D^2\lambda)_{k,l} &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{b}_{n-2k}\overline{a}_{t-2l}\lambda_{n,t}\\
(D^3\lambda)_{k,l} &= \frac{1}{2}\sum_{n,t\in\mathbb{Z}} \overline{b}_{n-2k}\overline{b}_{t-2l}\lambda_{n,t}
\end{aligned}
$$

decomposition and reconstruction analogous to the univariate case

2. Applications of Wavelets in Numerical Analysis

2.1 Introduction

numerical treatment of operator equations

$$Au = F$$
$$A: \; H^1 \longrightarrow H^2,$$

$H^1, \; H^2$ Sobolev spaces

Example 1:
Poisson equation

$$\begin{aligned} -\triangle u &= F \quad \text{in} \quad \Omega \subset \mathbb{R}^d \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ \triangle u &:= \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \end{aligned} \tag{4}$$

u: displacement of a membran
F: exterior force

Example 2:
double layer potential

$$(\frac{1}{2}I + \mathcal{K})v = g \quad \text{on } \partial\Omega, \ \Omega \subset I\!R^d$$

$$\mathcal{K}(f)(x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle x - y, \vec{n}(y) \rangle}{|x - y|^d} f(y) d\sigma(y)$$

Example 3:

$$\begin{aligned} -\frac{d^2 u}{dx^2} &= F \quad \text{in} \quad [0,1] \qquad (5) \\ u(0) &= u(1) = 0 \end{aligned}$$

in many cases: existence and uniqueness clarified
however: how does the solution look like?

Galerkin method

multiplication of (5) with test functions $\eta \in C_0^\infty$, supp $\eta \subset [0,1]$, integration by parts $\Longrightarrow$

$$\begin{aligned} -\int_0^1 u''(x)\eta(x)dx &= \int_0^1 F(x)\eta(x)dx \\ \Longleftrightarrow \int_0^1 u'(x)\eta'(x)dx &= \int_0^1 F(x)\eta(x)dx \\ a(u,\eta) &= \mathcal{F}(\eta) \qquad (6) \end{aligned}$$

Idea: project (6) onto a finite dimensional subspace $V_M = \text{span}\{\eta_1, \ldots, \eta_M\}$

Task: finde $u^M \in V_M$ such that

$$\begin{aligned} a(u^M, v) &= \mathcal{F}(v) \qquad \forall\, v \in V_M \\ \Longleftrightarrow a(u^M, \eta_i) &= \mathcal{F}(\eta_i), \qquad i = 1, \ldots, M \end{aligned} \tag{7}$$

$$u^M = \sum_{j=1}^{M} d_j \eta_j \quad \rightsquigarrow$$

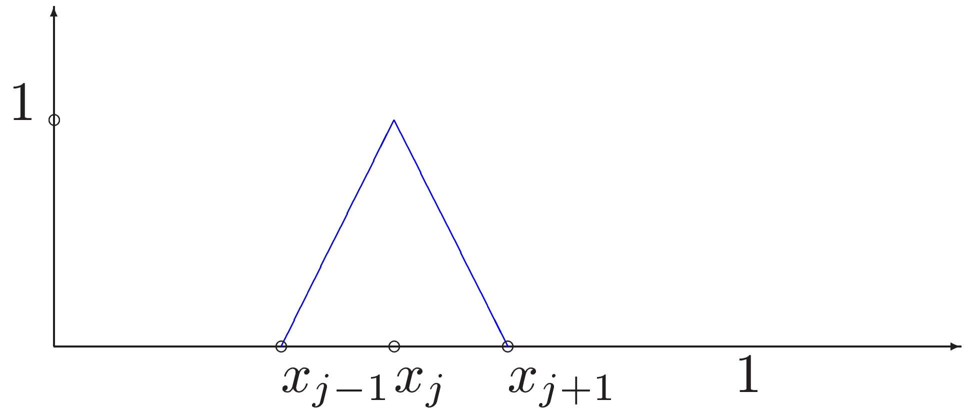
$$\begin{aligned} a(u^M, \eta_i) &= a(\sum_{j=1}^{M} d_j \eta_j, \eta_i) \\ &= \sum_{j=1}^{M} d_j a(\eta_j, \eta_i) = \mathcal{F}(\eta_i), \quad i = 1, \ldots, M \\ \Longleftrightarrow \mathbf{A}_M \mathbf{d} &= \mathbf{b} \\ \mathbf{A}_M &= (a(\eta_j, \eta_i))_{i,j=1,\ldots,M}, \qquad \mathbf{b} = (\mathcal{F}(\eta_i))_{i=1}^{M} \end{aligned}$$

$\mathbf{A}_M$ symmetric positive definite

Question: suitable choices of V_M?

one way: finite elements
$0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = 1$,
$x_i = ih,\ h = 1/(M+1),\ \eta_i$ piecewise linear with

$$\eta_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \quad j = 1, \ldots, N \end{cases}$$



$h \rightsquigarrow h/2 \rightsquigarrow h/4 \ldots$
one can show:

$$\mathbf{A}_M = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & & & \\ & 0 & & & & \\ & & & & & \\ & & & & 2 & -1 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{pmatrix}$$

in $\mathbb{R}^2$: use sequence of triangulations

advantage: $\mathbf{A}_M$ sparse $\longrightarrow$ iterative solvers

$$
\begin{aligned}
x^{j+1} &= Tx^j + c, \qquad \text{e.g.} \\
x^{j+1} &= x^j - D^{-1}(Ax^j - b), \qquad \mathbf{D} = \text{diag}(A)
\end{aligned}
$$

disadvantage: condition numbers of $\mathbf{A}_M = \kappa(\mathbf{A}_M) = \frac{\lambda_{\max}}{\lambda_{\min}} \sim \mathcal{O}(h^{-2})$

condition numbers determine

- order of convergence of iterative schemes
- robustness of the scheme

2.2 Preconditioning

Idea: use wavelets as basis functions for the Galerkin scheme $\longmapsto$ wavelet Galerkin scheme !
Choose the spaces $\{V_j\}_{j\geq 0}$ of the M.R.A. as approximation spaces!

$$V_j = \overline{\text{span}\{\varphi(\cdot - l), \psi_{i,j,k},\ k,l \in \mathbb{Z}^d, j \geq 0, i \in I\}}$$

Preconditioning possible!

$$\mathbf{A}_M\mathbf{d} = \mathbf{b} \rightsquigarrow \mathbf{D}^{-1}\mathbf{A}_M\mathbf{D}^{-1}\mathbf{D}\mathbf{d} = \mathbf{D}^{-1}\mathbf{b}$$

Choose, e.g., for the Poisson equation

$$\begin{aligned} \mathbf{D}_{(j,i,k),(j',i',k')} &= 2^j\delta_{j,j'}\delta_{i,i'}\delta_{k,k'} \Longrightarrow \\ \kappa(\mathbf{D}^{-1}\mathbf{A}_M\mathbf{D}^{-1}) &= \mathcal{O}(1)\ ! \end{aligned}$$

Diagonal preconditioning strategy is suitable for a huge class of problems!

2.3 Matrix Compression

integral equations

$$
\begin{aligned}
Au &= F \quad \text{in} \quad \Omega \subset I\!R^d \\
(Au)(x) &= \int_\Omega K(x,y)u(y)dy \\
|\partial_x^\alpha \partial_y^\beta K(x,y)| &\leq c_{\alpha,\beta}\text{dist}(x,y)^{(-d+2t+|\alpha|+|\beta|)} \qquad (8)
\end{aligned}
$$

Example: Hilbert transform

$$
K(x,y) = \frac{1}{x-y}, \qquad (Au)(x) = \int_{I\!R} \frac{u(y)}{x-y} dy
$$

Galerkin scheme with finite elements $\Longrightarrow \mathbf{A}_M$ not sparse!

We use again wavelets! Then many entries a priori small!

Observe: wavelets have vanishing moments,

$$
\int_{I\!R} x^\alpha \psi(x) dx = 0, \qquad 0 \leq \alpha \leq N-1
$$

typical entry:

$$\langle A(\psi_{i,j,k}), \psi_{i',j',k'}\rangle = \int_\Omega \int_\Omega K(x,y)\psi_{i',j',k'}(x)\psi_{i,j,k}(y)dxdy$$
$$= \int_\Omega \int_\Omega (K(x,y) - p(x,y))\psi_{i',j',k'}(x)\psi_{i,j,k}(y)dxdy$$

$p(x,y)$ polynomial

analogous to data compression: choose $p(x,y)$ as Taylor polynomial $\Longrightarrow$

$$K(x,y) - p(x,y) \sim$$
$$\sum_{|\alpha|,|\beta|=N} (x - x_{i',j',k'})^\alpha (y - y_{i,j,k})^\beta \partial_x^\alpha \partial_y^\beta K(x_{i',j',k'}, y_{i,j,k})$$

combine this with (8) $\Longrightarrow$

$$2^{-(j'+j)t}|\langle A(\psi_{i,j,k}), \psi_{i',j',k'}\rangle| \leq$$
$$C_{10} \frac{2^{-|j-j'|\sigma}}{(1 + 2^{\min(j,j')}\mathrm{dist}(\mathrm{supp}\psi_{i',j',k'}, \mathrm{supp}\psi_{i,j,k})^{d+2\tilde{m}+2t}}$$

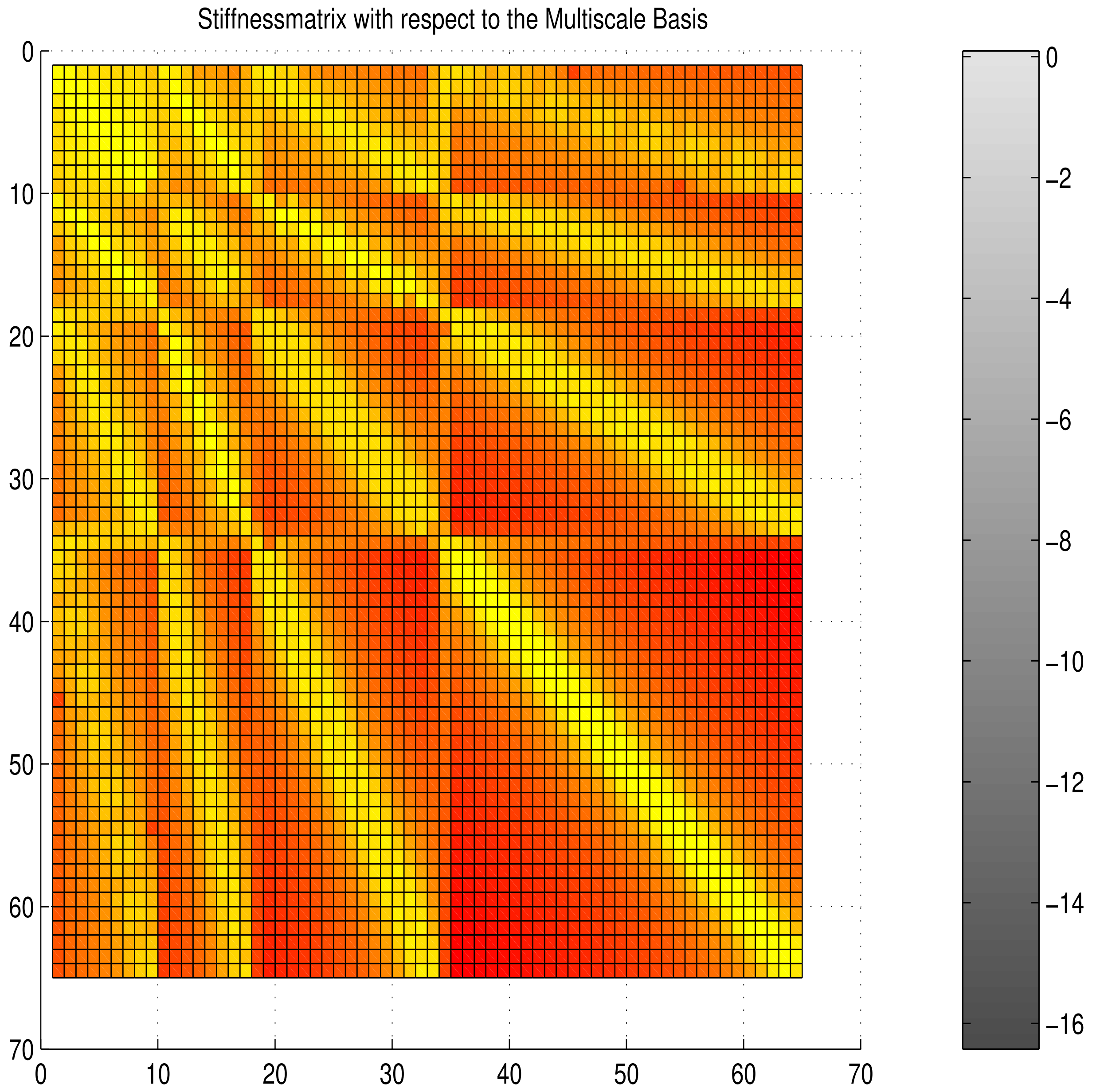
entries small, if

- distance of scales large
- distance of supports large

We throw them away!

rigorous error estimation is possible

Example: Hilbert transform



2.4 Adaptivity

Problem: $\mathbf{A}_N$ might become very large!

way out:
adaptive schemes: do not use the full linear spaces V_j, but only suitable parts $S_\Lambda \subset V_j$

updating strategy:

- S_Λ depend on u
- self–regulating

solve	–	estimate	–	refine
$\mathbf{A}_\Lambda \mathbf{c}_\Lambda = \mathbf{F}_\Lambda$		$\\|u - u_\Lambda\\| =?$ a posteriori error estimator		refine partition, add functions if necessary

one can show:

the use of wavelets leads to an adaptive strategy

- which is guaranteed to converge
- realizes optimal order of convergence

idea:

- error estimator

$$
\begin{aligned}
\|u - u_\Lambda\|_{H^1} &\leq C_{11} \|A(u - u_\Lambda)\|_{H^2} \\
&\leq C_{11} \|F - A(u_\Lambda)\|_{H^2} \\
&\leq C_{11} \|r_\Lambda\|_{H^2} \\
&\leq C_{12} \left(\sum_{(i,j,k) \notin \Lambda} \underbrace{2^{-2tj} |\langle r_\Lambda, \psi_{i,j,k} \rangle|^2}_{\delta^2_{i,j,k}} \right)^{1/2}
\end{aligned}
$$

$$
\delta_{i,j,k} = 2^{-tj} |F_{i,j,k} - \sum_{(i',j',k') \in \Lambda} \langle A\psi_{i',j',k'}, \psi_{i,j,k} \rangle d_{i',j',k'}|,
$$

$$
F_{i,j,k} = \langle F, \psi_{i,j,k} \rangle
$$

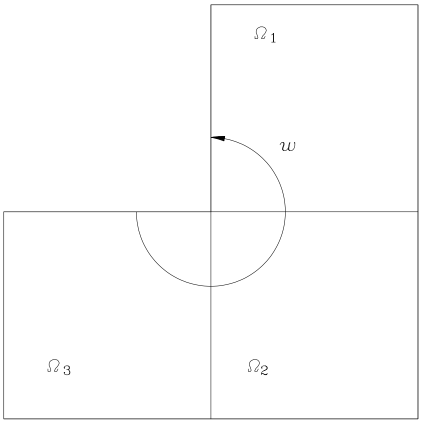
- adaptive strategy
  choose $\tilde{\Lambda}$ such that

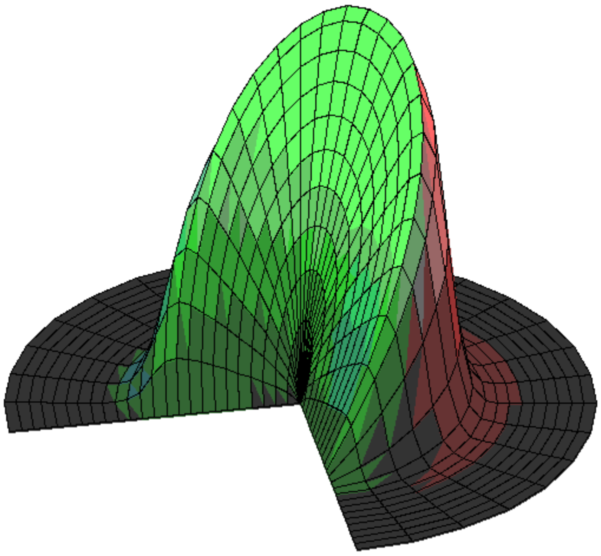
$$\sum_{(i,j,k)\in\tilde{\Lambda}\setminus\Lambda} 2^{-2tj}|\langle r_\Lambda, \psi_{i,j,k}\rangle|^2 \quad \geq \quad \gamma \sum_{(i,j,k)\notin\Lambda} 2^{-2tj}|\langle r_\Lambda, \psi_{i,j,k}\rangle|^2$$

$$\Longrightarrow \|u - u_{\tilde{\Lambda}}\| \quad \leq \quad \kappa\|u - u_\Lambda\|, \qquad \kappa \in (0,1)$$

Example:

$$\begin{aligned} -\triangle u &= F \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} &= 0 \end{aligned}$$





the right–hand side