Zentrum für Technomathematik

fachbereich 3
mathematik und informatik

Reconstruction of the Reflectivity Density by Wavelet Transforms

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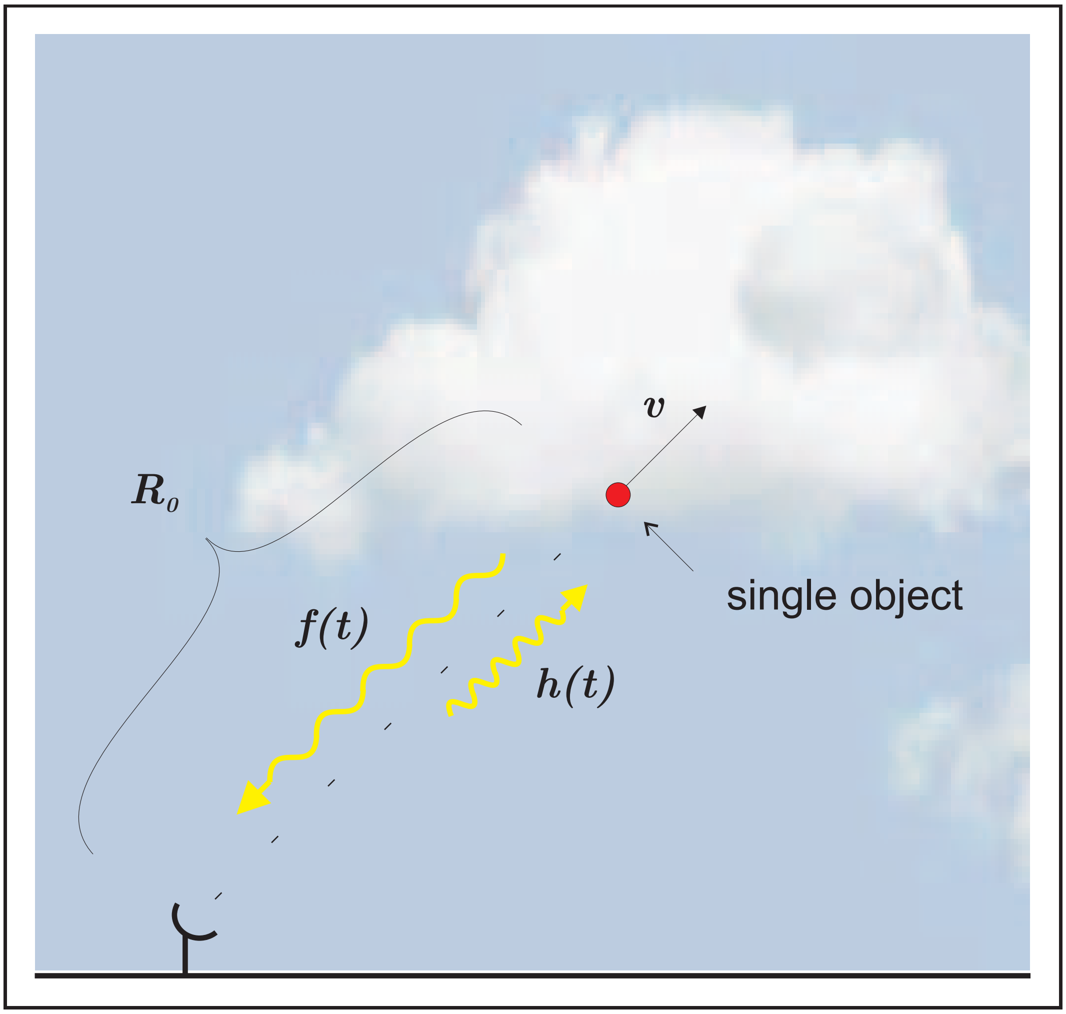
1. Introduction

- Analysis of radar data
- Goal: Obtain information by analyzing reflected waves!
- point object



$$R(t) = R_0 + vt$$

transmitted signal : $\quad h(t)$

at time t : $\quad \mathcal{S} = h\left(t - \frac{R_0 + tv}{c}\right)$

echo: $\quad f\left(t + \frac{R_0 + tv}{c}\right) = \gamma \mathcal{S}(t)$

$$s = \frac{c+v}{c-v}, \qquad \tau = \frac{2R_0}{c-v} \Longrightarrow$$

$$f(t) = \gamma h\left(\frac{t-\tau}{s}\right), \qquad \gamma = |s|^{-1/2}$$

- reflectivity density $D(\tau, s)$?

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} D(\tau, s)|s|^{-1/2} h\left(\frac{t-\tau}{s}\right) \frac{ds\,d\tau}{s^2}$$

Problem: How can $D(s, \tau)$ be reconstructed?

- continuous wavelet transform, mathematical microscope

$\psi(x) \in L_2(I\!R)$: analyzing wavelet

$$\psi_{a,b}(x) := |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$$

$$W_\psi(F)(a,b) := \int\limits_{I\!R} F(x)|a|^{-1/2}\overline{\psi\left(\frac{x-b}{a}\right)}dx$$

admissibility condition

$$\int\limits_{I\!R} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi < \infty \quad \Longrightarrow$$

$$F(x) = \frac{1}{c}\int\limits_{I\!R}\int\limits_{I\!R\setminus\{0\}} [(W_\psi F)(a,b)]|a|^{-1/2}\psi\left(\frac{x-b}{a}\right)\frac{da}{a^2}db$$

$$f(t) = \int\limits_{I\!R}\int\limits_{I\!R\setminus\{0\}} D(\tau,s)|s|^{-1/2}h\left(\frac{t-\tau}{s}\right)\frac{dsd\tau}{s^2}$$

Conjecture:

$$D(\tau,s) := \frac{1}{c}[(W_h f)(\tau,s)]$$

Problem: W_ψ not onto!

- Idea: Transmit family $\{h_m\}_{m\in\mathbb{Z}}$ of signals!

Naparst (1988): orthonormal basis
Rebollo-Neira/Plastino/Fernandez–Rubio (2000): frame theoretic approach

2. Reconstruction Formulas

a system $\{h_m\}_{m\in\mathbb{Z}}$ is called a *frame* if

$$A\|F\|^2_{L_2(\mathbb{R})} \leq \sum_{m\in\mathbb{Z}} |\langle F, h_m\rangle|^2 \leq B\|F\|^2_{L_2(\mathbb{R})}$$

$$T(F) := \sum_{m\in\mathbb{Z}} \langle F, h_m\rangle h_m \quad \text{frame operator} \quad T$$

Theorem 1. *Let $\{h_m\}_{m\in\mathbb{Z}}$ be a frame in $L_2(\mathbb{R})$. Then*

i) T is invertible and $B^{-1}I \leq T^{-1} \leq A^{-1}I$.

ii) $\{h^m\}_{m\in\mathbb{Z}}$, $h^m := T^{-1}h_m$ is a frame called the reciprocal frame *of $\{h_m\}_{m\in\mathbb{Z}}$.*

iii) Every $F \in L_2(\mathbb{R})$ can be written as

$$F = \sum_{m\in\mathbb{Z}} \langle F, h^m\rangle h_m = \sum_{m\in\mathbb{Z}} \langle F, h_m\rangle h^m.$$

Theorem 2. *Let $\{h_m\}_{m\in\mathbb{Z}}$ be a frame of outgoing signals in $L_2(\mathbb{R})$ and let f_m denote the corresponding echoes produced by $D(\tau, s)$,*

$$f_m(t) = \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} D(\tau, s)|s|^{-1/2} h_m\left(\frac{t-\tau}{s}\right)\frac{ds d\tau}{s^2}.$$

Integrability conditions $\Longrightarrow$ $D(\tau, s)$ *can be reconstructed as follows*

$$D(\tau, s) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \int_{-\infty}^{0} -\frac{1}{i}\widehat{f_m'}(\omega)\widehat{h^m(\frac{\cdot}{s})}(\omega)|s|^{1/2}e^{i\tau\omega}d\omega$$

$$+\frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \int_{0}^{\infty} \frac{1}{i}\widehat{f_m'}(\omega)\widehat{h^m(\frac{\cdot}{s})}(\omega)|s|^{1/2}e^{i\tau\omega}d\omega,$$

where $\{h^m\}_{m\in\mathbb{Z}}$ denotes the reciprocal frame of $\{h_m\}_{m\in\mathbb{Z}}$.

Idea of Proof: Fourier transform $\Longrightarrow$

$$\hat{f}_m(\omega) = \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} D(\tau, s) \int_{\mathbb{R}} |s|^{-1/2} h_m\left(\frac{t-\tau}{s}\right) e^{-i\omega t} dt \frac{ds\,d\tau}{s^2}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} D(\tau, s)|s|^{-1/2} \widehat{h_m\left(\frac{\cdot - \tau}{s}\right)}(\omega) \frac{ds\,d\tau}{s^2}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} D(\tau, s)|s|^{1/2} e^{-i\omega\tau} \hat{h}_m(s\omega) \frac{ds\,d\tau}{s^2}$$

$$= \int_{\mathbb{R}\setminus\{0\}} \left(\int_{\mathbb{R}} D(\tau, s) e^{-i\omega\tau} d\tau \right) \hat{h}_m(s\omega)|s|^{-3/2} ds$$

$$= \int_{\mathbb{R}\setminus\{0\}} \widehat{D(\cdot, s)}(\omega) \hat{h}_m(s\omega)|s|^{-3/2} ds$$

$\sigma = s\omega \Longrightarrow$

$$\hat{f}_m(\omega) = \int_{\mathbb{R}\setminus\{0\}} \widehat{D(\cdot, \frac{\sigma}{\omega})}(\omega) \hat{h}_m(\sigma) \left|\frac{\sigma}{\omega}\right|^{-3/2} |\omega|^{-1} d\sigma$$

$$= \langle \widetilde{D}(\omega, \cdot), \hat{h}_m(\cdot) \rangle_{L_2(\mathbb{R})},$$

$$\widetilde{D}(\omega, \sigma) := \widehat{D(\cdot, \frac{\sigma}{\omega})}(\omega)|\omega|^{1/2}|\sigma|^{-3/2}$$

Lemma 1. *Let* $\{h_m\}_{m\in\mathbb{Z}}$ *be a frame and let* $\{h^m\}_{m\in\mathbb{Z}}$ *denote the reciprocal frame* $\Longrightarrow$ $\{\hat{h}_m\}_{m\in\mathbb{Z}}$ *is also a frame and* $(\hat{h})^m = \widehat{h^m}$

Lemma 1 $\Longrightarrow$ $\hat{f}_m(\omega)$ are the coefficients of $\widetilde{D}(\omega,\cdot)$ with respect to the frame $\{\hat{h}_m\}_{m\in\mathbb{Z}}$! $\Longrightarrow$

we may reconstruct $\widetilde{D}(\omega,\sigma)$ as

$$\begin{aligned}\widetilde{D}(\omega,\sigma) &= \sum_{m\in\mathbb{Z}} \langle \widetilde{D}(\omega,\cdot), \hat{h}_m(\cdot)\rangle \widehat{h^m}(\sigma) \\ &= \sum_{m\in\mathbb{Z}} \hat{f}_m(\omega)\widehat{h^m}(\sigma)\end{aligned}$$

$\widetilde{D}(\omega,\sigma) \rightsquigarrow D(\tau,s)$! $\square$

3. Error Estimates

in practice: truncation necessary!

Question: Order of approximation?

suppose that $\{h_m\}_{m \in \mathbb{Z}}$ is generated by orthonormal wavelet basis:

$$\begin{aligned} h_m = h_{m(j,k)} &= \mathcal{F}^{-1}\psi_{j,k}, \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \\ \langle \psi_{j,k}, \psi_{j',k'} \rangle &= \delta_{j,j'}\delta_{k,k'}. \end{aligned}$$

I. Daubechies (1988): $\exists$ family ${}_N\psi$ of mother wavelets satisfying

$$\begin{aligned} &{}_N\psi \in C^{\beta N}(\mathbb{R}), \\ &|\mathrm{supp}\,{}_N\psi| = 2N - 1, \\ &\int_{\mathbb{R}} x^l\,{}_N\psi(x)dx = 0, \quad l \le N - 1 \end{aligned}$$

Theorem 3. *Suppose that for some fixed* α

$$G(\alpha, \omega) := |\widetilde{D}(\omega, \cdot)|^2_{H^\alpha(R^d)} < \infty$$

Then the following error estimate holds:

$$\int_{I\!R}\int_{I\!R}\left| D(\tau, s) - \sum_{j \leq J} \int_{I\!R} \hat{f}_m(\omega)\hat{h}_m(s\omega)|\omega||s|^{3/2} e^{i\tau\omega} d\omega \right|^2 d\tau \frac{ds}{|s|^3}$$
$$\lesssim 2^{-2J\alpha} \int_{I\!R} |\omega| G(\alpha, \omega) d\omega.$$

Proof: Use Jackson-type estimates from classical wavelet analysis!

Theorem 3 can be generalized to *biorthogonal* wavelet bases,

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'} \delta_{k.k'}$$

4. Narrow–Band Approach

- In practice: speed of object << speed much of light
  $|v|/c << 1$

$$s = \frac{c+v}{c-v}, \sim 1 + \frac{2v}{c}, \qquad \tau = \frac{2R_0}{c-v} \sim \frac{2R_0}{c}$$

remove high carrier frequency $\omega_c \Longrightarrow$

$$H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega_c)t} \hat{h}(\omega) d\omega)$$

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega_c)t} \hat{f}(\omega) d\omega.$$

- single point

$$\hat{f}(\omega) = |s|^{1/2} e^{-i\omega\tau} \hat{h}(s\omega) \approx e^{-i\tau\omega} \begin{cases} \hat{h}(\omega+\phi) & \omega > 0, \\ \hat{h}(\omega-\phi) & \omega < 0. \end{cases},$$

where $\phi = \dfrac{2\omega_c v}{c}$,

$$F(t) = e^{-i\phi(t-\tau)}\frac{1}{2\pi}e^{-i\omega_c t}\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\hat{h}_P(\omega)d\omega$$

$$+e^{i\phi(t-\tau)}\frac{1}{2\pi}e^{-i\omega_c t}\int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\hat{h}_R(\omega)d\omega$$

$$\hat{h}_P := \hat{h}(\omega)\chi_{[0,\infty)}(\omega), \quad \hat{h}_R(\omega) := \hat{h}(\omega)\chi_{(-\infty,0]}(\omega) \Rightarrow$$

$$F(t) = e^{-i\phi t}H_P(t-\tau) + e^{i\phi t}H_R(t-\tau)$$

- dense target environment:

$$F(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} (H_{P\phi,\tau}(t) + H_{R,-\phi,\tau}(t))D_{NB}(\phi,\tau)d\phi d\tau,$$

$$H_{P\phi,\tau}(t) = e^{-i\phi t}H_P(t-\tau)$$

$$L_2(\mathbb{R}) \simeq L_{2,P}(\mathbb{R}) \times L_{2,R}(\mathbb{R})$$

$$L_{2,P}(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \mid \hat{f}_P = \hat{f}\}$$

$$L_{2,R}(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \mid \hat{f}_R = \hat{f}\}$$

Theorem 4. *Let* $\{h_m\}_{m\in\mathbb{Z}}, \{g_m\}_{m\in\mathbb{Z}}$ *be the outgoing signals in* $L_{2,P}(\mathbb{R})$ *and* $L_{2,R}(\mathbb{R})$*, respectively. Assume that* $\{H_m\}_{m\in\mathbb{Z}}$ *and* $\{G_m\}_{m\in\mathbb{Z}}$ *form frames, and let* $\{H^m\}_{m\in\mathbb{Z}}$ *and* $\{G^m\}_{m\in\mathbb{Z}}$ *denote the reciprocal frames* $\Longrightarrow$

$$D_{NB}(\phi,\tau) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} F_{m,P}(t)H^m(t-\tau)e^{i\phi t}dt + \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} F_{m,R}(-t)G^m(-t-\tau)e^{i\phi t}dt$$

Proof: Follow the lines of the proof of Theorem 2!

4. Numerical Experiments

to test the algorithm: choose D as

$$D(\tau, s) := e^{i\tau\omega_0} e^{-\tau^2/2} 1_{[-s_1, s_2]}(s) \ ,$$

$\omega_0 \hat{=}$ shift in Fourier domain,
$s_1 = 0.90$ and $s_2 = 0.95$.

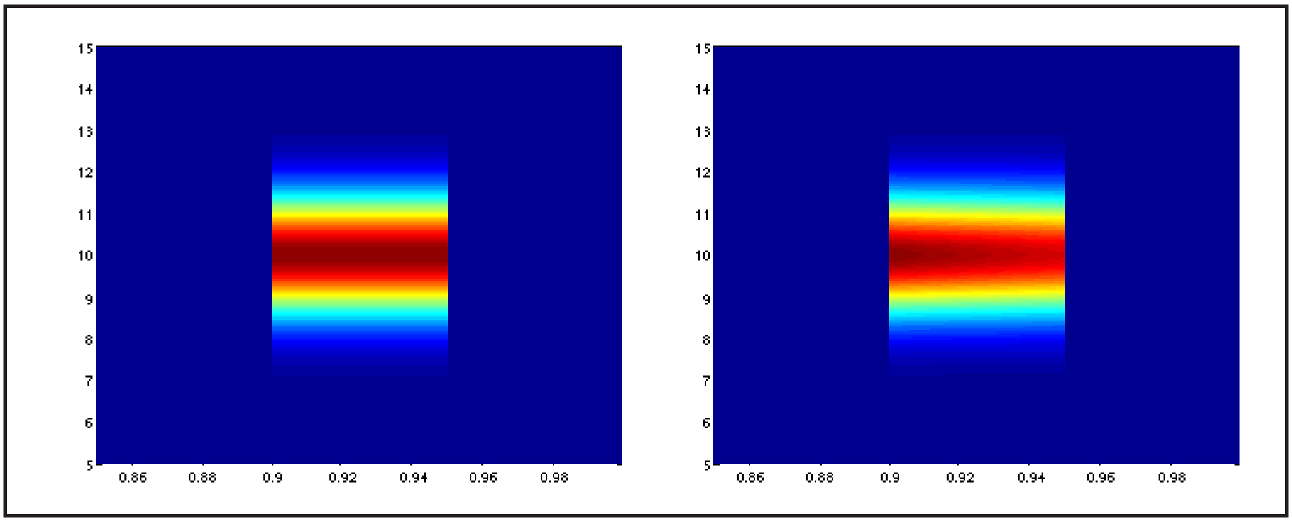


Figure 1: $\widehat{D(\cdot, s)}(\omega)$ and $\widehat{D(\cdot, s)}(\omega)|s|^{-3/2}$

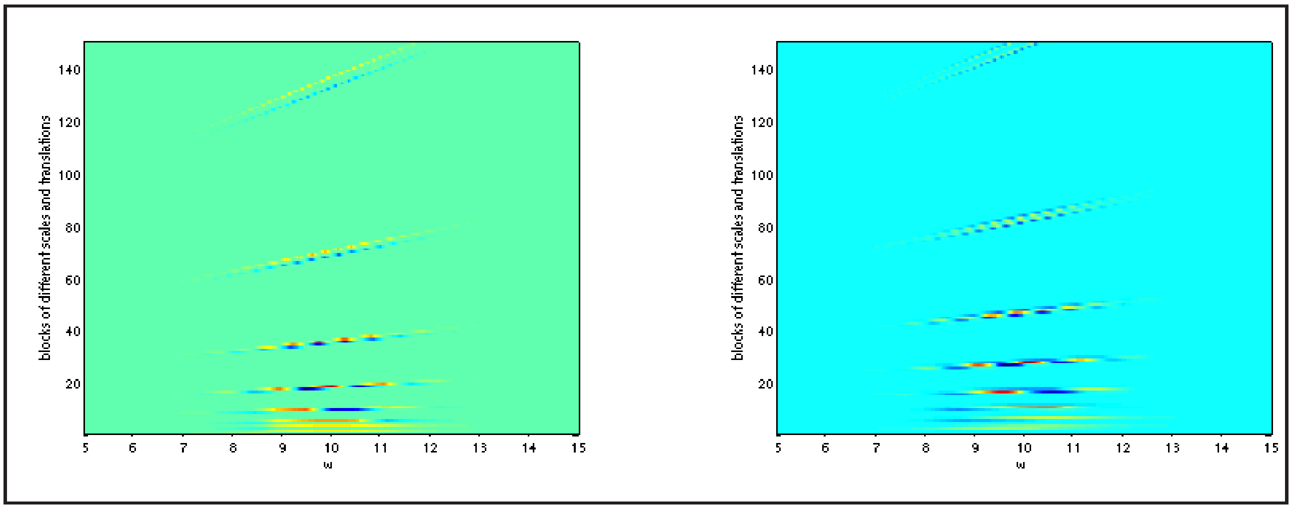


Figure 2: The echos $\{\hat{f}_m\}_{m\in\mathbf{Z}}$ for the Haar frame (left-hand side) and for the Daubechies-5-frame (right-hand side)

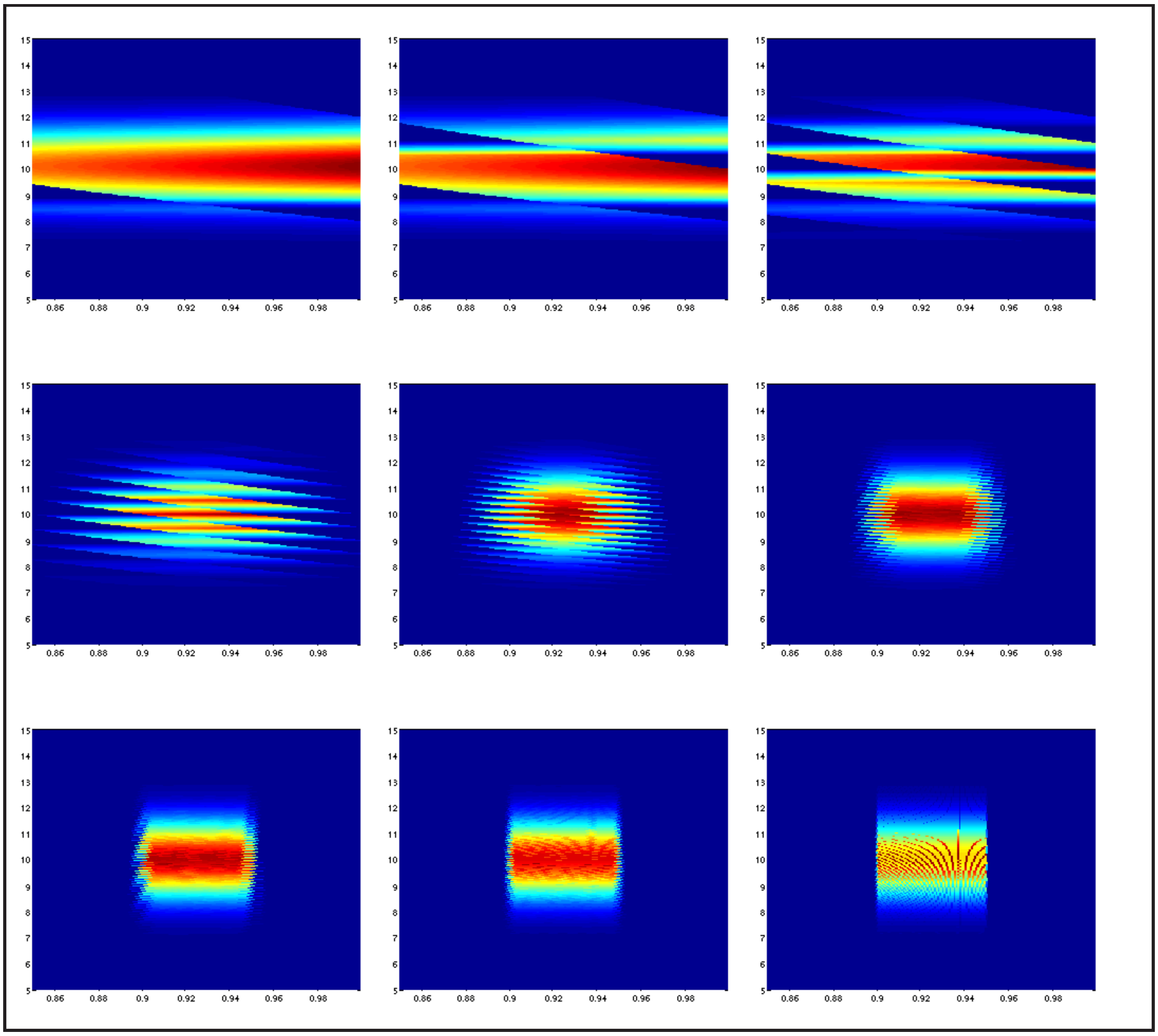


Figure 3: Partial reconstructions with respect to the Haar frame. The shown images correspond to the reconstructed densities for $J = -3, -2, -1, 0, 1, 2, 3, 4$ and 6 (from top left to bottom right).

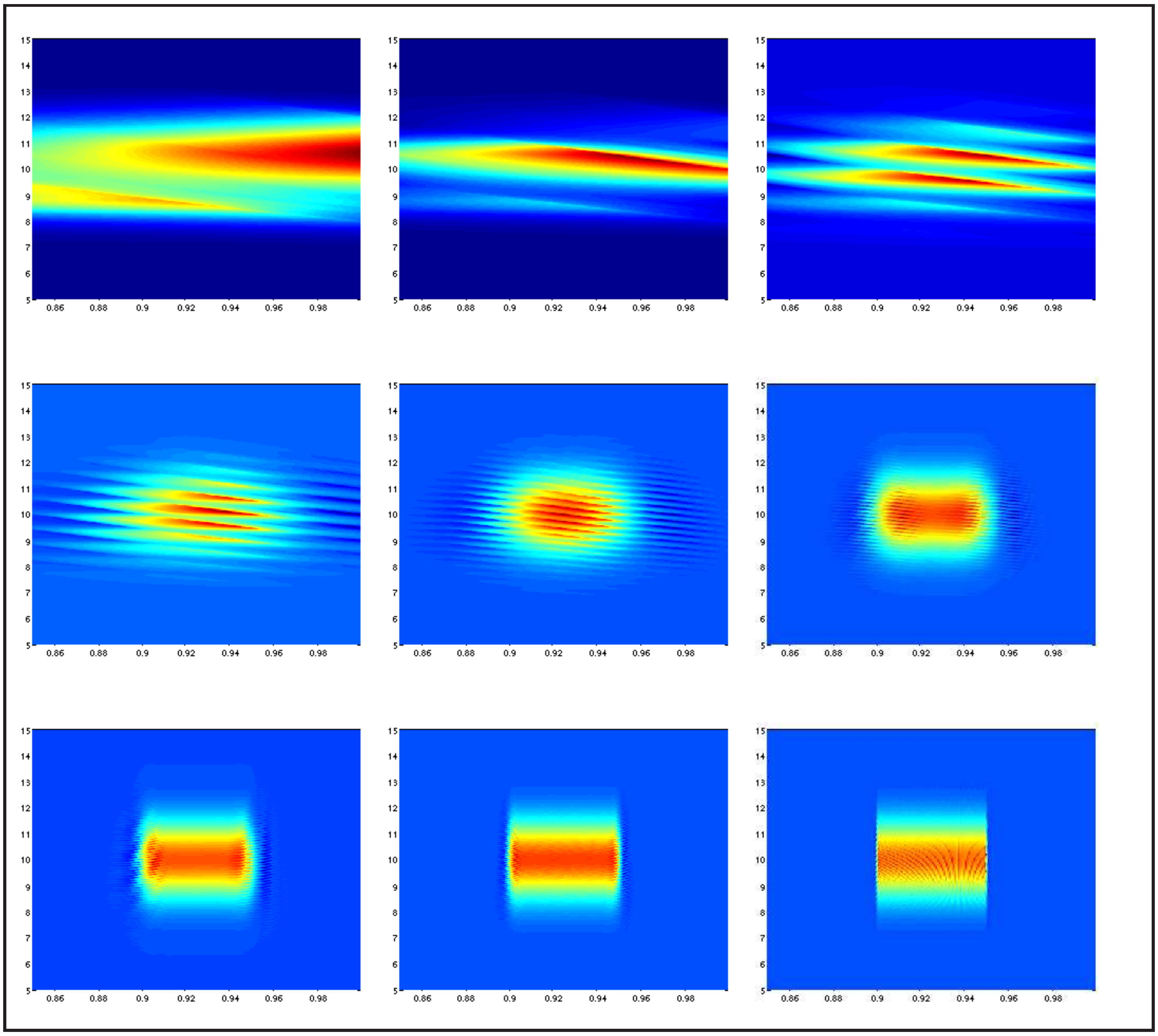


Figure 4: Partial reconstructions with respect to the Daubechies frame (N=2). The shown images correspond to the reconstructed densities for $J = -3, -2, -1, 0, 1, 2, 3, 4$ and 6 (from top left to bottom right).

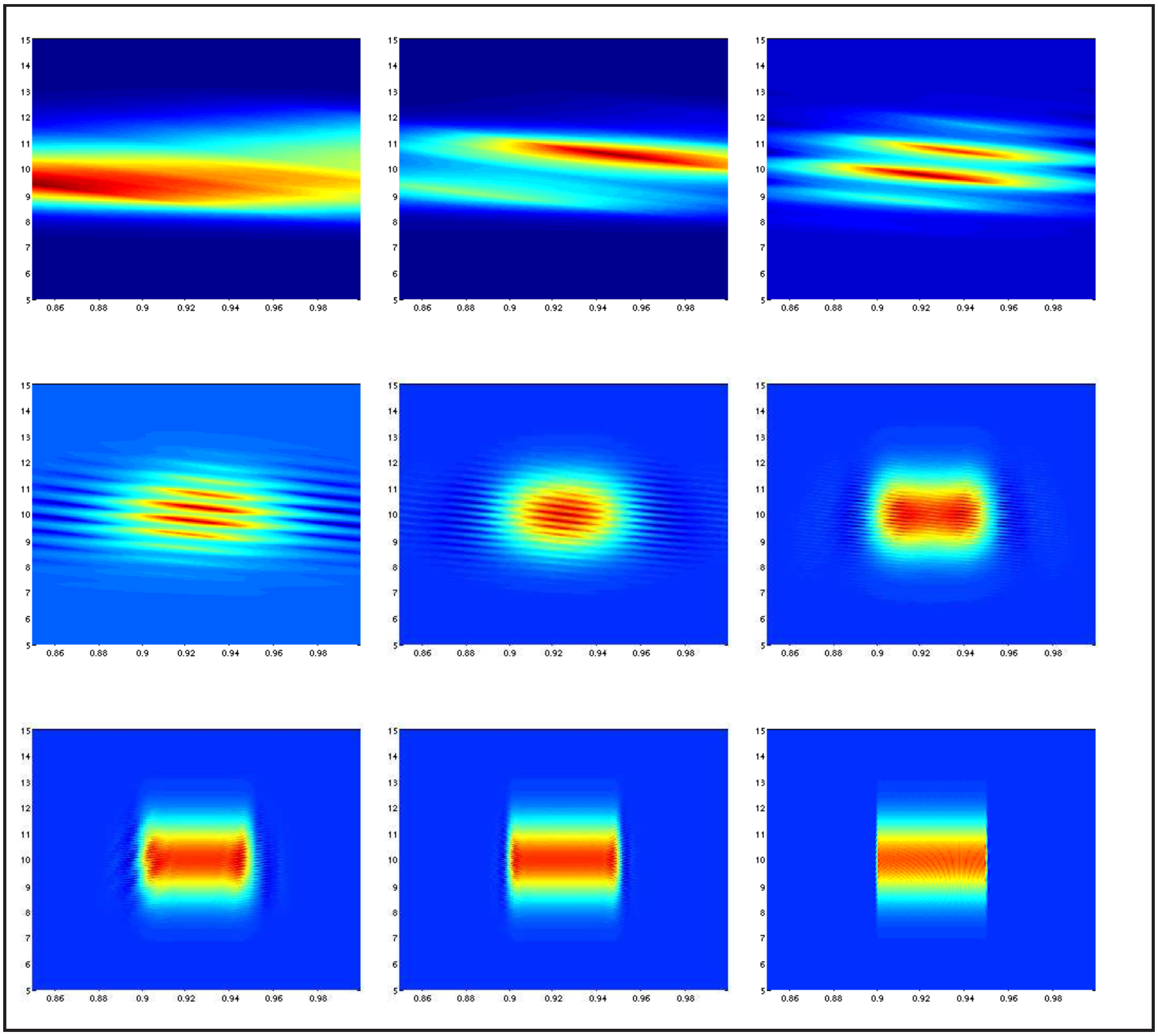


Figure 5: Partial reconstructions with respect to the Bior2.4 frame. The shown images correspond to the reconstructed densities for $J = -3, -2, -1, 0, 1, 2, 3, 4$ and 6 (from top left to bottom right).

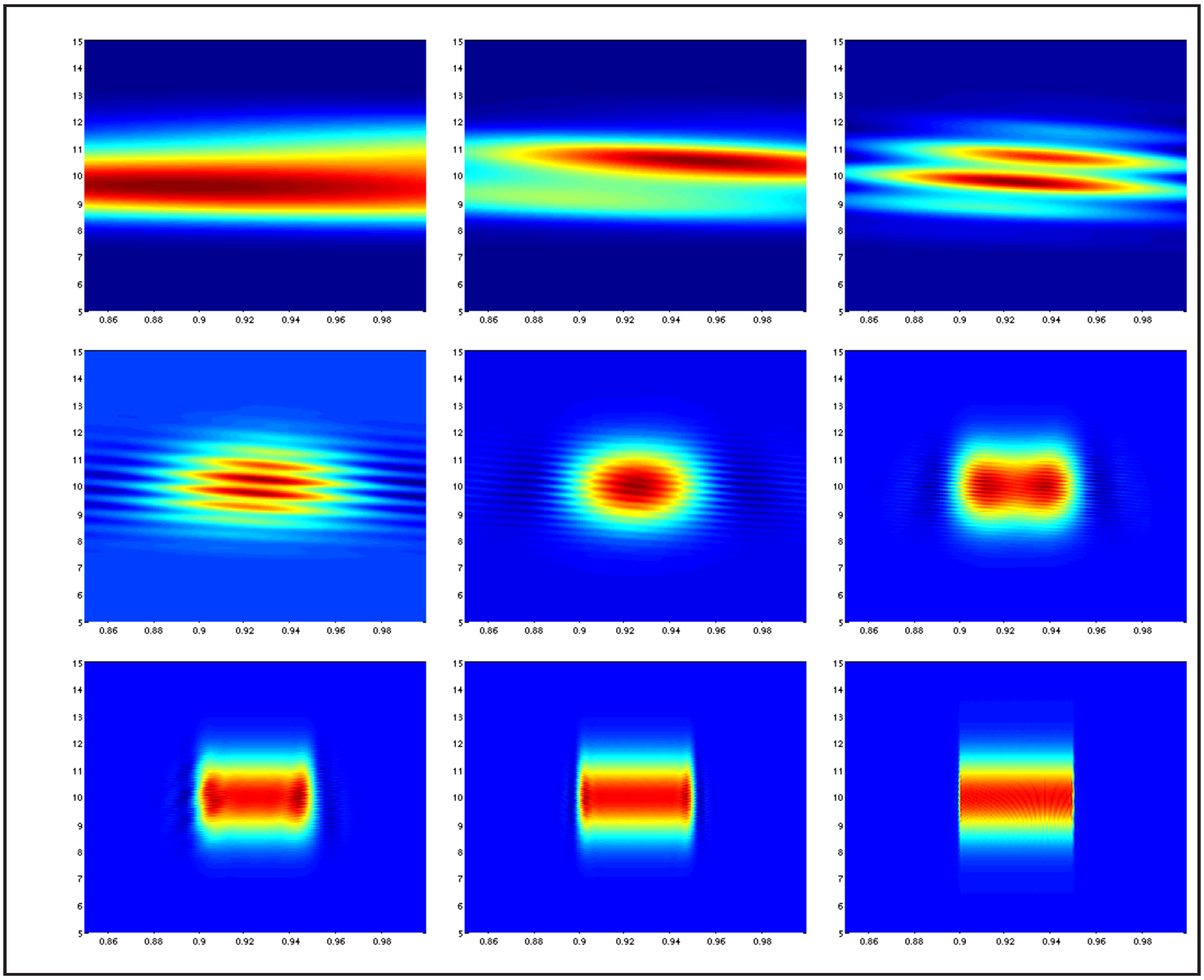


Figure 6: Partial reconstructions with respect to the Bior2.8 frame. The shown images correspond to the reconstructed densities for $J = -3, -2, -1, 0, 1, 2, 3, 4$ and 6 (from top left to bottom right).

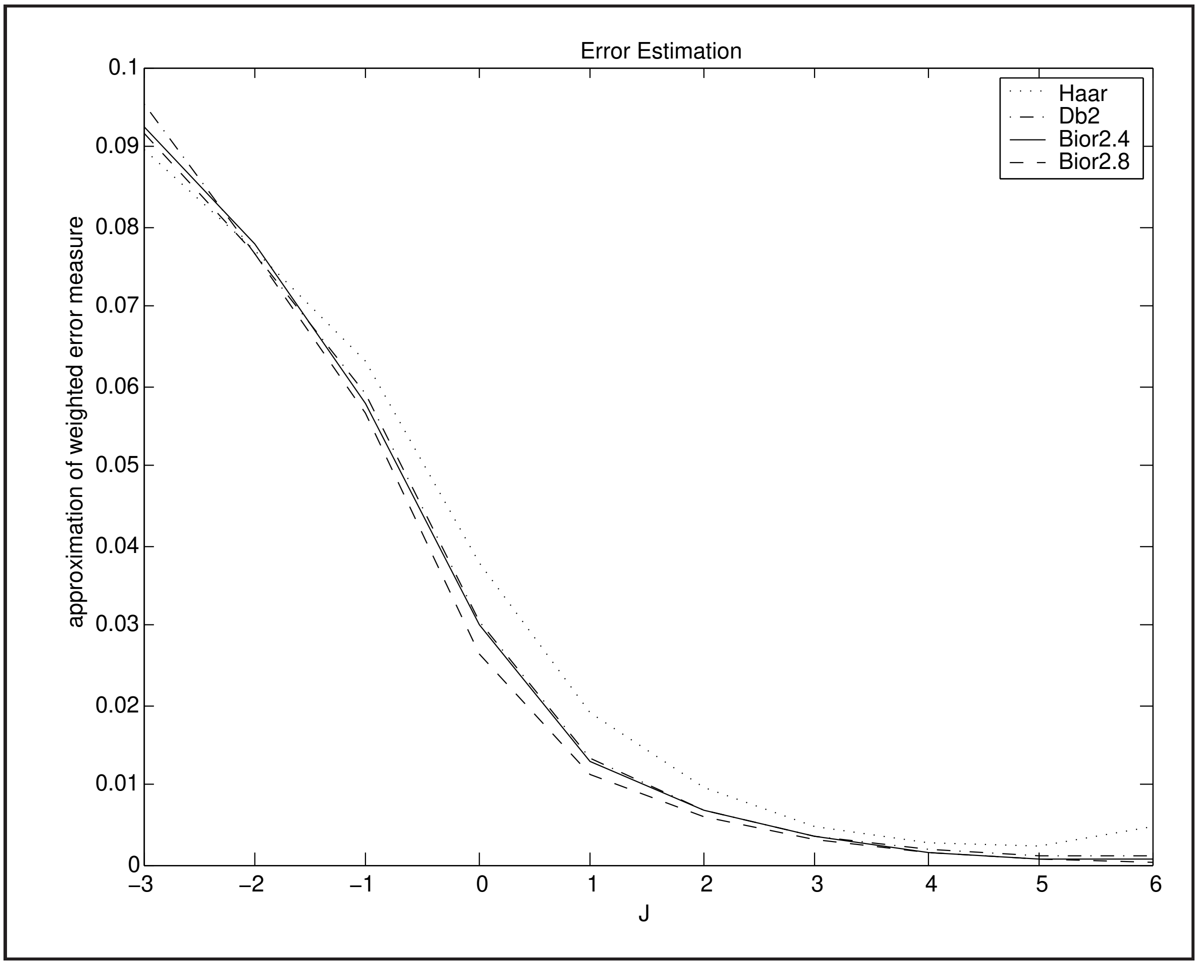


Figure 7: The weighted L_2-error

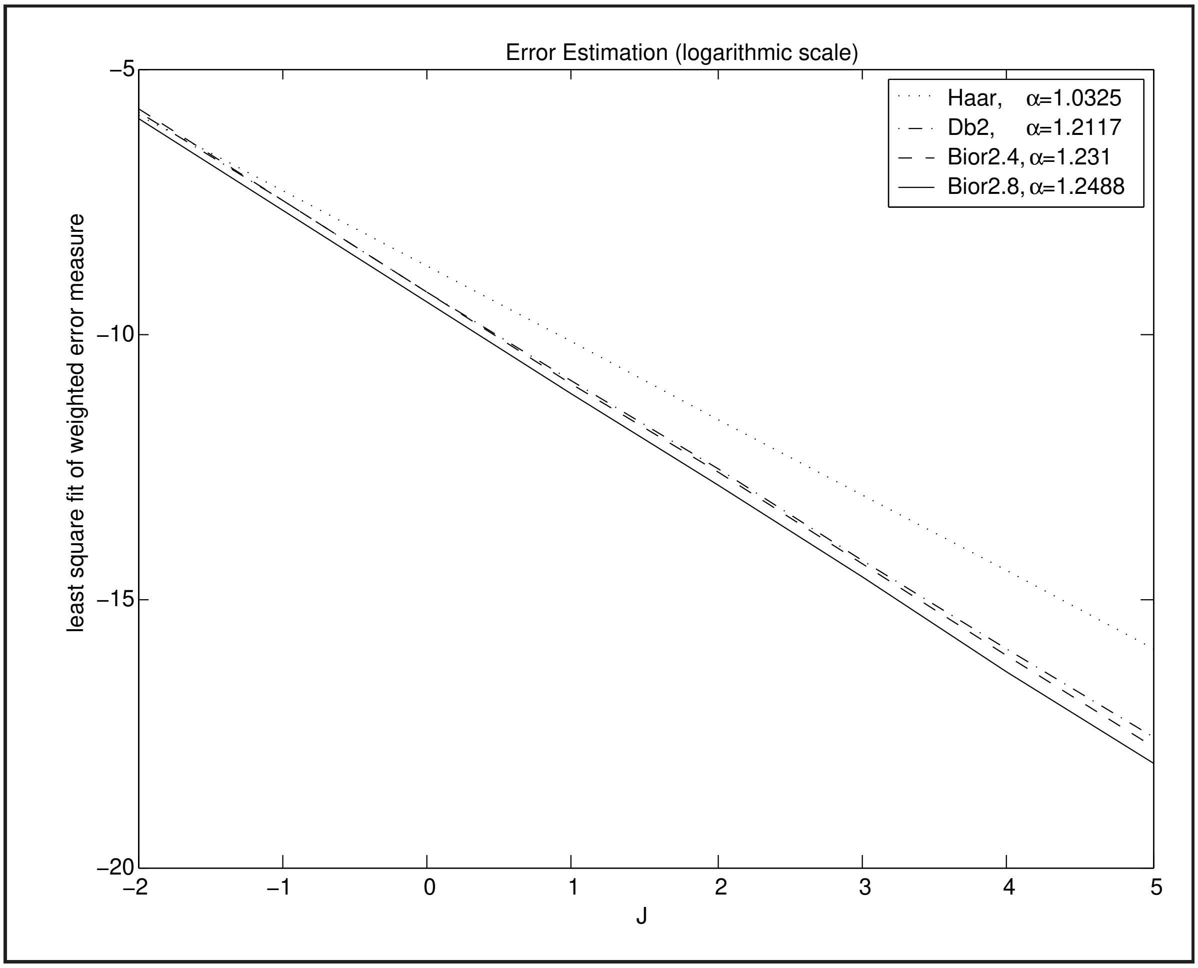


Figure 8: The weighted L_2-error, logarithmic scale