# Coorbit Spaces and Banach Frames on Homogeneous Spaces with Applications to Analyzing Functions on Spheres

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# 1. Motivation

- Classical problem: analyze/decompose/approximate... a given signal/function

  Basic step: decomposition into suitable building blocks

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

- common thread: square integrable group representations

  $\mathcal{H}$: Hilbert space, $\mathcal{G}$: Lie group,
  $U : \mathcal{G} \longrightarrow \mathcal{L}(\mathcal{H})$

  $U$ is square-integrable if $\exists\ \psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi\rangle_{\mathcal{H}}|^2 \, d\nu(g) < \infty$$

  Such a function $\psi$ is called admissible.

Examples:

- Weyl–Heisenberg group $\mathcal{G}_{WH}^{red}$, $\mathcal{H} = L_2(\mathbb{R})$
- affine group, $\mathcal{H} = L_2(\mathbb{R})$

- integral transform

$$V_\psi : \mathcal{H} \longrightarrow L_2(\mathcal{G})$$

$$V_\psi f(h) := \langle f, U(h^{-1})\psi \rangle$$

$\mathcal{G}_{WH}^{red} \rightsquigarrow$ Gabor transform

affine group $\rightsquigarrow$ wavelet transform

$$V_\psi(f) * V_\psi(\psi)(h) = V_\psi(f)(h)$$

- smoothness spaces, coorbit spaces

$$\mathcal{C}o(Y) := \{f \mid V_\psi(f) \in Y\}, \qquad \|f\|_{\mathcal{C}o(Y)} = \|V_\psi(f)\|_Y$$

$\mathcal{G}_{WH}^{red} : \mathcal{C}o(L_p(\mathcal{G}))$ modulation spaces $M_{pp}^0$

affine group : Besov spaces $B_p^{1/p-1/2}(L_p(\mathbb{R}))$

- Banach frames [Feichtinger/Gröchenig]

$$f = \sum_i c_i U(h_i^{-1})\psi = \sum_i c_i \psi_i$$

$$\frac{1}{B}\|f\|_{\mathcal{C}o(L_p)} \leq \|(\langle f, \psi_i\rangle)\|_{\ell_p} \leq \frac{1}{A}\|f\|_{\mathcal{C}o(L_p)}$$

Questions:

- How can Gabor/wavelet transforms on manifolds and associated coorbit spaces be defined?
- Is it possible to obtain Banach frames for these coorbit spaces?
- Convergence order of best $N$–term approximation?
- Applications to spheres?

Strategy: Combine the following building blocks:

- representations involving quotient groups [B. Torresani, J.–P. Antoine...]
- Feichtinger–Gröchenig theory
- nonlinear approximation by frames [K. Gröchenig/S. Samarah],[ R. DeVore/V. Temlyakov...]
- local Fourier analysis on spheres [B. Torresani]

# 2. Coorbit Spaces

$\mathcal{N}$ manifold, $\mathcal{H} = L_2(\mathcal{N})$

- First step: find group $\mathcal{G}$ with representation in $L_2(\mathcal{N})$

- In many cases (e.g. for the spheres): no square integrable representation available!

- Way out: quotients $\mathcal{G}/\mathcal{G}_{\mathcal{F}}$, $\mathcal{G}_{\mathcal{F}}$ closed subgroup
  $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\mathcal{F}}$, $\sigma : \mathcal{G}/\mathcal{G}_{\mathcal{F}} \to \mathcal{G}$ Borel section
  $U$ is strictly square integrable mod $(\mathcal{G}_{\mathcal{F}}, \sigma)$, if $\exists \psi \in L_2(\mathcal{N})$ such that

  $$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi \rangle_{L_2(\mathcal{N})}$$

  is an isometry.

- Problem: no group structure, no convolution, no reproducing kernel....

$$\begin{aligned} R(h,l) &:= \langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \\ &= \langle \psi, U(\sigma(h)\sigma(l)^{-1})\psi\rangle_{L_2(\mathcal{N})} \end{aligned}$$

$$C \geq \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(h,l)|d\mu(l) \qquad (*)$$

**Proposition 1.** *Let $U$ be a strictly square integrable* $\Longrightarrow$

$$\langle V_\psi f, R(h,\cdot)\rangle_{L_2(\mathcal{G}/\mathcal{G}_{\mathcal{F}})} = V_\psi f(h), \quad f \in L_2(\mathcal{N})$$

- Coorbit spaces

$$H_1 := \{f \in L_2(\mathcal{N}) : \ V_\psi f \in L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}})\},$$

$$\|f\|_{H_1} := \|V_\psi f\|_{L_1(\mathcal{G}/\mathcal{G}_{\mathcal{F}})}$$

$H_1 \hookrightarrow H \hookrightarrow H_1' \qquad H_1' :=$ dual space

$(*) \Longrightarrow \psi \in H_1 \Longrightarrow$

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle, \qquad f \in H_1'!$$

coorbit spaces:

$$M_p := \{f \in H_1' : \; V_\psi f \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})\}, \quad 1 \le p \le \infty$$

$$\|f\|_{M_p} := \|V_\psi f\|_{L_p(\mathcal{G}/\mathcal{G}_\mathcal{F})}$$

$$\mathcal{M}_p := \{F \in L_p(\mathcal{G}/\mathcal{G}_\mathcal{F}) \;:\; \langle F, R(h,\cdot)\rangle = F\}$$

**Proposition 2.** *(Correspondence Principle) Let $U$ be a strictly square integrable representation of $\mathcal{G}$ $\mathrm{mod}\ (\mathcal{G}_\mathcal{F}, \sigma) \Longrightarrow M_p$ and $\mathcal{M}_p$ are isometrically isomorphic, i.e., $\forall f \in M_p$*

$$\langle V_\psi(f), R(h,\cdot)\rangle = V_\psi(f)$$

*Moreover: $M_\infty = H_1'$, $M_2 = L_2(\mathcal{N})$*

# 3. Banach Frames

$e \in \mathcal{U} \subset \mathcal{G}, X = (x_i)_{i \in \mathcal{I}}$ is $\mathcal{U}$-dense if $\bigcup_{i \in \mathcal{I}} \mathcal{U} x_i = \mathcal{G}$

$$\mathcal{I}_\sigma := \{i \in \mathcal{I} : \ \sigma(\mathcal{G}/\mathcal{G}_\mathcal{F}) \cap \mathcal{U} x_i \neq \emptyset\}, \quad x_i = \sigma(h_i)$$

$$\mathrm{osc}_\mathcal{U}(l,h) :=$$
$$\sup_{u \in \mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi \rangle_{L_2(\mathcal{N})}|$$

**Theorem 1.** *Let $U$ be a strictly square integrable representation of $\mathcal{G}$ $\mathrm{mod}$ $(\mathcal{G}_\mathcal{F}, \sigma)$ in $L_2(\mathcal{N})$. Choose $\mathcal{U}$ so small that*

$$\int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \mathrm{osc}_\mathcal{U}(l,h)\,d\mu(l) < 1 \text{ and } \int_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \mathrm{osc}_\mathcal{U}(l,h)\,d\mu(h) < 1\,.$$

*Technical conditions $\Longrightarrow$ any $f \in M_p$, $1 \leq p < \infty$, has an expansion*

$$f = \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi.$$

*Moreover, the set*

$$\psi_i = U(\sigma(h_i)^{-1})\psi, \quad i \in \mathcal{I}_\sigma,$$

*is a Banach frame for $M_p$, i.e., there exist two constants $0 < A \leq B < \infty$ such that*

$$\frac{1}{B}\|f\|_{M_p} \leq \|(\langle f, \psi_i \rangle)_{i \in \mathcal{I}_\sigma}\|_{\ell_p} \leq \frac{1}{A}\|f\|_{M_p}.$$

# 4. Nonlinear Approximation

Quality of approximation schemes based on Banach frames? Especially, best $N$–term approximation

$$\Sigma_n := \{S \in M_p : \ S = \sum_{i \in J} a_i \psi_i, \ J \subseteq \mathcal{I}_\sigma, \ \mathrm{card} J \le n\}$$

$$E_n(f)_{M_p} := \inf_{S \in \Sigma_n} \|f - S\|_{M_p} \ .$$

- wavelet transform: [DeVore/Jawerth/Popov] $\rightsquigarrow$ Besov spaces

- Gabor transform: [Gröchenig/Samarah] $\rightsquigarrow$ Modulation spaces

**Theorem 2.** *Let* $\{\psi_i : \ i \in \mathcal{I}_\sigma\}$ *be a Banach frame for* $M_p$, $1 \le p \le \infty$. *If* $1 \le p < q$, $\alpha := 1/p - 1/q$ *and* $f \in M_p$, *then*

$$\left( \sum_{n=1}^{\infty} \frac{1}{n} \left( n^\alpha E_n(f)_{M_q} \right)^p \right)^{1/p} \le c \|f\|_{M_p}$$

*for a constant* $c < \infty$.

# 5. Applications to the Spheres

Aims:

- construction of a windowed Fourier transform on spheres [B. Torresani]
- modulations spaces
- associated Banach frames

$$\mathcal{G} := E(n) = SO(n) \ltimes \mathbb{R}^n$$
$$(R,p) \circ (\tilde{R},\tilde{p}) = (R\tilde{R}, R\tilde{p} + p)$$
$$(R,p)^{-1} = (R^{-1}, -R^{-1}p)$$
$$(U(R,p))f(s) := e^{i<s,p>} f(R^{-1}s)\ , \quad s \in S^{n-1}$$

not square integrable!

In the sequel: primarily $S^1$

$$\mathcal{G}_{\mathcal{F}} \cong \{(0,0,p_2) \in \mathcal{G}\}, \quad \Pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_{\mathcal{F}}$$

flat section $\sigma(\theta, p_1) = (\theta, p_1, 0)$

**Lemma 1. [B. Torresani]** $\operatorname{supp}\psi \subset [-\pi/2, \pi/2]$ *and*

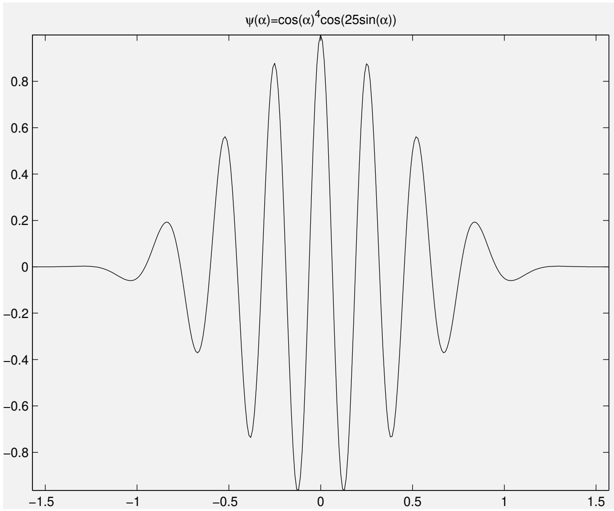
$$2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\alpha)|^2}{\cos\alpha} d\alpha = 1 \implies$$

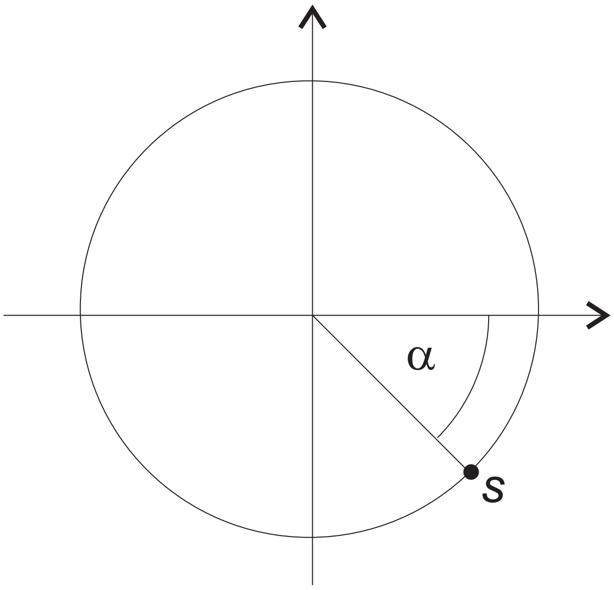
$$L_2(S^1) \ni g \mapsto V_\psi g \in L_2(\mathcal{G}/\mathcal{G}_F)$$

*is an isometry.*

Example:

$$\psi(\alpha) = \cos^4(\alpha) \cdot \chi_{[-\pi/2,\pi/2]}(\alpha)$$



not enough! Check

$$C \geq \int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(h,l)| d\mu(l) \qquad (*)$$

$$\int_{\mathcal{G}/\mathcal{G}_{\mathcal{F}}} |R(l,h)|\, d\mu(h) = \int_{-\pi}^{\pi} \int_{I\!R} |\hat{F}_{\theta,p_l}(p_h)|\, dp_h\, d\theta_h$$

$$\sigma(h) = (\theta_h, p_h, 0),\ \sigma(l) = (\theta_l, p_l, 0) \in \mathcal{G}/\mathcal{G}_{\mathcal{F}},\ \theta := \theta_h - \theta_l$$

$$\psi(\alpha) = \cos^6(\alpha) \cdot \chi_{[-\pi/2,\pi/2]}(\alpha)$$

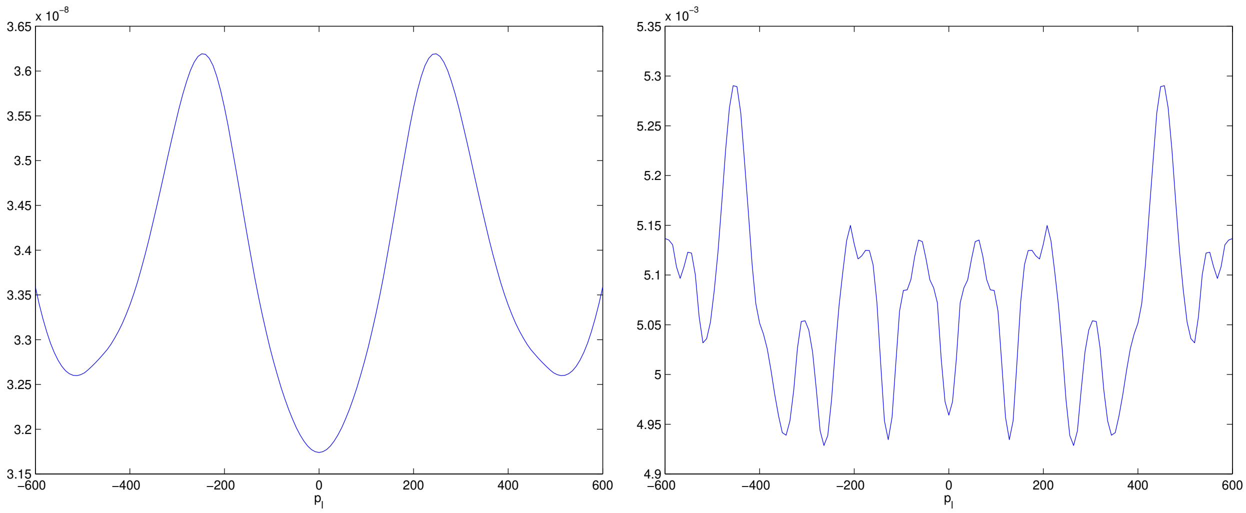



Figure 1: Left: $\int |\hat{F}_{\theta,p_l}(-p_h)| dp_h$ for $\theta = -2.7416$, right: $\int |\hat{F}_{\theta,p_l}(-p_h)| dp_h$ for $\theta = 2.0584$

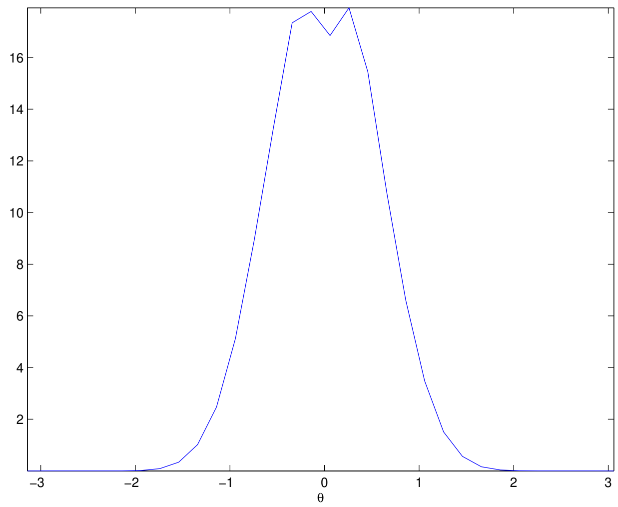



Figure 2: $\max_{p_l} \int |\hat{F}_{\theta,p_l}(-p_h)| dp_h$ as a function of $\theta$

$\Longrightarrow$ generalized modulation spaces well–defined!

moreover: $\mathcal{U}$ sufficiently small $\Longrightarrow$

$$\int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \mathrm{osc}_\mathcal{U}(l,h)d\mu(l) < 1 \text{ and } \int\limits_{\mathcal{G}/\mathcal{G}_\mathcal{F}} \mathrm{osc}_\mathcal{U}(l,h)d\mu(h) < 1$$

$\Longrightarrow$ Banach frames exist!

# Perspectives:

- higher–dimensional spheres
- numerical schemes
- more general manifolds
- more general coorbit spaces, weighted $L_p$–spaces

# Summary

- Banach frames for coorbit spaces
- integrability modulo subgroups
- generalized reproducing kernels, generalized coorbit spaces
- $\mathcal{U}$–oscillation $\Longrightarrow$ Banach frames
- applications to spheres