Symmetries of the Einstein-Hilbert action

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Abstract

Like the related paper *Grundbegriffe Differentialgeometrie für Gravitationstheorien* (in German) this talk was held at the seminar Geometrische Analysis (spring 2005) to clarify some basics of the mathematics in *General Relativity*. As an often ignored or mistaken topic which, too, was only grazed by the preceding presentation, the *gauge invariance* or *diffeomorphism invariance* of the *Einstein-Hilbert action* is shown explicitly and elementally. It turns out that the induced transformation of the involved geometrical objects as the *Levi-Civita connection* and the *curvature tensors* is each similar to its transformation under coordinate changes – a fact that originates in the same behavior of the *metric*, the dynamical variable, itself.

1 Introduction

In *General Relativity* the *metric* encodes the curvature of *space-time* M, thereby modelling the gravitational “force” on matter. So the metric itself is regarded as the dynamical *field* variable and we denote the space of (possible) fields

$$\begin{aligned}\mathscr{M} &:= \left\{\textit{Lorentz metric}\text{s } g \text{ on space-time } M \doteq \mathbb{R}^{1,3}\right\}\\ &= C^\infty(M; T^\# M \vee T^\# M).\end{aligned}$$

Since a metric g is (locally) described as a symmteric ‘matrix’ (more exactly a symmteric 2-tensor), the *degrees of freedom* for $M = \mathbb{R}^{1,3}$ are $\mathbb{R}^{10}$.

Whereas in the other common *field theories* (as *Yang-Mills theory* for *bosons* of spin < 2 or as *Dirac theory* for *fermions*) the *Lagrange action* on the fields

$$S : \mathscr{M} \longrightarrow \mathbb{R}$$

(sloppy as in physicist’s language) depends on the field and it’s first derivatives only, here one has to go to second derivatives to get a *covariant* action, i.e. invariant under the *action* (again this name!) of a certain *symmetry group* $\mathscr{G}$ – called *gauge group* (in generalization of *Maxwell theory* of *gauge fields*). The abstract *Lagrange mechanism* requires invariance of S under $\mathscr{G}$, and one actually has a mapping

$$S : \mathscr{M}/\mathscr{G} \longrightarrow \mathbb{R},$$

which is to be minimized (*principle of least action*) for a field to be a *physical field*. In general, that invariance thing is due to several reasons, especially the following ones:

1. Easier presentation and calculation: Often the space of fields $\mathscr{M}$ itself is linear or almost linear. In any case there is more algebraic structure than on $\mathscr{M}/\mathscr{G}$. This means one introduces additional degrees of freedom to get a nicer description, but has to take care of invariant formulation under some selecting conditions, captured by the gauge group.

2. Easily deducing conservation laws: Via *Noether theorem* each symmetry induces a *conservation law*.

We want to describe symmetries of the *Einstein-Hilbert action*

$$S_{\mathrm{EH}}(g) = \int_M \mathrm{Scal}^g \, dV_g \tag{1.1}$$

of General Relativity and show that S_{EH} is invariant under the *gauge group*

$$\mathscr{G} := \mathrm{Diff}^+(M)$$

of orientation preserving *diffeomorphisms* ϕ of space-time M. In general such symmetry groups for fields $M \longrightarrow E$ with degrees of freedom E split into

$$\mathscr{G} = \mathrm{Diff}(M) \times \mathrm{Weyl}(E) \doteq \mathrm{Aut}(M) \times \mathrm{Aut}(E)$$

where the latter *Weyl group* factor operates on the image space E, e.g. as $U(N)$ or $SU(N)$ on $\mathbb{C}^n$ in Yang-Mills theory. More exactly, in the Yang-Mills situation $E = T^\# M \otimes \mathbb{C}^n$ and the so called 'internal symmetries' $U(N)$ operate only on the $\mathbb{C}^n$-part, whereas the actual 'gauge symmetries' (in a tighter sense) act on the cotangent space $T^\# M$, leading to the direct product of both for $\mathrm{Weyl}(E)$. The former factor $\mathrm{Diff}(M)$ is to be regarded as kind of reparametrizations and (for $M = R^{1,d-1}$) is usually reduced to the *Poincaré group*

$$\mathrm{Poin}(d) \ := \ O(1,d-1) \ltimes \mathbb{R}^d \ = \ \mathrm{Aut}(\mathbb{R}^{1,d-1}) \ \sqsubset \ \mathrm{Aff}(\mathbb{R}^d)$$

due to the additional affine linear structur. The second factor $\mathrm{Weyl}(E)$ in the Einstein-Hilbert situation is trivial – except for the case of dimension $1+1$, where we really have invariance of the action under multiplications.

Remark. Because of the reparametrization character one usually has an additional group operation of the $\mathrm{Diff}(M)$ factor on the second $\mathrm{Weyl}(E)$ factor leading to the *semidirect product* as the true gauge group:

$$\mathscr{G} = \mathrm{Diff}(M) \ltimes \mathrm{Weyl}(E),$$

as was similarly the case for the Poincaré group above as a semi-directly splitted $\mathrm{Aut}(M)$ itself. Usually the Weyl group operation is the simple action of multiplications on the target space E: For fields $\mathscr{M} \ni \varphi : M \longrightarrow E$ we have the action of $\mathrm{Weyl}(E) = C^\infty(M, \mathbb{R}_+)$ given as

$$C^\infty(M, \mathbb{R}_+) \times \mathscr{M} \longrightarrow \mathscr{M}, \ (\rho, \varphi) \mapsto \rho\varphi$$

with $\big(\rho\varphi\big)(o) = \rho(o)\varphi_o$. As explained just before, $\mathrm{Diff}^+(M)$ acts on $C^\infty(M, \mathbb{R}_+)$, namely by pullback and we end up at the semidirect product $(\phi, \rho)(\phi', \rho') = \big(\phi\phi', (\rho \circ \phi'^{-1})\rho'\big)$.

This splitting fact of the gauge group $\mathscr{G}$ may be expressed in terms of exact sequences as

$$0 \longrightarrow \mathrm{Weyl} \longrightarrow \mathscr{G} \longrightarrow \mathrm{Diff} \longrightarrow 0$$

which is analogous to the well-known Quantum mechanics picture of *Heisenberg group* and *symplectic group*,

$$0 \longrightarrow \mathrm{Heis}(\mathbb{R}^{2n+1}) \longrightarrow \mathscr{G} \longrightarrow \mathrm{Sp}(2n+1,\mathbb{R}) \longrightarrow 0$$

or in pure mathematics to the splitting of the automorphism group into translations and multiplications

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Aut}(\mathbb{C}) \longrightarrow \mathbb{C}^{\times} \longrightarrow 1$$

or more generally

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \mathrm{Aff}(\mathbb{R}^n) \longrightarrow \mathrm{GL}(n,\mathbb{R}) \longrightarrow 1.$$

Note that for the full invariance (as we will obtain later by transformation rule) the semidirect product structure, i.e. the action of the reparametrizations $\mathrm{Diff}(M)$ on the simple multiplications, is essential.

But in our case, the Einstein-Hilbert action does not admit further symmetries than diffeomorphisms. Thus this more complicated structure is not important for us and only should have aided us to find our position.

2 The gauge group of diffeomorphisms

Proposition. *The group* $\mathscr{G} = \mathrm{Diff}^{+}(M)$ *of diffeomorphisms on space-time* M *acts on the space of fields* $\mathscr{M}$*, the Lorentz metrics on* M*, by* pullback*:*

$$\mathscr{G} \ltimes \mathscr{M} \longrightarrow \mathscr{M},\ (\phi, g) \mapsto \phi^* g.$$

Remark. Here for $o \in M$ and tangent vectors $X, Y \in T_o M$ at o, the pullback g' of g under fixed ϕ^{-1} is given by

$$\big((\phi^{-1})^* g\big)_o(X,Y) = g_{\phi(o)}(\Phi X, \Phi Y) \tag{2.1}$$

with the shorthand $\Phi := T_o\phi$, it's inverse then being $\Phi^{-1} = T_{\phi(o)}(\phi^{-1})$. We took the inverse on the left hand side to avoid three times inverse on the right hand side and further more.

Proof. For $\phi, \psi \in \mathscr{G}$ and $g \in \mathscr{M}$ we simply have

$$\begin{aligned}
\big(\phi^*(\psi^* g)\big)_o(X,Y) &= \psi^* g_{\phi^{-1}(o)}(\Phi^{-1}X, \Phi^{-1}Y) \\
&= g_{\psi^{-1}\left(\phi^{-1}(o)\right)}(\Psi^{-1}\Phi^{-1}X, \Psi^{-1}\Phi^{-1}Y) \\
&= g_{(\phi\circ\psi)^{-1}(o)}\big((\Phi\Psi)^{-1}X, (\Phi\Psi)^{-1}Y\big) \\
&= \big((\phi\circ\psi)^* g\big)_o(X,Y).
\end{aligned}$$

□

Remark. We want to present two further descriptions of this action.

1. In local coordinates with metric components $g_{mn} := g(\partial_m, \partial_n)$ and again the tangent or *Jacobian* $\Phi^{\mu}_{m}\partial_\mu := \Phi\partial_m = T_o\phi\,\partial_m$

$$g'_{mn}(o) = \Phi^{\mu}_{m} g_{\mu\nu}\big(\phi(o)\big)\Phi^{\nu}_{n}.$$

2. To abbreviate notation we may regard the metric as a matrix identifying the 2-*tensor* g_{mn} with matrix components g^m_n. We then have $g(X,Y) = X^t gY$ as matrix multiplication where X now denotes a column with coefficients X^m locally given by $X = X^m \partial_m$. Finally, with $\widetilde{g} := g \circ \phi$ we can write the above equation as

$$g' = \Phi^t \widetilde{g}\,\Phi.$$

In general, we want to distinguish between "*inner*" and "*outer*" transformations of the following objects and abbreviate the "inner" ones by a tilde. In each case they will be the usual pullback of the object (under ϕ^{-1}) but without the "outer" ones emerging for matrix valued fields as connection and curvature. Here in the last notation, g has become a (matrix valued) scalar, thus its pullback reduces to simply composing with ϕ: the "inner" transformation, and, in addition, multiplying with Φ^t and Φ from each side, respectively: the "outer" transformation.

3 Einstein-Hilbert action

We describe the ingredients of the Einstein-Hilbert action

$$S_{\mathrm{EH}}(g) = \int_M \mathrm{Scal}^g \, dV_g. \tag{3.1}$$

For a metric $g \in \mathscr{M}$, i.e. on $M = \mathbb{R}^{1,3}$ and of Minkowski type (signature $(1,3)$), we subsequently construct the following objects:

- Levi-Civita connection (*metric connection*) $\nabla^g : \Gamma(TM) \longrightarrow \mathrm{End}\big(\Gamma(TM)\big)$ of g
- Riemann curvature $R^g : \Gamma(TM) \times \Gamma(TM) \longrightarrow \mathrm{End}\big(\Gamma(TM)\big)$ of ∇^g
- Ricci curvature $\mathrm{Ric}^g : \Gamma(TM) \times \Gamma(TM) \longrightarrow \mathbb{R}$ of R^g
- Scalar curvature $\mathrm{Scal}^g : M \longrightarrow \mathbb{R}$ of R^g by means of Ric^g

Here we only want to recall some collected formulas for each of them and refer to [Eckert] (in German) as a more detailed presentation or to standard texts in differential geometry, e.g. [Hicks].

- The *Levi-Civita connection* ∇ is given by the (local) *Christoffel symbols*

$$\Gamma^i_{jk} := \partial^i \nabla_{\partial_k} \partial_j = \tfrac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} + \partial_l g_{jk} = \Gamma^i_{kj} \tag{3.2}$$

  where g^{ij} are the so-called *inverse metric* or *dual metric* components with $g^{ij} g_{jk} = \delta^i_j$. Besides we want to work with the *connection-1-form* A, being endomorphism-valued, i.e. we have to apply $A(X)$ to a *vector field* yielding again a vector field:

$$A(X)Y := \nabla_X Y. \tag{3.3}$$

- The *Riemann curvature* of the connection is given by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -R(Y,X) \tag{3.4}$$

or in local expressions by the curvature coefficients

$$R^m_{nkl} := \partial^m R(\partial_k, \partial_l)\partial_n = \partial_k \Gamma^m_{nl} - \partial_l \Gamma^m_{nk} + \Gamma^m_{rk}\Gamma^r_{nl} - \Gamma^m_{rl}\Gamma^r_{nk}, \tag{3.5}$$

but we do not need the latter term. The curvature is in fact a *2-form* again endomorphism-valued and now called *field strength*

$$F := R = d_A A := dA + A \wedge A = (dA^i_j + A^i_k \wedge A^k_j)_{i,j}. \tag{3.6}$$

We shall prove this formula as an example at the end of this listing. Note that the wedge product is a matrix wedge product where A is viewed as a matrix of 1-forms A^i_j which of course is squared by summing up wedge products of its entries.

- The *Ricci curvature* is given by $\mathrm{Ric}(X,Z) = \mathrm{tr}\big(R(X, \mathbf{.})Z\big)$, globally, and locally by its components

$$R_{nk} := \mathrm{Ric}(\partial_n, \partial_k) = R^m_{nkm}. \tag{3.7}$$

- The *scalar curvature* Scal is most commonly described using metric *contraction* or *raising* and *lowering* of indices (*Ricci calculus*):

$$\mathrm{Scal} = R^k_k := g^{kn} R_{nk} = g^{kn} R^m_{nkm}. \tag{3.8}$$

Of course it also may be defined globally (or invariantly): $\mathrm{Scal} = \mathrm{tr}\, S$ where the endomorphism S is given by the formula $\mathrm{Ric}(X,Z) = g(X,SZ)$ possible at least for symmetric Ricci tensor, also called *torsion free*. This is always the case when constructed from the metric via Levi-Civita connection.

Lemma. *As mentioned above, the curvature-2-form is given by*

$$F = dA + A \wedge A = d_A A. \tag{3.9}$$

Proof. With $F(X,Y) = R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ on basis vectors we have

$$\begin{aligned} F(\partial_k, \partial_l)\partial_j &= \nabla_{\partial_k}(\Gamma^i_{jl}\partial_i) - \nabla_{\partial_l}(\Gamma^i_{jk}\partial_i - \nabla_{[\partial_k,\partial_l]}\partial_j \\ &= (\partial_k \Gamma^i_{jl})\partial_i + \Gamma^i_{jl}\Gamma^m_{ik}\partial_m - (\partial_l \Gamma^i_{jk})\partial_i - \Gamma^i_{jk}\Gamma^m_{il}\partial_m - 0. \end{aligned}$$

Then by (3.10) and(3.11) beneath, for $A =: A_m \partial^m$

$$\begin{aligned} \big(dA + A \wedge A\big)(\partial_k, \partial_l)\partial_j &= dA(\partial_k, \partial_l)\partial_j + (A \wedge A)(\partial_k, \partial_l)\partial_j \\ &= (\partial_k A_l - \partial_l A_k)\partial_j + A(\partial_k)\big(A(\partial_l)\partial_j\big) - A(\partial_l)\big(A(\partial_k)\partial_j\big) \\ &= (\partial_k \Gamma^i_{jl})\partial_i - (\partial_l \Gamma^i_{jk})\partial_i + A(\partial_k)\Gamma^i_{jl}\partial_i - A(\partial_l)\Gamma^i_{jk}\partial_i \\ &= F(\partial_k, \partial_l)\partial_j \end{aligned}$$

as shown above. For the missing equations we write

$$dA(\partial_k, \partial_l) = (dA_m \wedge \partial^m)(\partial_k, \partial_l) = \partial_k A_m \partial^m \partial_l - \partial_l A_m \partial^m \partial_k \tag{3.10}$$

where then $\partial^m \partial_l = \delta^m_l$ etc. and besides

$$(\partial_k A_l)\partial_j = \big(\partial_k A(\partial^l)\big)\partial_j = \partial_k\big(A(\partial^l)\partial_j\big) = \partial_k(\Gamma^i_{jl}\partial_i) = (\partial_k \Gamma^i_{jl})\partial_i. \tag{3.11}$$

□

Finally there is left one ingredient of the Einstein-Hilbert action, namely the *measure*

$$dV_g := \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n \tag{3.12}$$

which by construction of integration on manifolds is simply the standard measure on (M,g), i.e. M equipped with the metric g.

4 Gauge invariance of the Einstein-Hilbert action

We want to show that the Einstein-Hilbert action S_{EH} is (covariant or) invariant under the gauge group $\mathscr{G}$ of diffeomrphisms, that is $\phi^* S_{\mathrm{EH}} = S_{\mathrm{EH}}$ for any $\phi \in \mathscr{G}$, or for $g \in \mathscr{M}$,

$$S_{\mathrm{EH}}(\phi^* g) = S_{\mathrm{EH}}(g).$$

To start with, we first inspect the measure part of the action. With

$$|\det\big((\phi^{-1})^* g\big)| = |\det(\Phi^t \widetilde{g}\,\Phi)| = \det{}^2\Phi |\det g| \circ \phi$$

we obtain

$$dV_{(\phi^{-1})^* g} = \sqrt{|\det(\phi^{-1})^* g} dx = |\det \Phi| \sqrt{|\det g|} \circ \phi\ dx.$$

And since $\Phi = T\phi$, to apply trafo-formula it remains to show

$$\mathrm{Scal}^{(\phi^{-1})^* g} = \mathrm{Scal}^g \circ \phi, \tag{4.1}$$

which is the same formula as for the fact that the scalar curvature “transforms as” a *scalar* (tensor). But be careful, here we don’t deal with coordinate changes but with transformations of the metric g and the induced changes of the respective objects as Scal in the end. Compare next remark.

The tail of this section shall prove this transformation formula. Its main portion, at least technically, is the transformation of the Levi-Civita which is covered by the following

Proposition. *Under a diffeomorphism transformation of the metric g the Levi-Civita connection in local terms is changed into*

$$\Gamma'^k_{mn} = (\partial_m \Phi^\kappa_n)(\Phi^{-1})^k_\kappa + \Phi^\mu_m \Phi^\nu_n \widetilde{\Gamma}^\kappa_{\mu\nu} (\Phi^{-1})^k_\kappa. \tag{4.2}$$

Again, the prime indicates the components of the connection for $(\phi^{-1})^* g$, and for the simple functions $\Gamma^\kappa_{\mu\nu}$ the tilde again only means composition with ϕ.

Remark. 1. If we investigate that formula we find the following structure:

$$\Gamma' = \Phi^{-1}\Gamma\Phi + \Phi^{-1} d\Phi = \Phi^{-1}(\Gamma + d)\Phi.$$

Of course this is quite sloppy: Firstly in dropping the “inner” transformations, namely the tilde of (the function) Γ^k_{mn} together with the transformation of the n index, which in form notation is the argument X of $A(X)$. And secondly it is sloppy in treating the exterior derivative d the same as Γ. Observe that $d\Phi = TT\phi$ is second derivative of ϕ and $(d\Phi)^k_{mn} = \partial_m \Phi^k_n = \partial_m \partial_n \phi^k$ in charts.

2. After this we can read off the transformation rule for the connection-1-form $A(X) = \nabla_X$ as

$$A' = \Phi^{-1}\widetilde{A}\Phi + \Phi^{-1} d\Phi = \Phi^{-1}(\widetilde{A} + d)\Phi \tag{4.3}$$

where now the “inner” transformation $\widetilde{A}$ means $\widetilde{A}_o(X) := A_{\phi(o)}(\Phi X)$. This formula in turn adds up to clarify the first topic.

3. The formula is the same as for transformation under a change of coordinates. In fact, the action of a gauge ϕ on a metric g is the same, as one would have expressed a change of charts by the diffeomorphism ϕ, at first for g but apparently it pulls out. Thereafter, the transformation law of the proposition implies that the connection is not a tensor because of the additional summand with second derivatives.

Proof. By definition of the Christoffel symbols we find it easier to multiply by the "inverses", and thus calculate

$$\begin{aligned}
2\Gamma'^k_{mn} g'_{kl} &= \partial_m g'_{nl} + \partial_n g'_{lm} - \partial_l g'_{mn} \\
&= \partial_m(\Phi^\nu_n \widetilde{g}_{\nu\lambda}\Phi^\lambda_l) + \partial_n(\Phi^\lambda_l \widetilde{g}_{\lambda\mu}\Phi^\mu_m) - \partial_l(\Phi^\mu_m \widetilde{g}_{\mu\nu}\Phi^\nu_n) \\
&= (\partial_m\Phi^\nu_n)\widetilde{g}_{\nu\lambda}\Phi^\lambda_l + \Phi^\nu_n(\widetilde{\partial_\mu g_{\nu\lambda}})\Phi^\mu_m\Phi^\lambda_l + \Phi^\nu_n\widetilde{g}_{\nu\lambda}(\partial_m\Phi^\lambda_l) \\
&\quad + (\partial_n\Phi^\lambda_l)\widetilde{g}_{\lambda\mu}\Phi^\mu_m + \Phi^\lambda_l(\widetilde{\partial_\nu g_{\lambda\mu}})\Phi^\nu_n\Phi^\mu_m + \Phi^\lambda_l\widetilde{g}_{\lambda\mu}(\partial_n\Phi^\mu_m) \\
&\quad - (\partial_l\Phi^\mu_m)\widetilde{g}_{\mu\nu}\Phi^\nu_n - \Phi^\mu_m(\widetilde{\partial_\lambda g_{\mu\nu}})\Phi^\lambda_l\Phi^\nu_n - \Phi^\mu_m\widetilde{g}_{\mu\nu}(\partial_l\Phi^\nu_n)
\end{aligned}$$

where some terms cancel by Schwarz rule of commuting second partial derivatives, remembering the preceding remark that $\partial_m\Phi^k_n = \partial_m\partial_n\phi^k$: namely third with seventh and fourth with the last. In addition, by the same reason the first term is equal to the sixth such that we get

$$\begin{aligned}
2\Gamma'^k_{mn} g'_{kl} &= 2(\partial_m\Phi^\eta_n)\widetilde{g}_{\eta\lambda}\Phi^\lambda_l + \Phi^\mu_m\Phi^\nu_n\Phi^\lambda_l(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})^{\widetilde{}} \\
&= 2(\partial_m\Phi^\eta_n)\widetilde{g}_{\eta\lambda}\Phi^\lambda_l + 2\Phi^\mu_m\Phi^\nu_n\Phi^\lambda_l(\Gamma^\kappa_{\mu\nu}g_{\kappa\lambda})^{\widetilde{}}.
\end{aligned}$$

Now, inserting a $\delta^\kappa_\eta = (\Phi^{-1})^k_\eta\Phi^\kappa_k$ in front of either g-term and adapting its indices, in both summands we obtain a factor $\Phi^\kappa_k\widetilde{g}_{\kappa\lambda}\Phi^\lambda_l = g'_{kl}$ which can be pulled out to end up with

$$2\Gamma'^k_{mn} g'_{kl} = 2\big((\partial_m\Phi^\eta_n)(\Phi^{-1})^k_\eta + \Phi^\mu_m\Phi^\nu_n\widetilde{\Gamma}^\eta_{\mu\nu}(\Phi^{-1})^k_\eta\big)g'_{kl}.$$

□

Once this result is established that transforming g by diffeomorphisms induces changes of the metric and consequently of the connection similar to changes of coordinates, we can continue with standard proofs for the transformation of the curvature, which contrary to the connection again becomes a tensor, since the ill terms cancel. We want to avoid the many indices, hence switch to form notation which shall be demonstrated anyway. In addition, this separates the different roles of "outer" and "inner" transformations.

Proposition. *Transforming the metric g by the gauge group, the curvature F is changed into*

$$F' = \Phi^{-1}\widetilde{F}\Phi \tag{4.4}$$

where now, of course, $\widetilde{F}_o(X,Y) := F_{\phi(o)}(\Phi X, \Phi Y)$.

Proof. Expanding the expression

$$\begin{aligned}
F' &= dA' + A'\wedge A' \\
&= d(\Phi^{-1}\widetilde{A}\Phi + \Phi^{-1}d\Phi) + \Phi^{-1}(\widetilde{A}+d)\Phi\wedge\Phi^{-1}(\widetilde{A}+d)\Phi
\end{aligned}$$

by product rule for exterior derivative (observe the minus when passing a form of odd degree) and by contraction of $B\Phi^{-1}\wedge\Phi C = B\wedge\Phi^{-1}\Phi C = B\wedge C$, we obtain

$$\begin{aligned}
F' = d\Phi^{-1}\wedge\widetilde{A}\Phi + \Phi^{-1}(d\widetilde{A})\Phi - \Phi^{-1}\widetilde{A}\wedge d\Phi + d\Phi^{-1}\wedge d\Phi + \Phi^{-1}d^2\Phi \\
+ \Phi^{-1}\widetilde{A}\wedge\widetilde{A}\Phi + \Phi^{-1}\widetilde{A}\wedge d\Phi - d\Phi^{-1}\wedge\widetilde{A}\Phi - d\Phi^{-1}\wedge d\Phi
\end{aligned}$$

where for the last two terms (carefully!) we used $\Phi^{-1}d\Phi = -d(\Phi^{-1})\Phi$ which is easily seen from the product rule again:

$$d(\Phi^{-1})\Phi + \Phi^{-1}d\Phi = d(\Phi^{-1}\Phi) = d\,\mathbf{1} = 0.$$

From the preceeding sum three pairs of terms cancel, and with $d^2 = 0$ we arrive at

$$F' = \Phi^{-1}(d\widetilde{A} + \widetilde{A} \wedge \widetilde{A})\Phi = \Phi^{-1}\widetilde{F}\Phi.$$

Note that actually $d\widetilde{A} = \widetilde{dA}$ and similarly $\widetilde{A} \wedge \widetilde{A} = \widetilde{A \wedge A}$, which has to be checked by direct computation, i.e. in local expressions: If again $A = A_k\partial^k$ then $\widetilde{A} = \widetilde{A}_k\partial^k\Phi = (A_k \circ \phi)\Phi^k_m\partial^m$ and by definition of exterior derivative one has

$$\begin{aligned} d\widetilde{A} &= \partial_n(\widetilde{A}_k\Phi^k_m)\partial^n \wedge \partial^m \\ &= \left[(\partial_n\widetilde{A}_k)\Phi^k_m + \widetilde{A}_k\partial_n\Phi^k_m\right]\partial^n \wedge \partial^m \\ &= \widetilde{\partial_l A_k}\Phi^l_n\Phi^k_m\partial^n \wedge \partial^m \end{aligned}$$

since the second summand $\partial_n\Phi^k_m = \partial_n\partial_m\phi^k$ vanishes in the wedge product because of Schwarz rule. Note that $\widetilde{\partial_l A_k}$ is a convenient shortcut for $(\partial_l A_k) \circ \phi$, coming in from chain rule. Now, by the same computations that is exactly the same as

$$\widetilde{dA} = (\partial_l A_k\partial^l \wedge \partial^k)^{\widetilde{}} = \widetilde{\partial_l A_k}\Phi^l_n\Phi^k_m\partial^n \wedge \partial^m.$$

The similar wedge statement is trivial. Of course, for this 2-form $A \wedge A$ the same definition of the tilde is being in mind as for the 2-from F of curvature. □

Finally, we can conclude the proof of invariance by the following, already stated result:

Corollary. *Under a diffeomorphism change of the metric g the scalar curvature* Scal^g *is transformed into*

$$\mathrm{Scal}^{(\phi^{-1})^*g} = \mathrm{Scal}^g \circ \phi.$$

Proof. As common in Ricci calculus, let $R = \mathrm{Scal}$ and $R' = \mathrm{Scal}^{(\phi^{-1})^*g}$. Then, from $F' = \Phi^{-1}\widetilde{F}\Phi$ we read off the transformation of the curvature components as

$$R'^m_{nkl} = (\Phi^{-1})^m_\mu \widetilde{R}^\mu_{\nu\kappa\lambda}\Phi^\nu_n\Phi^\kappa_k\Phi^\lambda_l \tag{4.5}$$

where the *conjugation* with Φ, the "outer" transformation, is encoded in the summation over μ and ν, and the two other products with Φ arise from the "inner" transformation of the two vector slots of R, or F at will. This way, of the inner transformation there is still left the composition with ϕ, which is denoted $\widetilde{R}^\mu_{\nu\kappa\lambda}$.

Besides $g'^{ab} = (\Phi^{-1})^a_\alpha \widetilde{g}^{\alpha\beta}(\Phi^{-1})^b_\beta$, for, if we multiply the right hand side with the well-known $g'_{bc} = \Phi^\eta_b\widetilde{g}_{\eta\gamma}\Phi^\gamma_c$, this reduces to

$$(\Phi^{-1})^a_\alpha\widetilde{g}^{\alpha\beta}\delta^\eta_\beta\widetilde{g}_{\eta\gamma}\Phi^\gamma_c = (\Phi^{-1})^a_\alpha\delta^\alpha_\gamma\Phi^\gamma_c = \delta^a_c.$$

But then, we get

$$R' = R'^k_k = g'^{kn}R'^m_{nkm} = (\Phi^{-1})^k_\alpha\widetilde{g}^{\alpha\beta}(\Phi^{-1})^n_\beta(\Phi^{-1})^m_\mu\widetilde{R}^\mu_{\nu\kappa\lambda}\Phi^\nu_n\Phi^\kappa_k\Phi^\lambda_l$$

where all the Φ-terms cancel over m, n, and k, yielding

$$R' = \widetilde{g}^{\kappa\nu}\widetilde{R}^\mu_{\nu\kappa\mu} = (g^{\kappa\nu}R^\mu_{\nu\kappa\mu}) \circ \phi = R^\kappa_\kappa \circ \phi = R \circ \phi,$$

the assertion. □

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(The last two references being only useful here in few aspects.)

List of Symbols

A	connection-1-form **4**
Aff	affine transformations **2**
$\mathrm{Aut}(\mathbb{R}^{1,d-1})$	automorphisms preserving the Lorentz metric **2**
$\mathrm{Diff}^+(M)$	orientation preserving diffeomorphisms **2**
dV_g	measure **5**
F	field strength **5**
$\mathscr{G}$	gauge group **2**
g	Lorentz metric **1**
g^{ij}	dual metric **4**
g_{mn}	metric components **3**
Γ^i_{jk}	Christoffel symbols **4**
$\mathrm{Heis}(\mathbb{R}^{2n+1})$	Heisenberg group **3**
M	space-time **1**
$\mathscr{M}$	space of fields **1**
∇^g	Levi-Civita connection **4**
$O(1,d-1)$	pseudo-orthogonal group **2**
$o \in M$	space-time points **3**

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