A Proof of Theorem 1

Lemma 1: Let $s_\imath$, $\imath = 1 \ldots m$, be real numbers such that $0 \leq s_1 \leq s_2 \ldots \leq s_m$. Then, for all permutations $\tau \in \mathcal{S}_m$,

$$\sum_{\imath=1}^{m} (\imath - s_\imath)^2 \;\leq\; \sum_{\imath=1}^{m} (\imath - s_{\tau(\imath)})^2 \tag{11}$$

Proof: We have

$$\begin{aligned}\sum_{\imath=1}^{m} (\imath - s_{\tau(\imath)})^2 &= \sum_{\imath=1}^{m} (\imath - s_\imath + s_\imath - s_{\tau(\imath)})^2 \\ &= \sum_{\imath=1}^{m} (\imath - s_\imath)^2 + 2\sum_{\imath=1}^{m} (\imath - s_\imath)(s_\imath - s_{\tau(\imath)}) + \sum_{\imath=1}^{m} (s_\imath - s_{\tau(\imath)})^2 .\end{aligned}$$

Expanding the last equation and exploiting that $\sum_{\imath=1}^{m} s_\imath^2 = \sum_{\imath=1}^{m} s_{\tau(\imath)}^2$ yields

$$\sum_{\imath=1}^{m} (\imath - s_{\tau(\imath)})^2 = \sum_{\imath=1}^{m} (\imath - s_\imath)^2 + 2\sum_{\imath=1}^{m} \imath\, s_\imath - 2\sum_{\imath=1}^{m} \imath\, s_{\tau(\imath)} .$$

On the right-hand side of the last equation, only the last term $\sum_{\imath=1}^{m} \imath\, s_{\tau(\imath)}$ depends on τ. Since $s_\imath \leq s_\jmath$ for $\imath < \jmath$, this term becomes maximal for $\tau(\imath) = \imath$. Therefore, the right-hand side is larger than or equal to $\sum_{\imath=1}^{m} (\imath - s_\imath)^2$, which proves the lemma. □

Lemma 2: Let $\mathsf{Pr}(\cdot \,|\, x)$ be a probability distribution over $\mathcal{S}_m$ and let $p(\tau) \overset{\mathrm{df}}{=} \mathsf{Pr}(\tau \,|\, x)$. Moreover, let

$$s_\imath \overset{\mathrm{df}}{=} m - \sum_{\jmath \neq \imath} \mathsf{Pr}(\lambda_\imath \succ_x \lambda_\jmath) \tag{12}$$

with

$$\mathsf{Pr}(\lambda_\imath \succ_x \lambda_\jmath) = \sum_{\tau \,:\, \tau(\jmath) < \tau(\imath)} \mathsf{Pr}(\tau \,|\, x). \tag{13}$$

Then, $s_\imath = \sum_{\jmath \neq \imath} p(\tau)\, \tau(\imath)$.

Proof: We have

$$
\begin{aligned}
s_\imath &= m - \sum_{\jmath \neq \imath} \mathsf{Pr}(\lambda_\imath \succ_x \lambda_\jmath) \\
&= 1 + \sum_{\jmath \neq \imath} (1 - \mathsf{Pr}(\lambda_\imath \succ_x \lambda_\jmath)) \\
&= 1 + \sum_{\jmath \neq \imath} \mathsf{Pr}(\lambda_\jmath \succ_x \lambda_\imath) \\
&= 1 + \sum_{\jmath \neq \imath} \sum_{\tau \,:\, \tau(\jmath) < \tau(\imath)} p(\tau) \\
&= 1 + \sum_{\tau} p(\tau) \sum_{\jmath \neq \imath} \begin{cases} 1 & \text{if} \quad \tau(\imath) > \tau(\jmath) \\ 0 & \text{if} \quad \tau(\imath) < \tau(\jmath) \end{cases} \\
&= 1 + \sum_{\tau} p(\tau)(\tau(\imath) - 1) \\
&= \sum_{\tau} p(\tau)\, \tau(\imath)
\end{aligned}
$$

Under the assumption that the base learners' estimates correspond exactly to the probabilities of pairwise preference, i.e.,

$$
\mathcal{R}_x(\lambda_\imath, \lambda_\jmath) = \mathcal{M}_{\imath\jmath}(x) = \mathsf{Pr}(\lambda_\imath \succ_x \lambda_\jmath), \tag{14}
$$

$s_\imath \leq s_\jmath$ is equivalent to $S(\lambda_\imath) \geq S(\lambda_\jmath)$. Thus, ranking the alternatives according to $S(\lambda_\imath)$ (in decreasing order) is equivalent to ranking them according to $s_\imath$ (in increasing order).

Theorem 1: The expected distance

$$
E(\tau') = \sum_{\tau} \; p(\tau) \cdot D(\tau', \tau) = \sum_{\tau} p(\tau) \sum_{\imath=1}^{m} (\tau'(\imath) - \tau(\imath))^2
$$

becomes minimal by choosing τ' such that $\tau'(\imath) \leq \tau'(\jmath)$ whenever $s_\imath \leq s_\jmath$, with $s_\imath$ given by (12).

Proof: We have

$$\begin{aligned}
E(\tau'_x) &= \sum_{\tau} p(\tau) \sum_{\imath=1}^{m} (\tau'_x(\imath) - \tau(\imath))^2 \\
&= \sum_{\imath=1}^{m} \sum_{\tau} p(\tau)(\tau'_x(\imath) - \tau(\imath))^2 \\
&= \sum_{\imath=1}^{m} \sum_{\tau} p(\tau)(\tau'_x(\imath) - s_\imath + s_\imath - \tau(\imath))^2 \\
&= \sum_{\imath=1}^{m} \sum_{\tau} p(\tau) \left[(\tau(\imath) - s_\imath)^2 - 2(\tau(\imath) - s_\imath)(s_\imath - \tau'(\imath)) \right. \\
&\qquad\qquad\qquad \left. + (s_\imath - \tau'(\imath))^2 \right] \\
&= \sum_{\imath=1}^{m} \left[\sum_{\tau} p(\tau)(\tau(\imath) - s_\imath)^2 - 2(s_\imath - \tau'(\imath)) \cdot \right. \\
&\qquad \left. \cdot \sum_{\tau} p(\tau)(\tau(\imath) - s_\imath) + \sum_{\tau} p(\tau)(s_\imath - \tau'(\imath))^2 \right]
\end{aligned}$$

In the last equation, the mid-term on the right-hand side becomes 0 according to Lemma 2. Moreover, the last term obviously simplifies to $(s_\imath - \tau'(\imath))$, and the first term is a constant $c = \sum_{\tau} p(\tau)(\tau(\imath) - s_\imath)^2$ that does not depend on τ'. Thus, we obtain $E(\tau'_x) = c + \sum_{\imath=1}^{m} (s_\imath - \tau'(\imath))^2$ and the theorem follows from Lemma 1. □