A Proof of Theorem 1

Lemma 1: Let $s_i, i = 1 \ldots m$, be real numbers such that $0 \leq s_1 \leq s_2 \ldots \leq s_m$. Then, for all permutations $\tau \in S_m$,

\[
\sum_{i=1}^{m} (i - s_i)^2 \leq \sum_{i=1}^{m} (i - s_{\tau(i)})^2
\]  \hspace{1cm} (11)

Proof: We have

\[
\sum_{i=1}^{m} (i - s_{\tau(i)})^2 = \sum_{i=1}^{m} (i - s_i + s_i - s_{\tau(i)})^2
\]
\[
= \sum_{i=1}^{m} (i - s_i)^2 + 2 \sum_{i=1}^{m} (i - s_i)(s_i - s_{\tau(i)}) + \sum_{i=1}^{m} (s_i - s_{\tau(i)})^2.
\]

Expanding the last equation and exploiting that $\sum_{i=1}^{m} s_i^2 = \sum_{i=1}^{m} s_{\tau(i)}^2$ yields

\[
\sum_{i=1}^{m} (i - s_{\tau(i)})^2 = \sum_{i=1}^{m} (i - s_i)^2 + 2 \sum_{i=1}^{m} i s_i - 2 \sum_{i=1}^{m} i s_{\tau(i)}.
\]

On the right-hand side of the last equation, only the last term $\sum_{i=1}^{m} i s_{\tau(i)}$ depends on $\tau$. Since $s_i \leq s_j$ for $i < j$, this term becomes maximal for $\tau(i) = i$. Therefore, the right-hand side is larger than or equal to $\sum_{i=1}^{m} (i - s_i)^2$, which proves the lemma. \qed

Lemma 2: Let $Pr(\cdot | x)$ be a probability distribution over $S_m$ and let $p(\tau) \overset{\text{df}}{=} Pr(\tau | x)$. Moreover, let

\[
s_i \overset{\text{df}}{=} m - \sum_{j \neq i} Pr(\lambda_i \succ x \lambda_j)
\]  \hspace{1cm} (12)

with

\[
Pr(\lambda_i \succ x \lambda_j) = \sum_{\tau : \tau(j) < \tau(i)} Pr(\tau | x).
\]  \hspace{1cm} (13)

Then, $s_i = \sum_{j \neq i} p(\tau) \tau(i)$.  

Proof: We have

\[ s_i = m - \sum_{j \neq i} \Pr(\lambda_i \succ_x \lambda_j) \]
\[ = 1 + \sum_{j \neq i} (1 - \Pr(\lambda_i \succ_x \lambda_j)) \]
\[ = 1 + \sum_{j \neq i} \Pr(\lambda_j \succ_x \lambda_i) \]
\[ = 1 + \sum_{j \neq i} \sum_{\tau : \tau(j) < \tau(i)} p(\tau) \]
\[ = 1 + \sum_{\tau} p(\tau) \sum_{j \neq i} \begin{cases} 
1 & \text{if } \tau(i) > \tau(j) \\
0 & \text{if } \tau(i) < \tau(j) 
\end{cases} \]
\[ = 1 + \sum_{\tau} p(\tau)(\tau(i) - 1) \]
\[ = \sum_{\tau} p(\tau) \tau(i) \]

Under the assumption that the base learners’ estimates correspond exactly to the probabilities of pairwise preference, i.e.,

\[ \mathcal{R}_x(\lambda_i, \lambda_j) = \mathcal{M}_{ij}(x) = \Pr(\lambda_i \succ_x \lambda_j), \quad \text{(14)} \]

\( s_i \leq s_j \) is equivalent to \( S(\lambda_i) \geq S(\lambda_j) \). Thus, ranking the alternatives according to \( S(\lambda_i) \) (in decreasing order) is equivalent to ranking them according to \( s_i \) (in increasing order).

**Theorem 1:** The expected distance

\[ E(\tau') = \sum_{\tau} p(\tau) \cdot D(\tau', \tau) = \sum_{\tau} p(\tau) \sum_{i=1}^{m} (\tau'(i) - \tau(i))^2 \]

becomes minimal by choosing \( \tau' \) such that \( \tau'(i) \leq \tau'(j) \) whenever \( s_i \leq s_j \), with \( s_i \) given by (12).
Proof: We have

$$E(\tau'_x) = \sum_{\tau} p(\tau) \sum_{i=1}^{m} (\tau'_x(i) - \tau(i))^2$$

$$= \sum_{i=1}^{m} \sum_{\tau} p(\tau) (\tau'_x(i) - \tau(i))^2$$

$$= \sum_{i=1}^{m} \sum_{\tau} p(\tau) (\tau'_x(i) - s_i + s_i - \tau(i))^2$$

$$= \sum_{i=1}^{m} \sum_{\tau} p(\tau) \left[ (\tau(i) - s_i)^2 - 2(\tau(i) - s_i)(s_i - \tau'(i)) + (s_i - \tau'(i))^2 \right]$$

$$= \sum_{i=1}^{m} \left[ \sum_{\tau} p(\tau)(\tau(i) - s_i)^2 - 2(s_i - \tau'(i)) \cdot \sum_{\tau} p(\tau)(\tau(i) - s_i) + \sum_{\tau} p(\tau)(s_i - \tau'(i))^2 \right]$$

In the last equation, the mid-term on the right-hand side becomes 0 according to Lemma 2. Moreover, the last term obviously simplifies to \((s_i - \tau'(i))^2\), and the first term is a constant \(c = \sum_{i=1}^{m} p(\tau)(\tau(i) - s_i)^2\) that does not depend on \(\tau'\). Thus, we obtain \(E(\tau'_x) = c + \sum_{i=1}^{m} (s_i - \tau'(i))^2\) and the theorem follows from Lemma 1. 