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A cancellation theorem for finite algebras

H. Peter Gumm

If $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are finite algebras with $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$ then Lovász [3] showed that $\mathfrak{B}\cong\mathfrak{C}$ in case that $\mathfrak{A}$ has a one-element subalgebra.

What can be said about the relationship of $\mathfrak{B}$ and $\mathfrak{C}$ if $\mathfrak{A}$ does not have a one-element subalgebra? Trivial examples show that $\mathfrak{B}$ and $\mathfrak{C}$ need not be isomorphic.

The aim of this note is to show that in this case $\mathfrak{B}$ and $\mathfrak{C}$ have to be *isotopic,* a notion defined as follows:

Definition: Let $\mathfrak{A}=(A, (f_i)_{i\in I})$ and $\mathfrak{B}=(B, (g_i)_{i\in I})$ be universal algebras of the same type $(n_i)_{i\in I}$. $\mathfrak{A}$ and $\mathfrak{B}$ are called *isotopic* if there exist bijective mappings Ψ_i, $i\in I$ and a bijection Φ such that

$$\Psi_i(f_i(x_1, \ldots, x_{n_i})) = g_i(\Phi(x_1), \ldots, \Phi(x_{n_i}))$$

for all $i\in I$ and $x_1, \ldots, x_{n_i}\in A$.

To visualize the concept let us give an equivalent definition: Algebras $\mathfrak{A}=(A, (f_i)_{i\in I})$ and $\mathfrak{B}=(B, (g_i)_{i\in I})$ are isotopic iff there exists a family $(\sigma_i)_{i\in I}$ of permutations of B such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}^\sigma:=(B, (\sigma_i\circ g_i)_{i\in I})$.

Obviously isotopy is an equivalence relation on the class of all algebras of a given type, and is a generalization of isomorphy. We write $\mathfrak{A}\triangleq\mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ are isotopic.

A weaker version of isotopy has been studied to a great extent in the theory of quasigroups, see [1]. Our definition has the advantage that it is closer to isomorphy, such that, for example, isotopic universal algebras with idempotent fundamental operations are isomorphic, see the corollary below. The concept of isotopy arises very naturally at the study of algebras in permutable varieties, as we have shown in [2]. Namely if $\mathfrak{A}=(A, (f_i)_{i\in I})$ is an algebra (of arbitrary cardinality) in a permutable variety and if θ is a congruence relation on $\mathfrak{A}\times\mathfrak{A}$ which is a complement of the factor-congruences θ_{π_1} and θ_{π_2} then for $\mathfrak{B}:=\mathfrak{A}\times\mathfrak{A}/\theta$ we have $\mathfrak{A}\times\mathfrak{B}\cong\cong\mathfrak{A}\times\mathfrak{A}$. θ can always be chosen so that $\mathfrak{B}$ has a one-element subalgebra. Moreover we have that $\mathfrak{A}$ and $\mathfrak{B}$ are affine, i.e. there is an abelian group $\mathfrak{G}$ both defined on $\mathfrak{A}$ and on $\mathfrak{B}$ such that any n-ary polynomial p satisfies:

$$p(x_1, \dots, x_n)+p(y_1, \dots, y_n) = p(x_1+y_1, \dots, x_n+y_n)+p(0, \dots, 0).$$

Then there are two cases:

If $\mathfrak{A}$ has a one-element subalgebra then $\mathfrak{A}\cong\mathfrak{B}$, giving an infinite version of Lovász's theorem in this special case. Moreover the isomorphism can be given explicitly.

If $\mathfrak{A}$ has no one-element subalgebra then if we define $\mathfrak{A}^\nabla:=(A, (f_i^\nabla)_{i\in I})$ with

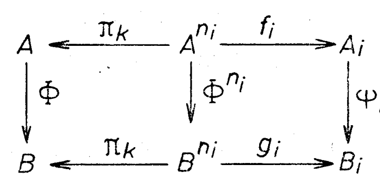
$$f_i^\nabla(x_1, \dots, x_{n_i}):=f_i(x_1, \dots, x_{n_i})-f_i(0, \dots, 0),$$

we have that $\mathfrak{A}^\nabla$ is isomorphic to $\mathfrak{B}$, hence $\mathfrak{A}$ is isotopic to $\mathfrak{B}$.

Similarly under the hypothesis that $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are finite and are contained in a permutable variety it has been shown by Smith [4] that $\mathfrak{B}$ and $\mathfrak{C}$ are isotopic in case $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$.

For arbitrary finite algebras one cannot expect a structure theory to yield us the desired isotopy. So our proof does not explicitly produce the isotopy, it only shows the existence. It is based on the observation that Lovász's proof of his cited result can be carried out in every category satisfying some special conditions which can be easily extracted from his proof. So after choosing a category suitable for our problem we will simply imitate Lovász's proof in this new category and interpret the results back into the category of all finite algebras of a given type.

We choose as objects of our category H certain "heterogeneous" algebras: An *H-algebra* of type $(n_i)_{i\in I}$ will be a triple $\mathfrak{A}=(A, (A_i)_{i\in I}, (f_i)_{i\in I})$ where each f_i is a map f_i: $A^{n_i}\to A_i$. A morphism between H-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a pair $(\Phi, (\Psi_i)_{i\in I})$ where Φ and Ψ_i are maps such that the following diagram of mappings commutes for every $i\in I$ and $k\leqq n_i$. (π_k denotes the k-th canonical projection from the n_i-th power A^{n_i} of A onto A.)



For every universal algebra $\mathfrak{A}=(A, (f_i)_{i\in I})$ we obtain a corresponding H-algebra $\underline{\mathfrak{A}}:=(A, (A)_{i\in I}, (f_i)_{i\in I})$.

Thus we get that two universal algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isotopic if and only if $\underline{\mathfrak{A}}$ and $\underline{\mathfrak{B}}$ are H-isomorphic. Moreover $\underline{\mathfrak{A}}$ gives a functor which preserves products, i.e.

$$\underline{\prod_{i\in I}\mathfrak{A}_i} \quad \text{and} \quad \prod_{i\in I}\underline{\mathfrak{A}_i} \quad \text{are } H\text{-isomorphic.}$$

A last fact to be easily verified is that in H every morphism h can be written as a composition $h_1 \circ h_2$ where h_1 is mono and h_2 is epi.

Let from now on I be finite and consider the subcategory of finite H-algebras. i.e. those $\mathfrak{A}=(A, (A_i)_{i\in I}, (f_i)_{i\in I})$ where the disjoint union of A and the A_i has finite cardinality, denoted by $|\mathfrak{A}|$. We procede now as in [3] and choose a family $({}_jQ)_{j\in J}$ of finite H-algebras such that every finite H-algebra is H-isomorphic to exactly one ${}_jQ$.

For $t\in J_n:=\{j\in J \mid n \geqq |{}_jQ|\}$ and for arbitrary finite H-algebras $\mathfrak{A}$ we get:

$$|\mathrm{Hom}\,({}_tQ, \mathfrak{A})| = |\mathrm{Mono}\,({}_tQ, \mathfrak{A})| + \sum_{j\in J_{n-1}} \frac{|\mathrm{Epi}\,({}_tQ, {}_jQ)|}{|\mathrm{Iso}\,({}_jQ, {}_jQ)|} \cdot \mathrm{Mono}\,|({}_jQ, \mathfrak{A})|.$$

Lemma 1. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are finite H-algebras and for all* $j\in J$ $|\mathrm{Hom}\,({}_jQ, \mathfrak{A})| = |\mathrm{Hom}\,({}_jQ, \mathfrak{B})|$ *then* $\mathfrak{A}$ *is H-isomorphic to* $\mathfrak{B}$.

Proof. Using the above formula show by induction on the cardinality of the ${}_jQ$'s that $|\mathrm{Mono}\,({}_jQ, \mathfrak{A})| = |\mathrm{Mono}\,({}_jQ, \mathfrak{B})|$ for all $j\in J$. It follows that there is a monomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and one from $\mathfrak{B}$ to $\mathfrak{A}$. This clearly implies that $\mathfrak{A}$ is H-isomorphic to $\mathfrak{B}$ since both are finite.

Lemma 2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be H-algebras then* $|\mathrm{Hom}\,(\mathfrak{A}, \mathfrak{B})| \neq \emptyset$.

Proof. For an arbitrary $b\in B$ define $\Phi(x):=b$ for all $x\in A$ and $\Psi_i(x):=g_i(b, \ldots, b)$ for all $i\in I$ and $x\in A_i$. Now we are ready to prove our cancellation theorem:

Theorem. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *and* $\mathfrak{C}$ *be finite universal algebras of the same finite type. If* $\mathfrak{A}\times\mathfrak{B}$ *and* $\mathfrak{A}\times\mathfrak{C}$ *are isotopic then* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are isotopic.*

Proof. $\mathfrak{A}\times\mathfrak{B} \triangleq \mathfrak{A}\times\mathfrak{C}$ implies that $\underline{\mathfrak{A}\times\mathfrak{B}}$ and $\underline{\mathfrak{A}\times\mathfrak{C}}$ are H-isomorphic. Hence $\underline{\mathfrak{A}}\times\underline{\mathfrak{B}}$ and $\underline{\mathfrak{A}}\times\underline{\mathfrak{C}}$ are H-isomorphic. Hence for all $j\in J$ we have: $|\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}}\times\underline{\mathfrak{B}})| = |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}}\times\underline{\mathfrak{C}})|$. Thus

$$\begin{aligned}|\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}})| \cdot |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{B}})| &= |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}}\times\underline{\mathfrak{B}})| \\ &= |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}}\times\underline{\mathfrak{C}})| \\ &= |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{A}})| \cdot |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{C}})|.\end{aligned}$$

By Lemma 2 we can cancel yielding $|\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{B}})| = |\mathrm{Hom}\,({}_jQ, \underline{\mathfrak{C}})|$, so by virtue of Lemma 1 $\underline{\mathfrak{B}}$ and $\underline{\mathfrak{C}}$ are H-isomorphic. Hence $\mathfrak{B}$ and $\mathfrak{C}$ are isotopic.

Corollary. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *and* $\mathfrak{C}$ *be finite universal algebras of the same finite type with* $\mathfrak{A}\times\mathfrak{B}$ *and* $\mathfrak{A}\times\mathfrak{C}$ *isotopic. If the fundamental operations of* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are idempotent, then* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are isomorphic.*

Proof. By the theorem $\mathfrak{B}$ and $\mathfrak{C}$ are isotopic. For arbitrary $x \in B$ we get by idempotency:

$$\Psi_i(x) = \Psi_i(f_i(x, \ldots, x)) = f_i(\Phi(x), \ldots, \Phi(x)) = \Phi(x).$$

Hence $\Phi = \Psi_i$ and Φ is an isomorphism.

We can sharpen our theorem in some cases to shorten the list of mappings Ψ_i which are needed to establish isotopy.

In particular let $\mathfrak{A}$ and $\mathfrak{B}$ be *principally isotopic* if there exist two bijections Ψ and $\Phi: A \to B$ such that for all $i \in I$ we get

$$\Psi f_i(x_1, \ldots, x_{n_i}) = g_i(\Phi(x_1), \ldots, \Phi(x_{n_i})).$$

Then a slight modification of our category H yields:

Corollary. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be finite algebras with $\mathfrak{A} \times \mathfrak{B} \cong \mathfrak{A} \times \mathfrak{C}$. If there exists an element $a \in A$ such that for all $i, j \in I$ we have $f_i(\underline{a}) = f_j(\underline{a})$ then $\mathfrak{B}$ and $\mathfrak{C}$ are principally isotopic.*

In the same fashion one may define *n-isotopic*, meaning that there are only n different Ψ_i needed to get a corresponding condition on the algebra $\mathfrak{A}$.

References

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Technische Hochschule Darmstadt,
Fachbereich Mathematik,
D–6100 Darmstadt,
Schloßgartenstr. 7

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