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ALGEBRAS IN MODULAR VARIETIES: BAER REFINEMENTS, CANCELLATION AND ISOTOPY

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Dedicated to Professor Jürgen Schmidt on his sixtieth birthday.

The purpose of the present note is to prove (in Section 5) a Cancellation Theorem for algebras in congruence modular varieties satisfying the ascending chain condition on congruences and the descending chain condition on subcongruences of the center – generalizing J. D. H. Smith [16; 424] for finite algebras in congruence permutable varieties. Here, the concept of commutation used to define the center is that introduced in [11].

In the proof we need the existence of "special refinements" in the sense of R. Baer [1,2] for direct decompositions $\mathbf{A} \times \mathbf{B} \cong \mathbf{C} \times \mathbf{D}$. In the congruence permutable case such a Special Refinement Theorem can be proven by direct use of "Fitting Methods" (see R. Baer [1] and A. W. Goldie [6]) or by appealing to lattice theoretic versions such as those of M. Grayev [17], L. A. Hostinsky [18], or E. N. Močul'skiĭ [19]. Unfortunately, these results are stated with hypotheses making them not applicable in our context. Thus we base our approach on the analysis of modular lattices generated by elements a,b,c,d with $a \oplus b = c \oplus d = 1$ which has been given in [12]. Such a lattice has a cartesian direct decomposition inducing special refinements of the pair {a,b}, {c,d} of "direct decompositions of 1," as long as the decomposition center

$$(a+c)(a+d)(b+c)(b+d)$$

has finite rank - weaker conditions modelling those in R. Baer [1] suffice.

Additional advantage is taken from the fact that the refinements are given via lattice words in a,b,c,d: We can use results on permutability in congruence modular varieties from [9] and [13] to show (in Section 2) that in the case of a congruence lattice the special refinements yield direct decompositions of the algebra indeed. We point out that the tools provided in this paper suffice to prove the existence of exchange isomorphic refinements of finite direct decompositions of algebras in congruence modular varieties: Modify e.g. the proof in A. G. Kurosh [15; §47]. On the other hand it seems hard to establish results as powerful as P. Crawley, B. Jónsson and A. Tarski [3,14] did for their kind of algebras.

Returning to the Cancellation Theorem, we have to observe that, if there are no 1-element subalgebras, then cancellation is only possible up to "isotopy." Here, we define two algebras **A** and **B** to be *C*-isotopic if there is an algebra **C** in a given class *C* of algebras and an isomorphism between **A** × **C** and **B** × **C** which commutes with the projections onto **C**. A characterization of *C*-isotopy in terms of congruence lattices is given in Section 3. The structure of **A**, **B** and **C** is then described in more detail in the following Section 4: **C** can always be chosen an affine algebra and the structure of **C** is inherited in **A** and in **B**. In fact, the latter may be linearized and their linearizations are isomorphic.

There are two different methods we use to study algebras in (congruence) modular varieties. One is an algebraic approach as introduced in [11] and in [12] which is rather technical in some parts; the other is a geometric approach developed in [8] and in [9] which often is more intuitive. Basically, most results could be obtained either way. However, in Chapters 2 and 5 we used the algebraic method, and in Chapters 3 and 4 the geometrical one, thus choosing the technique that seemed most natural to us in the specific circumstances.

§1. **Baer refinements in modular lattices.** In a lattice we write ab for the meet and a+b for the join of a and b. Brackets are omitted as usual. 0 and 1 stand for the smallest and the greatest element. If a+b = 1 and ab = c then we write a × b = c. If $b \leqslant a$ we denote by a/b the interval sublattice $\{x \mid b \leqslant x \leqslant a\}$. We write $a/b \nearrow c/d$ (a/b transposes upwards to c/d) and $c/d \searrow a/b$ (c/d transposes downwards to a/b) if a+d = c and ad = b; a/b and c/d are projective ($a/b \approx c/d$) if they are connected by a sequence of $\nearrow$ and $\searrow$.

DEFINITION. For a × b = c × d = 0 we define

$$Z(a,b,c,d) := (a+c)(a+d)(b+c)(b+d),$$

and call it the *decomposition center* of a,b,c,d.

Elements p_1,p_2 induce the special refinements a_1,a_2,b_1,b_2 and c_1,c_2,d_1,d_2 if the following hold:

(1) $p_1 + p_2 \leqslant Z(a,b,c,d)$;

(2) $q_1 \times q_2 = 0$ with

$$q_1 := (a+p_1)(d+p_1) + (b+p_1)(c+p_1)$$

and

$$q_2 := (a+p_2)(c+p_2) + (b+p_2)(d+p_2);$$

(3) $a_i = a+q_i$, $b_i = b+q_i$, $c_i = c+q_i$ and $d_i = d+q_i$ for $i \in \{1,2\}$;

(4) $a = a_1 \times a_2$, $b = b_1 \times b_2$, $c = c_1 \times c_2$ and $d = d_1 \times d_2$;

(5) $q_1 = a_1 \times b_1 = c_1 \times d_1 = a_1 \times d_1 = b_1 \times c_1$;

(6) $q_2 = a_2 \times b_2 = c_2 \times d_2 = a_2 \times c_2 = b_2 \times d_2$.

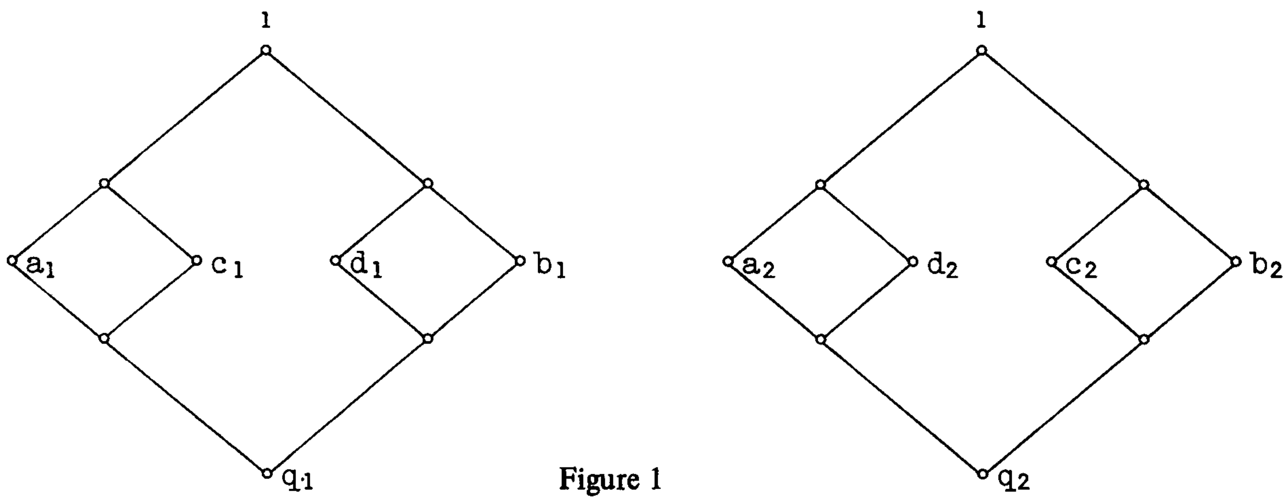


Figure 1

The relations in (5) and (6) are visualized by the partial lattice diagrams in Figure 1: Nontrivial joins and meets are defined only for elements of height 2.

DEFINITION. A quotient z/a in a lattice is said to satisfy *Condition* (B) if a/0 satisfies the ascending chain condition and if, for all $u \leqslant v \leqslant a$ with u/0 projective to v/0 in z/0, we get $u = v$; z/a satisfies *Condition* (C) if a/0 satisfies the descending chain condition and, for all $w \leqslant v \leqslant a$ with v/w projective to v/0 in z/0, we have $w = 0$.

1.1. THEOREM. *In a modular lattice* M, *if* $a \times b = c \times d = 0$ *and, for* $Z := Z(a,b,c,d)$, *the interval* Z/aZ *satisfies* (B) *or* (C), *then there are* p_1,p_2 *inducing special refinements.*

A basic tool in the proof are the lattice terms $g_n = g_n(w,x,y,z)$ defined inductively by

$$g_o(w,x,y,z) := w+x+y+z, \quad g_{n+1}(w,x,y,z) := (wg_n+xg_n)(yg_n+zg_n),$$

as well as

$$h_n(w,x,y,z):= g_n(w,x,y,z)g_n(w,y,x,z)g_n(w,z,x,y)$$

and their duals $g_n^*(w,x,y,z)$ and $h_n^*(w,x,y,z)$.

In Day and Wille [4], the lattice $FM(J_1^4)$ of Figure 2 has been introduced as the modular lattice freely generated by a_o,b_o,c_o,d_o satisfying

$$a_o \times b_o = a_o \times c_o = a_o \times d_o = b_o \times d_o = c_o \times d_o = b_o c_o = 0.$$

In particular this lattice is subdirectly irreducible and $\{a_o,b_o,c_o,d_o\}$ is the only four element generating set. With $e_n:= h_n(a_o,b_o,c_o,d_o)$ one sees that

$$\{a_oe_n,b_oe_n,c_oe_n,d_oe_n\}$$

generates a sublattice isomorphic to $FM(J_1^4)$.

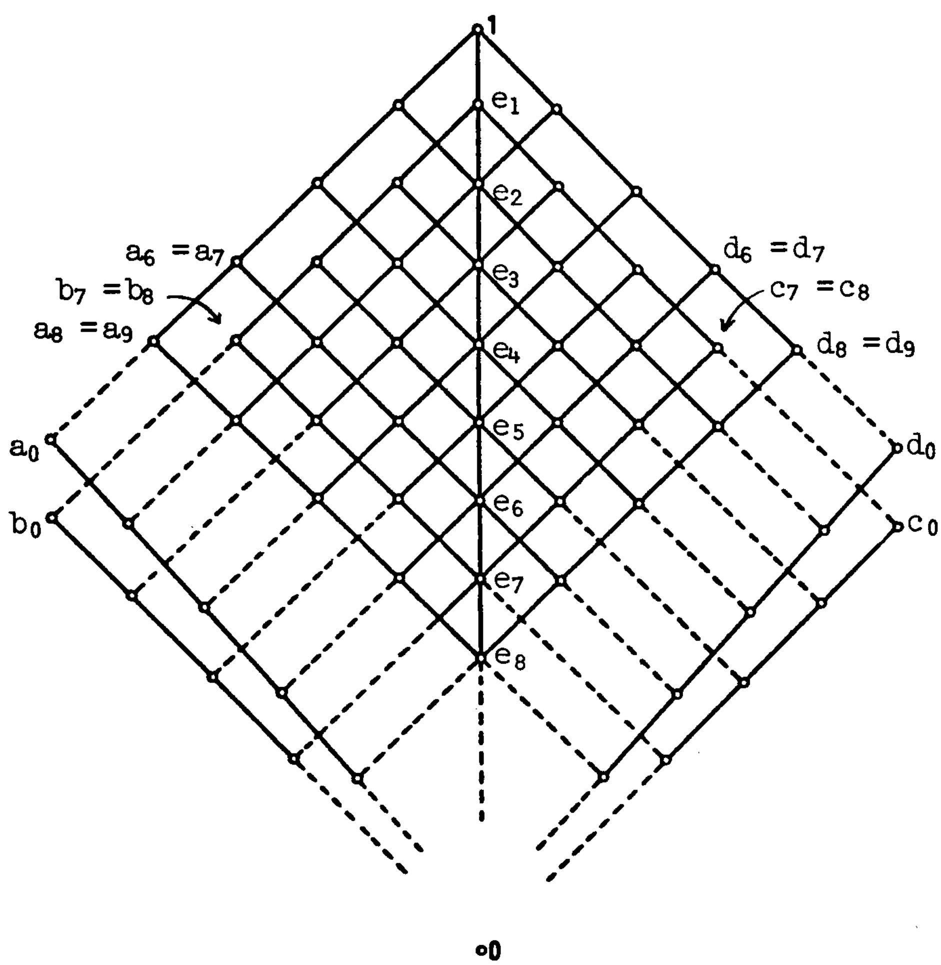


Figure 2.

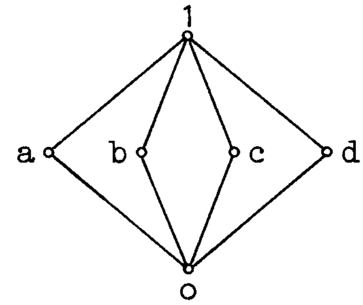


Figure 3.

On the other hand one gets, for each n, the interval sublattice $S(n,4) = 1/e_n$, which is selfdual and subdirectly irreducible, and has

$$a_n = a_o + e_n, b_n = b_o + e_n, c_n = c_o + e_n, d_n = d_o + e_n$$

as its unique four-element generating set. Evidently, if $\{w,x,y,z\}$ is this set then

$$h_m(w,x,y,z) = 0 \ (h_m^*(w,x,y,z) = 1) \text{ if and only if } n \leqslant m.$$

Let now M_4 be the lattice given in Figure 3.

1.2. CLAIM. [12] The lattices M_4, S(n,4) (for $n \in \mathbf{N}$), $FM(J_1^4)$, its dual $FM(J_1^4)^*$ form a complete list of subdirectly irreducible modular lattices generated by elements w,x,y,z such that $w \times x = y \times z = 0$.

1.3. CLAIM. Assume the hypothesis of 1.1 and, in addition, that **M** is generated by a,b,c,d. Then there is a positive integer N such that the subdirectly irreducible homomorphic images of **M** are among M_4 and the S(n,4) with $n \leqslant N$.

PROOF. Let **L** be the sublattice generated by aZ,bZ,cZ,dZ. Let φ be a homomorphism from **M** onto a subdirectly irreducible **S**. If **S** is M_4 or $FM(J_1^4)^*$ then $\varphi(Z) = 1_S$, whence $\varphi(L) = S$. If $S = FM(J_1^4)$ or $S = S(n,4)$ with $n \geqslant 1$, then $\varphi(Z) = e_1$ whence $\varphi(L) \cong FM(J_1^4)$ in the first and $\varphi(L) \cong S(n\text{-}1,4)$ in the second case. It suffices to show that there is an N′ such that the subdirectly irreducible homomorphic images of **L** are among M_4 and the S(n,4) with $n \leqslant N'$. In other words, to simplify notation we may suppose that 1/a satisfies (B) or (C). We write $h_n = h_n(a,b,c,d)$ and $h_n^* = h_n^*(a,b,c,d)$.

CASE 1. Let 1/a satisfy (C). Due to the descending chain condition, $FM(J_1^4)$ is not a homomorphic image of **M**. Moreover, since the h_n ($n \in \mathbf{N}$), form a descending chain, there is an N such that $h_n = h_N$ for all $n \geqslant N$. Now consider the homomorphisms φ onto S, S subdirectly irreducible. If $S = S(n,4)$ then $\varphi(h_N) = 0$ and $n \leqslant N$.

On the other hand $\varphi(h_N) = 1$ if $S = M_4$ or $S = FM(J_1^4)^*$. Therefore, defining $e' := eh_N$ for $e \in E := \{a,b,c,d\}$, we get $e'+f' = h_N$ for $e \neq f$ in E, and $a'b' = c'd' = 0$. Assume that there is a homomorphism $\varphi: M \to S \cong FM(J_1^4)^*$. Then without loss of generality $\varphi(ad) \neq 0$ or $\varphi(bc) \neq 0$. Since $\varphi(e') = \varphi(e)$ for $e \in E$, it follows that $a'(d'+b'c') \neq 0$. But in view of (C) and the projectivities

$$a'/0 \nearrow a'+b'/b' \searrow c'/b'c' \nearrow d'+b'/d'+b'c'$$
$$\searrow a'/a'(d'+b'c')$$

this is a contradiction.

CASE 2. Let $1/a$ satisfy (B). Due to the ascending chain condition, $FM(J_1^4)^*$ is not a homomorphic image of M. Moreover, since the h_n^*, $n \in \mathbf{N}$, form an ascending chain there is an N such that $h_n^* = h_N^*$ for all $n \geqslant N$. Now consider homomorphisms φ of M onto S, S subdirectly irreducible. If $S = S(n,4)$ then $\varphi(h_N) = 0$ and $n \leqslant N$. Therefore, with e' as before we get $e'f' = 0$ for $e \neq f$ in E, and $a'+b' = c'+d' = h_N$. Assume that there is a homomorphism

$$\varphi: M \twoheadrightarrow S = FM(J_1^4).$$

Without loss of generality, $\varphi(a+d) \neq 1$ or $\varphi(b+c) \neq 1$. Let M' be the sublattice generated by $\{a',b',c',d'\}$. Then $\varphi(M') \cong FM(J_1^4)$ and

$$\varphi(a'(b'+c'(a'+d'))) < \varphi(a').$$

But this is impossible due to (B) and the projectivities

$$a'/0 \nearrow a'+d'/d' \searrow c'(a'+d')/0 \nearrow b'+c'(a'+d')/b'$$
$$\searrow a'(b'+c'(a'+d'))/0.$$

PROOF OF 1.1. Obviously, it suffices to consider the case where M is generated by a,b,c,d. Let S_1 be the subdirect product over the $\varphi: M \twoheadrightarrow S = S(n,4)$ such that $\varphi(a) \times \varphi(d) = \varphi(b) \times \varphi(c) = 0$, S_2 the subdirect product over those with $\varphi(a) \times \varphi(c) = \varphi(b) \times \varphi(d) = 0$, and $S_3 = M_4$ if this is a homomorphic image and trivial otherwise. By Claim 1.3, M can be considered as a sublattice of $S_1 \times S_2 \times S_3$. We infer that $M \cong S_1 \times S_2 \times S_3$, since

$$g_N(a,c,b,d) = (0,1,1),\ g_N(a,d,b,c) = (1,0,1),\ h_N^*(a,b,c,d) = (1,1,0)$$

are in **M**. For π_i, the projection of **M** onto S_i, define $Z_i := \pi_i(Z)$. In order to construct our special refinement, we set $p_1 := (0,Z_2,Z_3)$ and $p_2 := (Z_1,0,0)$, and we define the q_i according to (2), i.e. $q_1 = (0,1,1)$ and $q_2 = (1,0,0)$, since $\varphi(q_1) = 0$ and $\varphi(q_2) = 1$ for all φ: $\mathbf{M} \twoheadrightarrow \mathbf{S}$ factoring through S_1, and $\varphi(q_1) = 1$ and $\varphi(q_2) = 0$ for all φ factoring through S_2 or S_3. To check this, observe firstly that $\varphi(Z)$ is either 1 or a dual atom. In particular, q_1 and q_2 are complementary central elements of M. (4) - (6) follow immediately.

§2. **Baer refinements for congruence modular algebras.** For this and the following sections, let all algebras be contained in a fixed modular variety, i.e. a variety **V** all of whose algebras have modular congruence lattices. We extend Baer's refinement Theorem [1] to algebras in **V**. To do so, we use the concept of commutation introduced in [11]. Given **A** in **V** and b,c in the congruence lattice $\mathcal{L}(\mathbf{A})$ of **A**, there is a smallest a in $\mathcal{L}(\mathbf{A})$ having the following property: There are **B** in HSP(**A**), a homomorphism φ from **B** onto **A**, and s,t in $\mathcal{L}(\mathbf{B})$ such that $st \leqslant \hat{\varphi}a$, $s+\hat{\varphi}a \geqslant \hat{\varphi}b$, and $t+\hat{\varphi}a \geqslant \hat{\varphi}c$ where $\hat{\varphi}x$ is the inverse image $\varphi^{-1} \times \varphi^{-1}(x)$ of x. We write $a = [b,c]$ and call a the *commutator* of b and c.

Commutators distribute over joins, $[b,\Sigma c_i] = \Sigma[b,c_i]$, whence, for each **A**, there is a greatest congruence $z_\mathbf{A}$ (the *center* of **A**) such that $[z_\mathbf{A},1_\mathbf{A}] = 0_\mathbf{A}$ where $1_\mathbf{A}$ and $0_\mathbf{A}$ denote the greatest and smallest element of $\mathcal{L}(\mathbf{A})$.

We write $a \circ b$ for the relational product of a and b, and $a \otimes b = ab$ if $a+b = 1_\mathbf{A}$ and a and b permute, i.e. if $\mathbf{A}_{/ab}$ is canonically isomorphic to $\mathbf{A}_{/a} \times \mathbf{A}_{/b}$.

2.1. THEOREM. *Let* **A** *be an algebra in a modular variety, and let* a,b,c,d *be congruences on* **A** *such that* $a \otimes b = c \otimes d = 0_\mathbf{A}$. *Assume that the center of* $\mathbf{A}_{/b}$ *has finite rank in* $\mathcal{L}(\mathbf{A}_{/b})$ *or that the quotient* $z_\mathbf{A}/az_\mathbf{A}$ *of* $\mathcal{L}(\mathbf{A})$ *satisfies one of the conditions* (B) *or* (C) *from Section 1. Then the direct decompositions* a,b *and* c,d *have "special refinements," i.e. there exist congruences* a_1,a_2,b_1,b_2 *and* c_1,c_2,d_1,d_2 *on* **A** *such that*

$$a = a_1 \otimes a_2,\ b = b_1 \otimes b_2,\ c = c_1 \otimes c_2,\ d = d_1 \otimes d_2,$$

$$a_1 \otimes b_1 = c_1 \otimes d_1 = a_1 \otimes d_1 = b_1 \otimes c_1,$$

$$a_2 \otimes b_2 = c_2 \otimes d_2 = a_2 \otimes c_2 = b_2 \otimes c_2.$$

In order to apply Theorem 1.1, we have to show that the decomposition center is contained in the center, and that permutability is granted wherever it is needed. Concerning the latter we use Korollar 5 from [9]:

2.2. PROPOSITION ([9]). *Let* a,b,c *be congruences of* **A** *in* **V** *such that* a *and* b *permute and* $ab \leqslant c \leqslant a+b$. *Then any two congruences in the sublattice of* $\mathcal{L}(\mathbf{A})$ *generated by* a,b,c *permute with each other.*

Moreover, we define for a congruence a of **A**, inductively:

$$a^{o} := a, \quad a^{n+1} := [a^{n}, a^{n}].$$

We call a *solvable* if there is an n such that $a^n = 0_{\mathbf{A}}$.

2.3. LEMMA. *Any solvable congruence of* **A** *in* **V** *permutes with every congruence of* **A**.

PROOF. We proceed by induction on the number n such that $a^n = 0_{\mathbf{A}}$. For $n = 1$, let $\mathbf{V}'$ be the idempotent reduct of **V**, i.e. the variety with fundamental operations corresponding to the algebraic terms idempotent in **V** and satisfying all identities valid in **V**. Due to A. Day [4], $\mathbf{V}'$ is congruence modular, too. Consider **A** as a member of $\mathbf{V}'$. Since any congruence with respect to **V** is also a congruence with respect to $\mathbf{V}'$, $[a,a] = 0$ is still valid. Surely, we will be done if we prove the claim with respect to $\mathbf{V}'$. Let b be an arbitrary congruence of **A**. Then $[a+b,a+b] \leqslant b$ by distributivity. Let B be a class of a+b. B is a subalgebra of **A**, and for the restrictions to B we find

$$(a+b)|_{B} = 1_{B} \text{ and } [1_{B}, 1_{B}] \leqslant b|_{B}.$$

Thus $\mathbf{B}_{/b}$ is abelain in the sense of [13], and by Corollary 5 loc.cit., $b|_{B}$ permutes with every congruence of **B**. In particular $b|_{B}$ permutes with $a|_{B}$. Since this is true for every class B of a+b, we conclude that a and b permute. Now, for $n \geqslant 2$, we have $a^n = (a^1)^{n-1} = 0_{\mathbf{A}}$, hence a^1 permutes with every b in $\mathcal{L}(\mathbf{A})$ due to the inductive hypothesis. By distributivity, we have

$$[b+a, b+a] \leqslant b + [a,a] = b+a^{1}.$$

Applying the case $n = 1$ to the images of a and $b+a^1$ in $\mathbf{A}_{/a^1}$, we get the permutability of a and $b+a^1$. It then follows that $a \circ b = a \circ a^1 \circ b = a \circ (a^1+b) = a+a^1+b = a+b$. 2.2 can also be obtained from this lemma.

PROOF OF 2.1. First we show that the decomposition center

$$Z = (a+c)(a+d)(b+c)(b+d)$$

lies in the center z_A of A. By distributivity we have $[a+c,1] = [a+c,a+b] \leqslant a + [c,b] \leqslant a+cb$. Hence $[Z,1] \leqslant a+cb$ and, by symmetry, $[Z,1] \leqslant m$ where

$$m = (a+bc)(a+bd)(b+ac)(b+ad).$$

But, looking at the subdirectly irreducible homomorphic images of the sublattice generated by a,b,c,d (Claim 1.2), one sees that $m = 0_A$.

Now let t in 1/b correspond to the center of $A_{/b}$. Then at $= az_A$, since the center of a product is the product of the centers. In particular, if t/b has finite length, then so has az_A, and (B) or (C) hold for z_A/az_A trivially. Thus, for a,b,c,d in $\mathbf{M} = \mathcal{L}(A)$, the hypotheses of 1.1 are satisfied. Let p_i,q_i,a_i,b_i,c_i,d_i $(i = 1,2)$ be given according to 1.1. Due to 2.2, any two elements e,f from {a,b,c,d} permute, hence so do e_i and f_i. It remains to show that, for any $e \in \{a,b,c,d\}$, e_1 and e_2 permute. For this, it is clearly enough to show that q_1 and q_2 permute. Clearly, we are done if we know that any two of the summands of q_1 and q_2 (as given in (2) of the first definition in § 1) permute. Since p_1 and p_2 are in $Z \leqslant z_A$, they permute with every congruence. Thus the given summands are of the form $h = (1+p)(s+p)$ and $k = (m+q)(t+q)$, with p and q permuting with every congruence, with $st = 0$, and $s \circ t = s+t$. First we easily check that $(1+p)(s+p)$ is equal to $((1 \circ p) \cap s) \circ p = ((1+p)s) \circ p$, simply because 1,p and s are equivalence relations, and p permutes with 1 and with s. Similarly, $(m+q)(t+q) = (m+q)t \circ q$. Since $0 = st \leqslant (1+p)s$, $(m+q)t \leqslant s \circ t = s+t$, we get by 2.2 that $(m+q)t$ permutes with s. Again by 2.2, since $s(m+q)t \leqslant (1+p)s \leqslant s \circ (m+q)t$, we find that $(m+q)t$ permutes with $(1+p)s$. And finally,

$$h \circ k = (1+p)s \circ p \circ (m+q)t \circ q = (1+p)s \circ (m+q)t \circ p \circ q$$
$$= (m+q)t \circ q \circ (1+p)s \circ p = k \circ h.$$

Hence q_1 and q_2 permute.

§3. **Isotopy.** Let all algebras be of a fixed type Δ. Let *C* be a class of algebras, closed under formation of direct products. Let **C** be an algebra in *C*.

For arbitrary algebras **A** and **B**, not necessarily in *C*, we define:

3.1. DEFINITION. **A** and **B** are *isotopic via* **C** if for every $c \in C$ there exists a bijective mapping $\alpha_c \colon A \to B$ such that, for every n-ary fundamental operation *f*, and for any $a_1,\dots,a_n \in A$, $c_1,\dots,c_n \in C$, we have

$$f(\alpha_{c_1}(a_1),\dots,\alpha_{c_n}(a_n)) = \alpha_{f(c_1,\dots,c_n)}(f(a_1,\dots,a_n)).$$

A and **B** are *affine isotopic via* **C** if additionally, for any two $c \neq c'$ and any $a \in A$, we have

$$\alpha_c(a) \neq \alpha_{c'}(a).$$

In other words: **A** and **B** are isotopic via **C** if there is a homomorphism α: **C** × **A** → **B** such that, for every $c \in C$, the mapping $\alpha(c,\text{-})$ is a bijective mapping from A to B. **A** and **B** are affine isotopic via **C** if, additionally, $\alpha(\text{-},a)$ is injective for every $a \in A$.

Notice that this definition of isotopy is much more restrictive than the one given in [7]. In particular, the above definition does not distinguish fundamental operations from term-functions.

Clearly the above definitions are symmetric, i.e. if **A** and **B** are (affine-) isotopic via **C**, then **B** and **A** are (affine-) isotopic via **C**, using the inverse mappings α_c^{-1}. And **A** and **B** are isomorphic iff they are isotopic via a one-element algebra **C**.

3.2. DEFINITION. **A** and **B** are *C-isotopic* (*affine-C-isotopic*), and we write

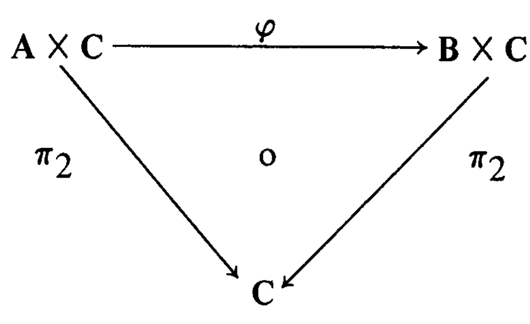
$$\mathbf{A} \underset{\mathcal{C}}{\sim} \mathbf{B}\ (\mathbf{A} \underset{\mathcal{C}}{\simeq} \mathbf{B}),$$

if, for some $\mathbf{C} \in \mathcal{C}$, **A** and **B** are isotopic (affine-isotopic) via **C**.

Clearly $\underset{\mathcal{C}}{\sim}$ is an equivalence relation on classes of isomorphic algebras. For transitivity note that, if **A** and **B** are isotopic via **C**, and if **B** and **D** are isotopic via **C**′, then **A** and **D** are isotopic via **C** × **C**′, using the mapping

$$\alpha_{(c,c')} := \alpha_{c'} \circ \alpha_c \text{ for } (c,c') \in C \times C'.$$

3.3. PROPOSITION. **A** *is isotopic to* **B** *via* **C** *iff there is an isomorphism* φ: **A** × **C** → **B** × **C** *which commutes with the second projections.*



PROOF. If **A** and **B** are isotopic via **C** by way of the mappings α_c, $c \in C$, define

φ: $\mathbf{A} \times \mathbf{C} \to \mathbf{B} \times \mathbf{C}$ by $\varphi(a,c) := (\alpha_c(a),c)$. It is easy to check that φ is an isomorphism commuting with the second projections. On the other hand, given such a mapping φ, define α: $\mathbf{A} \times \mathbf{C} \to \mathbf{B}$ by $\alpha := \pi_1 \circ \varphi$. If $\alpha_c(a) = \alpha_c(a')$, i.e. $\pi_1 \circ \varphi(a,c) = \pi_1 \circ \varphi(a',c)$, one has $\pi_2 \circ \varphi(a,c) = \pi_2(a,c) = c = \pi_2(a',c) = \pi_2 \circ \varphi(a',c)$ hence $\varphi(a,c) = \varphi(a',c)$ and therefore $a = a'$. So clearly α is an isotopy between **A** and **B** via **C**.

We will from now on assume that **A**, **B** and **C** are contained in a fixed congruence-modular variety. What we eventually are going to prove is that in this case **C** has to carry a module-structure which is in some specific sense inherited in **A** and in **B**. Intuitively (the size of) **C** measures how far apart **A** and **B** are from being isomorphic, cf. Theorem 4.3 and Corollary 4.4.

3.4. THEOREM. *For algebras* **A**, **B** *and* **C** *contained in a congruence-modular variety the following are equivalent:*

(i) **A** *and* **B** *are isotopic via* **C**.

(ii) **A** *and* **B** *are affine isotopic via a homomorphic image of* **C**.

(iii) *There is a congruence* β *on* $\mathbf{A} \times \mathbf{C}$ *such that* $(\mathbf{A} \times \mathbf{C})_{/\beta} \cong \mathbf{B}$, *and* β *is a complement of the kernel of* π_2.

(iv) *For a homomorphic image* $\mathbf{C}'$ *of* **C** *there is a congruence* β *on* $\mathbf{A} \times \mathbf{C}'$ *such that* $(\mathbf{A} \times \mathbf{C}')_{/\beta} \cong \mathbf{B}$, β *is a complement of* $\ker \pi_2$, *and* $\beta \wedge \ker \pi_1 = 0$.

PROOF. (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). Let **A** and **B** be isotopic via **C**, let α_c, $c \in C$, be the bijections between **A** and **B** establishing the isotopy. Define a congruence β on $\mathbf{A} \times \mathbf{C}$ by

$$(a,c)\beta(a',c') \textit{ iff } \alpha_c(a) = \alpha_{c'}(a').$$

By Proposition 3.3, $\beta = \ker(\pi_1 \circ \varphi)$ and therefore $(\mathbf{A} \times \mathbf{C})_{/\beta} \cong \mathbf{B}$. For arbitrary (a,c) and $(a',c') \in A \times C$ one has, by the surjectivity of α_c, that there is an $a'' \in A$ with $\alpha_c(a'') = \alpha_{c'}(a')$. Hence, one has $(a,c) \ker \pi_2 (a'',c) \beta (a',c')$, so $\ker \pi_2 \vee \beta = (A \times C)^2$. Suppose $(a,c) \ker \pi_2 \wedge \beta (a',c')$. Then $c = c'$ and $\alpha_c(a) = \alpha_c(a')$, so $(a,c) = (a',c')$ by injectivity of α_c, thus $\ker \pi_2 \wedge \beta = 0$.

(iii) $\Rightarrow$ (iv). Since $\ker \pi_1 \wedge \ker \pi_2 \leqslant \beta \leqslant \ker \pi_1 \vee \ker \pi_2$ and by modularity, the sublattice generated by $\ker \pi_1, \ker \pi_2$ and β must be a homomorphic image of the lattice in Figure 4. The fact that β is a complement of $\ker \pi_2$ collapses this lattice to the lattice in Figure 5.

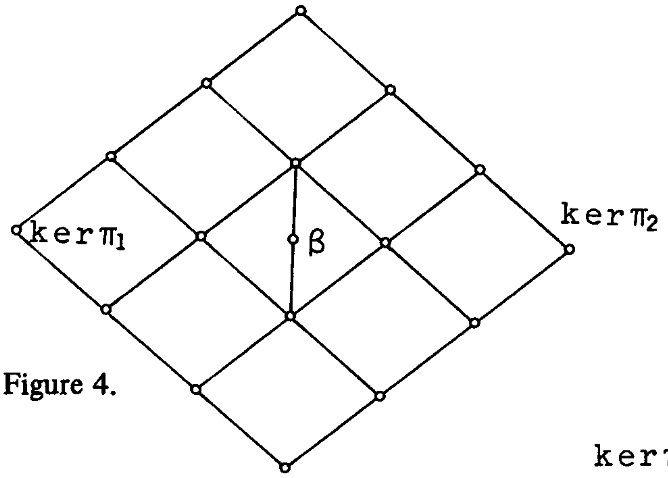
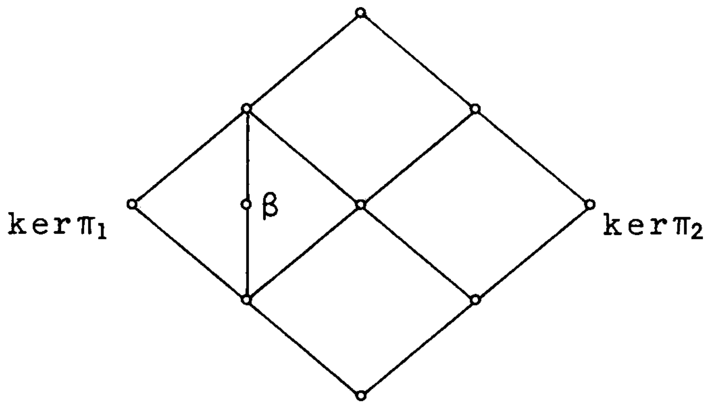


Figure 4.

Figure 5.



We define

$$\gamma := \ker \pi_2 \vee (\ker \pi_1 \wedge \beta),$$

and $\mathbf{C}' := (\mathbf{A} \times \mathbf{C})_{/\gamma}$. $\mathbf{C}'$ is a homomorphic image of $\mathbf{C}$ by the isomorphism theorem. Since γ permutes with $\ker \pi_1$ according to Korollar 5 of [9] (see also Proposition 2.2), we get: $(\mathbf{A} \times \mathbf{C})_{/\ker \pi_1 \wedge \beta} \cong \mathbf{A} \times \mathbf{C}'$.

(iv) ⇒ (ii). We may suppose $\beta \neq \ker \pi_1$, otherwise $\mathbf{A} \cong \mathbf{B}$ and with a one-element algebra $\mathbf{C}'$ our claim is true. Again, with Korollar 5 of [9] we have that β permutes with $\ker \pi_1$ and $\ker \pi_2$. For any arbitrary $c \in \mathbf{C}'$, define a map $\alpha_c: \mathbf{A} \to \mathbf{B}$ in the following way: $\alpha_c(a) = b$ iff $(a,c) \in b$ (here we consider the elements of B as β-classes). Obviously, α_c is a mapping from $\mathbf{A}$ to $\mathbf{B}$. $\alpha_c(a) = \alpha_c(a')$ would imply $(a,c)\beta(a',c)$. Since $\beta \wedge \ker \pi_2 = 0$ we conclude $a = a'$, hence α_c is one-to-one. For any $b \in \mathbf{B}$ there is an element $a \in \mathbf{A}$ with $(a,c) \in b$ since $\ker \pi_2$ permutes with β. Hence α_c is onto. Clearly, if $\alpha_{c_i}(a_i) = b_i$, i.e. $(a_i,c_i) \in b_i$ for $i = 1,2,\ldots,n$, and if f is an n-ary operation, it follows that

$$f((a_1,c_1),\ldots,(a_n,c_n)) = (f(a_1,\ldots,a_n),f(c_1,\ldots,c_n)) \in f(b_1,\ldots,b_n),$$

i.e. we have

$$f(\alpha_{c_1}(a_1),\ldots,\alpha_{c_n}(a_n)) = \alpha_{f(c_1,\ldots,c_n)}(f(a_1,\ldots,a_n)),$$

so $\mathbf{A}$ and $\mathbf{B}$ are isotopic via $\mathbf{C}$. Finally, suppose that $\alpha_c(a) = \alpha_{c'}(a)$ for some $a \in \mathbf{A}$ and $c,c' \in \mathbf{C}$. Then $(a,c)\beta(a,c')$, hence $c = c'$ since $\ker \pi_1 \wedge \beta = 0$. Thus $\mathbf{A}$ and $\mathbf{B}$ are affine isotopic as claimed.

(ii) ⇒ (i). If $\mathbf{C}'$ is a homomorphic image of $\mathbf{C}$, and if $\mathbf{A}$ is isotopic to $\mathbf{B}$ via $\mathbf{C}'$, then $\mathbf{A}$ is isotopic to $\mathbf{B}$ via $\mathbf{C}$. Namely, if φ is the epimorphism from $\mathbf{C}$ onto $\mathbf{C}'$, then define for each $c \in C$: $\alpha_c := \alpha_{\varphi(c)}$.

Let us finally show how the congruence relations of isotopic algebras have to be related.

3.5. PROPOSITION. *Let* **A** *be isotopic to* **B** *via* **C**, *with* **A**, **B** *and* **C** *contained in a modular variety. Then there is an isomorphism* ψ *between the congruence lattices of* **A** *and* **B** *which sends pairs of permuting congruences into pairs of permuting congruences. Moreover, for any congruence* θ *on* **A**, *we get that* $\mathbf{A}_{/\theta}$ *is isotopic to* $\mathbf{B}_{/\psi(\theta)}$ *via* **C**.

Before we prove this we need the following lemma:

3.6. LEMMA. *Let* b/a *and* d/c *be two intervals in a sublattice of the congruence lattice of an algebra in a modular variety. If* b/a *is transposed upwards (respectively downwards) to* d/c, *and if* b *and* c *(respectively* a *and* d*) permute, then this transposition carries permuting pairs of congruences in* b/a *into permuting pairs in* d/c.

PROOF. Suppose b/a is transposed upwards onto d/c and b and c permute. Let x_1, x_2 be permuting elements in b/a. Then, since $b \wedge c \leqslant x_1, x_2 \leqslant b+c$, and since b and c permute, by Korollar 5 of [9] or Proposition 2.2, we have that x_1 and x_2 permute with b and with c. We get:

$$(c+x_1) + (c+x_2) = c + (x_1 \circ x_2) = c \circ x_1 \circ x_2$$

$$= c \circ x_1 \circ c \circ x_2 = (c+x_1) \circ (c+x_2),$$

hence $c+x_1$ and $c+x_2$ permute, which we had to show.

Suppose now that b/a is transposed downwards to d/c, and choose $x_1, x_2 \in b/a$ with $x_1 \circ x_2 = x_2 \circ x_1$. Since $a \wedge d \leqslant x_1, x_2 \leqslant a+d$, we get that $x_1, x_2, d \wedge x_1$ and $d \wedge x_2$ generate a sublattice $\mathbf{D}_2^3$ as shown in the figure below. Moreover, another application of Proposition 2.2 shows that $d \wedge (x_1+x_2)$ permutes with $x_1 \wedge x_2$.

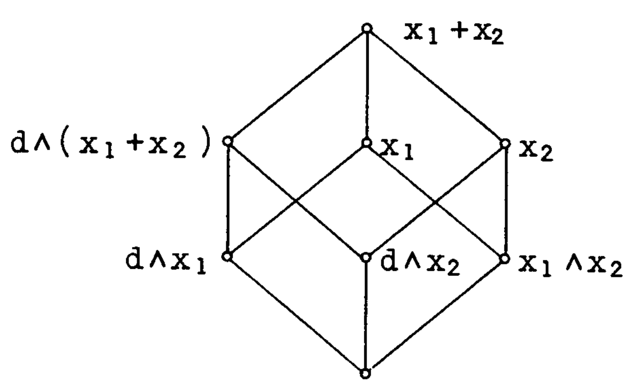


Figure 6.

Now we may forget the algebra we are working in and we may suppose, without loss of generality, that x_1+x_2 is the universal relation whilst $d \wedge x_1 \wedge x_2$ is the identity relation on the set A. Since $x_1 \wedge x_2$ and $d \wedge (x_1+x_2)$ are complementary and permute, they give rise to a product decomposition

$$A/_{d \wedge (x_1+x_2)} \times A/_{x_1 \wedge x_2}$$

of A. Moreover $A/_{x_1 \wedge x_2}$ is by the same reasons a direct product of $A/_{x_1}$ and $A/_{x_2}$. Hence $A \cong A_1 \times A_2 \times A_3$ with $d \wedge (x_1+x_2) = \ker \pi_1$, $x_1 = \ker \pi_2$ and $x_2 = \ker \pi_3$ with respect to this decomposition. Hence $d \wedge x_1 = \ker \pi_1 \wedge \ker \pi_2$ and $d \wedge x_2 = \ker \pi_1 \wedge \ker \pi_3$ with respect to this decomposition. Since $\ker \pi_1 \wedge \ker \pi_2$ commutes with $\ker \pi_1 \wedge \ker \pi_3$, so do $d \wedge x_1$ and $d \wedge x_2$ which was to be proved.

Finally to verify Proposition 3.5, we assume **A** and **B** to be isotopic via **C**. By Proposition 3.3 there is an isomorphism φ: $A \times C \to B \times C$ commuting with the second projections. φ induces in a natural way an isomorphism $\hat{\varphi}$ between the congruence lattices $\mathcal{L}(A \times C)$ and $\mathcal{L}(B \times C)$ such that $\hat{\varphi}(\ker \pi_2^{A\times C}) = \ker \pi_2^{B\times C}$. Clearly, $\hat{\varphi}$ carries pairs of permuting congruences into pairs of permuting congruences. Now let α be the mapping given by transposing the interval $(A \times C)^2/\ker \pi_1$ in the congruence lattice of $A \times C$ down onto the interval $\ker \pi_2/o$, and let β be the map transposing the interval $\ker \pi_2/o$ in the congruence lattice of $B \times C$ up onto the interval $(B \times C)^2/\ker \pi_1$. Then, together with Lemma 3.6, we get that the composition $\beta \circ \hat{\varphi} \circ \alpha$ is an isomorphism between $(A \times C)^2/\ker \pi_1$ in $\mathcal{L}(A \times C)$ and $(B \times C)^2/\ker \pi_1$ in $\mathcal{L}(B \times C)$ carrying pairs of permuting congruences into permuting pairs again. By the isomorphism theorem of course, the first interval is canonically isomorphic to $\mathcal{L}(A)$ whilst the second interval is canonically isomorphic to $\mathcal{L}(B)$. Let ψ be the isomorphism so constructed. For arbitrary θ in $\mathcal{L}(A)$ we have to show that $A/_\theta$ is isotopic via **C** to $B/_{\psi(\theta)}$. Viewing θ and $\psi(\theta)$ as congruences on $A \times C$ and $B \times C$, respectively, one gets product decompositions of $(A \times C)/_{\theta \wedge \ker \pi_2}$ and $(B \times C)/_{\psi(\theta) \wedge \ker \pi_2}$. Moreover, ψ agrees with $\hat{\varphi}$ on $\theta \wedge \ker \pi_2$ as well as on $\ker \pi_2$. (Use Proposition 2.2 and modularity to verify this.) Hence, there is an isomorphism between $(A \times C)/_{\theta \wedge \ker \pi_2}$ and $(B \times C)/_{\psi(\theta) \wedge \ker \pi_2}$, i.e. between $A/_\theta \times C$ and $B/_{\psi(\theta)} \times C$, so $\mathbf{A}/_\theta$ and $B/_{\psi(\theta)}$ are isotopic via **C**, by 3.3.

§4. **The structure of isotopic algebras.** We keep on working within a fixed congruence-modular variety **V**. Suppose **A**, **B** and **C** are algebras in **V** such that **A** is isotopic to **B** via **C**. By Theorem 3.4 we may as well suppose that **A** and **B** are affine isotopic via **C**. In the case that **C** has a one-element subalgebra, we immediately see that **A** and **B** have to be isomorphic; in particular, if **C** is a one-element algebra, this will be the case. If however **C** is bigger, then **A**, **B** and **C** will be equipped with an interesting structure, similar to a module structure. For this sake, let us define:

4.1. DEFINITION. An algebra **A** is *affine* with respect to an abelian group **G**, if A is the underlying set of an abelian group **G** such that, for every n-ary fundamental operation (every term-function) f of **A**, we have

$$f(x_1-y_1+z_1,\ldots,x_n-y_n+z_n) = f(x_1,\ldots,x_n)-f(y_1,\ldots,y_n)+f(z_1,\ldots,z_n),$$

for every x_i,y_i,z_i, A and $1 \leqslant i \leqslant n$.

Suppose **A** is an affine algebra. We define a new algebra $\mathbf{A}^\nabla$ on the same base set in the following way:

For any fundamental operation f on **A** define a new operation f^∇ by

$$f^\nabla(x_1,\ldots,x_n) := f(x_1,\ldots,x_n) - f(0,\ldots,0),$$

where 0 is the neutral element of **G**. For $\mathbf{A} = (A,(f_i)_{i\in I})$ we set now:

$$\mathbf{A}^\nabla := (A,(f_i^\nabla)_{i\in I}).$$

Obviously, $\mathbf{A}^\nabla$ is affine isotopic to **A** via **A**.

Let now a be an arbitrary element of A. Define a new group **G**′ on A by setting x+y: = x-a+y. Then a is the neutral element of **G**′. Clearly, **G** is isomorphic to **G**′ and, if we define

$$f^{\nabla'}(x_1,\ldots,x_n) := f(x_1,\ldots,x_n) - f(a,\ldots,a),$$

then we easily see that $\mathbf{A}^\nabla \cong \mathbf{A}^{\nabla'}$. Hence we will talk about $\mathbf{A}^\nabla$ without specifying which element we pick for a zero-element.

Let now **A** and **B** be affine isotopic via **C**. By Theorem 3.4, the congruence lattice of **U**: = **A** X **C** contains a 0-1-sublattice as indicated in the following figure:

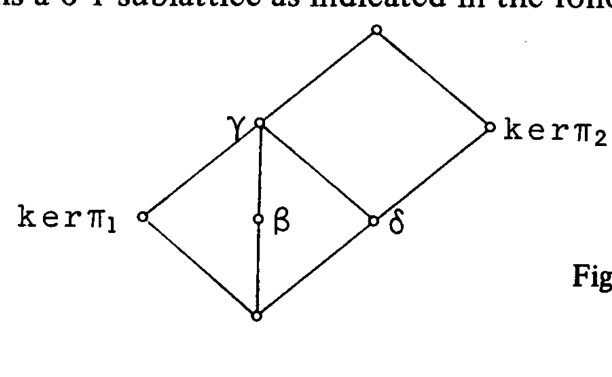


Figure 7.

Moreover, the congruences of this sublattice are pairwise permutable. Setting $U: = A \times C$ we see that γ partitions U into blocks U_i, $i \in U/\gamma$. Now, on every block of γ, the congruences $\ker \pi_1$, β and δ are inherited, so, on the block U_i, the equivalences $\alpha_i: = \ker \pi_1|_{U_i}$, $\beta_i: = \beta|_{U_i}$, $\delta_i: = \delta|_{U_i}$ are pairwise complementary and permutable. If, for a moment, we consider only those terms Ω on U which are idempotent on U/γ, then clearly $U_i: = (U_i,\Omega)$ is a subalgebra of $\bar{U}: = (U,\Omega)$, and α_i, β_i and δ_i are congruence relations on U_i. Hence M_3, the five-element modular nondistributive lattice, is a 0-1-sublattice of the congruence lattice of U_i. Since U_i still generates a modular variety we get, by [13] or by a minor adjustment of Satz 6 of [9], that

$$A_i = (A_i,\Omega): = U_{i/\alpha_i}, \; B_i = (B_i,\Omega): = U_{i/\beta_i}$$

and

$$\bar{C}_i = (C_i,\Omega): = U_{i/\delta_i}$$

are affine algebras. Note, that A_i is a subalgebra of (A,Ω), B_i a subalgebra of (B,Ω), whereas $\bar{C}_i$ is isomorphic to (C,Ω) via the isomorphism $\psi: (C,\Omega) \to \bar{C}_i$ sending $[x]\ker \pi_2$ to $([x]\ker \pi_2 \cap U_i) \in U_{i/\delta} = C_i$. Since A_i, B_i and $\bar{C}_i$ are affine with respect to the same abelian group G_i, it follows from the last remark that all G_i are isomorphic to one fixed abelian group G.

To proceed we have to step into the proof of the affineness of the last mentioned algebras and recall how the underlying group G was defined.

Take any U_i and choose an element $o_i \in U_i$. The elements of A_i, respectively B_i, respectively C_i are the classes of α_i, respectively β_i, respectively δ_i. The blocks of these congruences form, in a natural way, three parallel classes of a geometrical 3-net, see [8]. To introduce a geometrical language, we will draw the blocks of α_i, β_i, and δ_i as vertical, slanted and horizontal lines, respectively. Those blocks passing through o_i will be named o_i^A, o_i^B, o_i^C. Given $x,y,z \in A_i$ and $c \in C_i$ there exists a unique $c' \in C_i$ so that the following configuration holds:

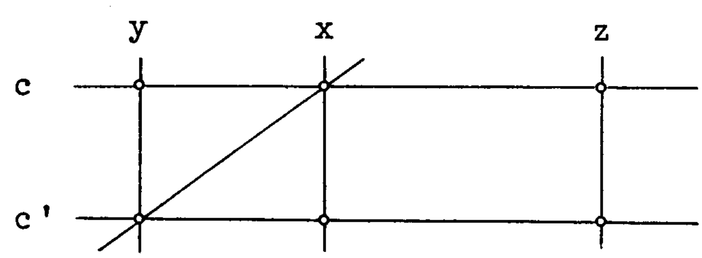


Figure 8.

Hence, applying the 6-ary polynomial exhibited in [9], there is a unique element $u \in A_i$ extending this configuration to the following one:

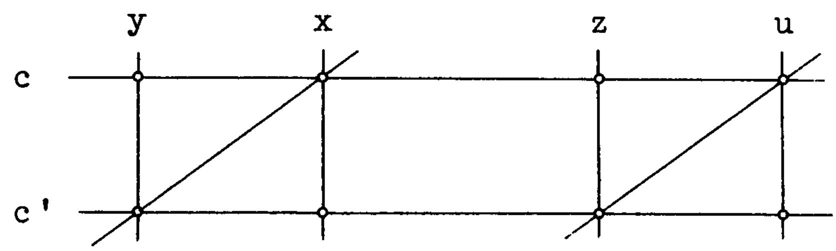


Figure 9.

In this case we set x-y+z: = u. This definition is independent from the choice of c because the Reidemeister condition is true in our 3-net, as was shown in [9]. In particular, choosing $y = o_i^A$ we get an abelian group $G_i = (A_i,+,o_i^A)$, see [9]. (We neglect to index the symbol + as $+_i$; for one, because we already know that G_i is isomorphic to G_j for different i and j, and we hope this will not cause confusions.) Now note that the operation x-y+z on every A_i is given by the configuration in Figure 9, which is defined as certain points being congruent to others. Any polynomial on U will preserve congruences and, hence, this configuration. Thus we get immediately:

4.2. LEMMA. *Let* f *be an* n*-ary polynomial of* A, *and let*

$$x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n \in A$$

be such that, for all $1 \leqslant i \leqslant n$, *we have* $x_i,y_i \in [z_i]\gamma$. *Then*

$$f(x_1-y_1+z_1,\dots,x_n-y_n+z_n) = f(x_1,\dots,x_n)-f(y_1,\dots,y_n)+f(z_1,\dots,z_n).$$

With those $o_i \in U_i$ chosen arbitrarily, the corresponding $\{o_i^A\}$ need not be subalgebras of A. Hence, for any n-ary operation f, we define a new operation f^∇ by

$$f^\nabla(x_1,\dots,x_n) := f(x_1,\dots,x_n) - f(o_{k_1}^A,\dots,o_{k_n}^A),$$

where $o_{k_j}^A$ is the (unique) o_i^A congruent to x_j modulo γ. We do this for every $f \in F$ for the algebra $A = (A,F)$, and get a new algebra $A^\nabla = (A,F^\nabla)$ with $F^\nabla = \{f^\nabla | f \in F\}$. It is routine to check that up to isomorphy A^∇ does not depend on the choice of the o_i's, see the remark after 4.1. The so defined algebra will thus be called the *γ-linearization* of A.

For the sake of investigating the relation between A and C, we define a map $\varphi: A^\nabla \to C^\nabla$ as follows (at this point it becomes convenient to suppose that we have chosen the o_i's to be mutually congruent modulo ker π_2): For $x \in A$, say $x \in A_i$, we consider the intersection of the (vertical) line x with the (horizontal) line $o^C \in C$ and obtain a unique point $\bar{x}$. Take the (slanted) β_i-line through $\bar{x}$ and let its point of intersection with the vertical line o_i^A be called $\bar{\bar{x}}$. The (horizontal) line passing through $\bar{\bar{x}}$, i.e. $[\bar{\bar{x}}] \ker \pi_2$, is our $\varphi(x)$.

The geometric definition of φ is shown in the left half of the forthcoming figure, which also serves to demonstrate that φ is a homomorphism:

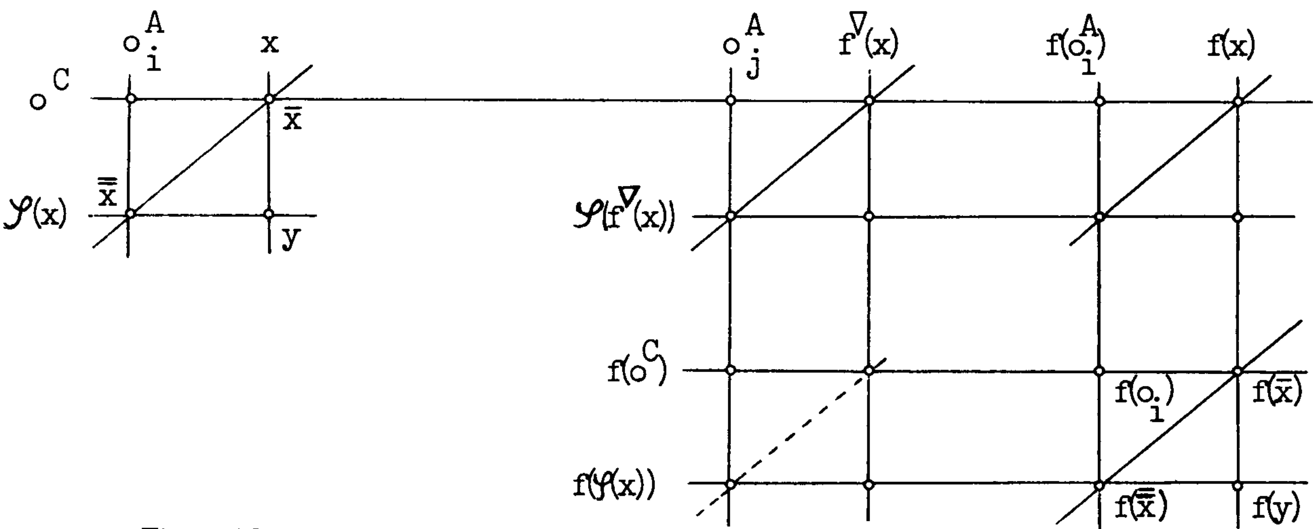


Figure 10.

The upper part of the right hand side of this figure gives the definition of f^∇ and of $\varphi(f^\nabla(x))$, the relations in the lower right hand corner are true because f preserves the relations of the left side of this figure. Hence, by the Reidemeister condition which holds according to [9], the relations of the lower left hand corner of the right side of this figure (indicated by a dotted slanted line) are true, which, in view of the definition of addition in C yields:

$$\varphi(f^\nabla(x)) = f^\nabla(\varphi(x)).$$

Thus φ is a homomorphism and clearly onto. Now consider $\ker \varphi$, the kernel of φ, and the congruence γ, (considered as a congruence on **A**). It is easy enough to see that γ and $\ker \varphi$ permute, their intersection and join are the trivial congruences on **A**. Since $\mathbf{A}_{/\gamma}$ and $\mathrm{U}_{/\gamma}$ are canonically isomorphic, we have thus established:

4.3. THEOREM. *Let* A *be affine isotopic to* **B** *via* C. *Then* **C** *is affine and* $\mathbf{A}^\nabla \cong \mathbf{C}^\nabla \times \mathbf{U}_{/\gamma}$, *where γ is as defined in 3.4 and* $\mathbf{A}^\nabla$ *is the γ-linearization of* **A**.

Noting that in the above formula **A** appears only on the left hand side but not on the right, we obtain readily, with γ as above:

4.4. COROLLARY. *Let* **A** *be isotopic to* **B** *then* **A** *and* **B** *have isomorphic*

γ-linearizations.

§5. Cancellation and remarks.

5.1. THEOREM. *Let* **A,B,C,D** *be algebras in a modular variety* **V** *such that the congruence lattice* $\mathcal{L}$(**A**) *satisfies the ascending chain condition and the center* $z_{\mathbf{A}}$ *of* **A** *has finite rank in* $\mathcal{L}$(**A**). *Then the affine isotopies* $\mathbf{A} \simeq \mathbf{C}$ *and* $\mathbf{A} \times \mathbf{B} \simeq \mathbf{C} \times \mathbf{D}$ *imply* $\mathbf{B} \simeq \mathbf{D}$.

PROOF. Consider **A** as above and assume that, for every direct factor $\mathbf{A}_1 \neq \mathbf{A}$ of **A**, it is already shown that, for all $\mathbf{B}_1,\mathbf{C}_1,\mathbf{D}_1 \in \mathbf{V}$, the affine isotopies $\mathbf{A}_1 \simeq \mathbf{C}_1$ and $\mathbf{A}_1 \times \mathbf{B}_1 \simeq \mathbf{C}_1 \times \mathbf{D}_1$ jointly imply $\mathbf{B}_1 \simeq \mathbf{D}_1$. Let **A**,**B**,**C** and **D** satisfy the conditions of the theorem, and set $\mathbf{E} := \mathbf{A} \times \mathbf{B}$. Denote the kernels of the projections onto **A** and **B** by a and b, respectively. Let c and d in $\mathcal{L}$(**E**) correspond to the kernels of the projections from $\mathbf{C} \times \mathbf{D}$ down to **C** and **D** as given through the isomorphism in 3.5. In particular, we have $a \otimes b = 0$, $c \otimes d = 0$, $\mathbf{F}_{/a} \cong \mathbf{A}$, $\mathbf{E}_{/b} \cong \mathbf{B}$ whereas $\mathbf{E}_{/c} \simeq \mathbf{C}$ and $\mathbf{E}_{/d} \simeq \mathbf{D}$. Let $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2$ be chosen according to 2.1. Then, by 3.4 the following isotopies hold:

$$\mathbf{E}_{/b_1} \simeq \mathbf{E}_{/d_1}, \mathbf{E}_{/a_1} \simeq \mathbf{E}_{/c_1}, \mathbf{E}_{/d_2} \simeq \mathbf{E}_{/a_2}, \mathbf{E}_{/c_2} \simeq \mathbf{E}_{/b_2}.$$

If $a_1 \otimes a_2 = a$ induces a proper direct decomposition of $\mathbf{A} \cong \mathbf{E}_{/a}$, then we may apply the inductive hypothesis to $\mathbf{A}_1 = \mathbf{E}_{/a_1}$, $\mathbf{B}_1 = \mathbf{E}_{/a_2}$, $\mathbf{C}_1 = \mathbf{E}_{/c_1}$ and $\mathbf{D}_1 = \mathbf{E}_{/c_2}$ to conclude from $\mathbf{A} \simeq \mathbf{C}$ and $\mathbf{A}_1 \simeq \mathbf{C}_1$ that

$$\mathbf{E}_{/d_2} \simeq \mathbf{E}_{/a_2} \cong \mathbf{B}_1 \simeq \mathbf{D}_1 \cong \mathbf{E}_{/c_2} \simeq \mathbf{E}_{/b_2}.$$

Thus $\mathbf{E}_{/b_i} \simeq \mathbf{E}_{/d_i}$ for $i = 1,2$, whence

$$\mathbf{B} \cong \mathbf{E}_{/b} \cong \mathbf{E}_{/b_1} \times \mathbf{E}_{/b_2} \simeq \mathbf{E}_{/d_1} \times \mathbf{E}_{/d_2} \cong \mathbf{E}_{/d} \simeq \mathbf{D}, \mathbf{B} \simeq \mathbf{D}.$$

If $a_1 \otimes a_2 = a$ is not a proper decomposition, then there are two possibilities. Firstly, suppose $a_1 = 1$ and $a_2 = a$. Then $b_1 = c_1 d_1 \leqslant c_1$ and $c_1 = c_1 + b_1 = 1$. Hence $c_2 = c$, and

$$\mathbf{E}_{/d_2} \simeq \mathbf{E}_{/a_2} = \mathbf{E}_{/a} \cong \mathbf{A} \simeq \mathbf{C} \simeq \mathbf{E}_{/c} = \mathbf{E}_{/c_2} \simeq \mathbf{E}_{/b_2}.$$

Now $\mathbf{B} \simeq \mathbf{D}$ follows as above. Alternately, suppose $a_2 = 1$ and $a_1 = a$. Then we have

$$\mathbf{E}_{/c} \simeq \mathbf{C} \simeq \mathbf{A} \cong \mathbf{E}_{/a} = \mathbf{E}_{/a_1} \simeq \mathbf{E}_{/c_1}.$$

If c would be properly contained in c_1, then an iterated application of 3.5 would yield an infinite ascending sequence in $\mathcal{L}(E_{/c}) \cong \mathcal{L}(A)$. Thus, $c = c_1$ and $c_2 = 1$. It follows that $b_2 = d_2 = 1$, $b = b_1$, $d = d_1$, and

$$B \cong E_{/b} = E_{/b_1} \simeq E_{/d_1} = E_{/d} \simeq D.$$

Now, since A satisfies the ascending chain condition, any direct factor F of A does so, too, and has a representation as a direct product of finitely many directly indecomposables. Moreover, since the center of a product is the product of the centers, the center z_F of F has finite rank in $\mathcal{L}(F)$. Thus, the above constitutes an inductive proof of the theorem.

5.2. REMARK. Theorem 2.1 can be easily used to prove a refinement theorem for finite direct decompositions of algebras in congruence modular varieties. One just has to go through one of the proofs in the literature (e.g. Kurosh [15; §47]) and to observe that permutability is granted wherever needed.

5.3. REMARK. The isomorphism of $\mathcal{L}(A)$ onto $\mathcal{L}(B)$ induced by an isotopy (Theorem 3.5) also preserves commutation. This is a trivial consequence of the distributivity law for commutation.

5.4. REMARK. Let A and B be affine isotopic via C and $E = A \times C$. Then, by 3.4, there are β,γ,δ in $\mathcal{L}(E)$ such that the relations of Figure 7 are valid. Obviously, γ is contained in the center of E. Any of the maps α_c^{-1} ($c \in C$) from 3.4 is a central shift in the sense of Smith [16; 4.1] - one just has to look at the construction of the centering congruence κ from the given congruences according to the proof of Theorem 1.4 in [11]. Thus, due to Theorem 421 in [16], the concept of central isotopy coincides with the concept of C-isotopy for algebras in a congruence permutable variety C.

5.5. REMARK. The compatibility result in 4.2 holds quite generally for congruences γ which are the biggest element of an M_3 in the congruence lattice of an algebra in a modular variety, provided the smallest element of M_3 is the smallest element of the congruence lattice, cf. [10; 2.3].

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