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An easy way to the commutator in modular varieties

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0. The purpose of this note is to present an elementary approach to the commutator in modular varieties and to derive some new results connected with this concept.

The theory of commutators in universal algebra was introduced and thoroughly studied in the framework of permutable varieties by J. D. H. Smith [8]. Then J. Hagemann and C. Herrmann in [6] studied this concept in modular varieties and were able to prove many of its important properties. Their results were used by R. Freese and R. McKenzie [1] to derive some deep results about modular varieties, also solving some longstanding problems.

Since we feel that not only the definitions and the proofs in [6] are unusually difficult but also the geometrical meaning and intuition still inherent in [8] is lost, we feel that a simplification of this concept is needed.

Our new approach was possible due to the translation of congruence modularity (a projective notion) into the affine geometries of the algebras involved, which is given mainly in [2].

1. Basic notions and prerequisites. We denote universal algebras with gothic letters and congruences with greek letters. Note that a congruence α on an algebra $\mathfrak{A}$ may be viewed as a special subalgebra of $\mathfrak{A} \times \mathfrak{A}$. If $\mathfrak{D}$ is a subalgebra of $\mathfrak{A} \times \mathfrak{B}$ then the kernels of the canonical projections will be denoted with π_1 and π_2. $\mathfrak{C}(\mathfrak{A})$ denotes the lattice of congruences on $\mathfrak{A}$. For $(x, y) \in \alpha$ we frequently write $x \alpha y$.

For a subset S of $\mathfrak{A} \times \mathfrak{A}$ we denote with $\langle S \rangle_{\mathfrak{A}}$ the smallest congruence relation on $\mathfrak{A}$ containing S. A description of $\langle S \rangle_{\mathfrak{A}}$ is given by Mal'cev in [7].

If $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ is a homomorphism and α is a congruence on $\mathfrak{A}$ then $\vec{\varphi}\alpha$ denotes the induced congruence on $\mathfrak{B}$, i.e. $\vec{\varphi}\alpha = \langle (\varphi \times \varphi)\alpha \rangle_{\mathfrak{A}}$. If β is a congruence on $\mathfrak{B}$ then $\overleftarrow{\varphi}\beta := \{(x, y) \in \mathfrak{A} \times \mathfrak{A} \mid (\varphi(x), \varphi(y)) \in \beta\}$ is a congruence on $\mathfrak{A}$. Note that for φ onto we have $\vec{\varphi}\overleftarrow{\varphi}\beta = \beta$ and $\overleftarrow{\varphi}\vec{\varphi}\alpha = \alpha \vee \ker \varphi$.

Now for the main (conceptual) tools we use: The first one is a simple observation, see [3]:

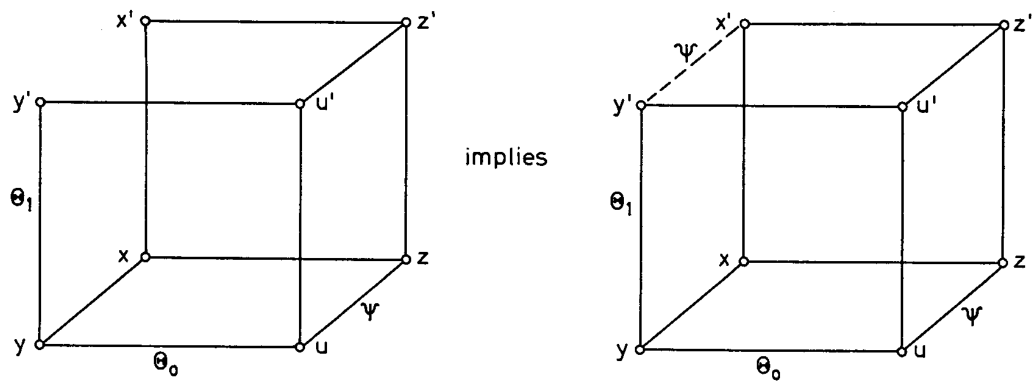
1.1. Shifting Lemma. *Let $\mathfrak{A}$ be an algebra with a modular congruence lattice. Let Θ_0, Θ_1 and Ψ be congruences on $\mathfrak{A}$ and x, y, z, u elements of $\mathfrak{A}$. Then*

with $\Phi = (\Theta_0 \wedge \Theta_1) \vee \Psi$.

(In the pictures we frequently use, points denote elements of $\mathfrak{A}$ and two points are joined with a line if they are congruent modulo a congruence α, in which case the line is labelled with the letter α. Parallel lines are always supposed to be equally labelled.)

Secondly we recall from [2] the

1.2. Cube Lemma. *Let $\mathfrak{A}$ be an algebra in a modular variety (i.e. all subalgebras of powers of $\mathfrak{A}$ have modular congruence lattices). Let $x, y, z, u, x', y', z', u'$ be elements of $\mathfrak{A}$ and Θ_0, Θ_1, Ψ congruences on $\mathfrak{A}$ with $\Theta_0 \wedge \Theta_1 \leqq \Psi$. Then*



2. The Commutator: Definition and simple properties. Let α and β be congruences on $\mathfrak{A}$. Define a congruence Δ_α^β on α (viewed as a subalgebra of $\mathfrak{A} \times \mathfrak{A}$) by

$$\Delta_\alpha^\beta := \langle\{((x, x), (y, y)) \mid x \beta y\}\rangle_\alpha .$$

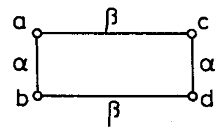
and

$$[\alpha, \beta] := \{(x, y) \mid (x, x) \Delta_\alpha^\beta (x, y)\}$$

$[\alpha, \beta]$ is called the commutator of α and β.

It is a simple exercise to show that this concept coincides with the well known commutator for groups, via the translation of congruences into normal subgroups.

2.1. Properties of Δ_α^β.

(i) $(a, b) \Delta_\alpha^\beta (c, d)$ *implies* ,

(ii) $(a, b) \Delta_\alpha^\beta (c, d)$ *implies* $(b, a) \Delta_\alpha^\beta (d, c)$

(iii) $a \beta b$ *implies* $(a, a) \Delta_\alpha^\beta (b, b)$. '

Proof. (iii) being part of the definition, (ii) follows immediately from the symmetry of α, or fancier, note that $(x, y) \to (y, x)$ yields an automorphism of α, leaving invariant the generating set of Δ_α^β. For (i) note that $\Delta_\alpha^\beta \leqq \beta \times \beta|_\alpha$ where $\beta \times \beta|_\alpha$ is the congruence on α given by $(x, y)\beta \times \beta|_\alpha(z, u)$ iff $(x, z) \in \beta$ and $(y, u) \in \beta$.

2.2. Properties of $[\alpha, \beta]$.

(i) $[\alpha, \beta] = \{(x, y) \mid (x, x) \Delta_\alpha^\beta (y, x)\}$,

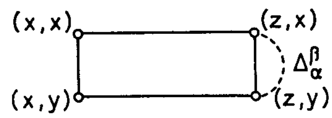
(ii) $[\alpha, \beta] = \{(x, y) \mid \exists z(\ (z, x) \Delta_\alpha^\beta (z, y)\ )\}$
$= \{(x, y) \mid \exists z(\ (x, z) \Delta_\alpha^\beta (y, z)\ )\}$.

(iii) $[\alpha, \beta]$ *is a congruence relation on* $\mathfrak{A}$.

(iv) $[\alpha, \beta] \leqq \alpha \wedge \beta$.

Proof. (i) follows from 2.1 (ii).

(ii) follows with the Shifting lemma applied to



(iv) is immediate from the definition and from 2.1 (i). For (iii): All properties of a congruence relation are immediate with 2.1. For transitivity we use 2.2, namely $x[\alpha, \beta]y[\alpha, \beta]z$ implies $(x, y)\Delta_\alpha^\beta(y, y)\Delta_\alpha^\beta(z, y)$ hence $x[\alpha, \beta]z$ with 2.2.

From Mal'cevs description of congruences generated by a binary (symmetric) relation on an algebra [7] we obtain

2.3. An alternative description of the commutator. *The statement* "$(x, y) \in [\alpha, \beta]$" *in equivalent to "there exist unary algebraic functions* $\tau_0, \ldots, \tau_n$ *on* α *and* (s_0, t_0), $\ldots, (s_n, t_n) \in \beta$ *with*

$$\tau_0(s_0, s_0) = (x, x),$$
$$\tau_i(t_i, t_i) = \tau_{i+1}(s_{i+1}, s_{i+1}), \quad 0 \leqq i < n,$$
$$\tau_n(t_n, t_n) = (x, y)\text{''}.$$

With the foregoing description at hand, the verification of the following proposition is routine:

2.4. Proposition. *Let* $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism and* α, β *congruences on* $\mathfrak{A}$. *Then* $\vec{\varphi}[\alpha, \beta] \leqq [\vec{\varphi}\alpha, \vec{\varphi}\beta]$.

Proof. Consider the homomorphism $\varphi \times \varphi\colon \alpha \to \vec{\varphi}\alpha$ and apply it to the equations of 2.3. Each τ_i, which is an algebraic function of (the algebra) α will be transformed by $\varphi \times \varphi$ into an algebraic function $\bar{\tau}_i$ of (the algebra) $\vec{\varphi}\alpha$. Thus we get

$$\bar{\tau}_0(\varphi(s_0), \varphi(s_0)) = (\varphi(x), \varphi(x)),$$
$$\bar{\tau}_i(\varphi(t_i), \varphi(t_i)) = \bar{\tau}_{i+1}(\varphi(s_{i+1}), \varphi(s_{i+1})), \quad \text{for} \quad 0 \leqq i < n,$$
$$\bar{\tau}_n(\varphi(t_n), \varphi(t_n)) = (\varphi(x), \varphi(y)).$$

Hence with 2.3 we have $(\varphi(x), \varphi(y)) \in [\vec{\varphi}\alpha, \vec{\varphi}\beta]$.

The following corollary is well known for groups:

2.5. Corollary. *The commutator of fully invariant congruences is again fully invariant.*

As we go on we need the following technical result:

2.6. Theorem. *Let* $\mathfrak{D}$ *be a subalgebra of* $\mathfrak{A} \times \mathfrak{A}$ *(with* $\mathfrak{C}(\mathfrak{D})$ *modular). Let* $\varkappa_i$, $i \in I$ *be a family of congruence relations on* $\mathfrak{D}$ *with the property:*

$$(x, y)\,\varkappa_i\,(z, u) \quad \textit{implies} \quad (x, x)\,\varkappa_i\,(z, z).$$

Then for all $x, y, z \in \mathfrak{A}$ *we have:*

$$(x, x) \bigvee \varkappa_i\,(y, z) \quad \textit{implies} \quad (y, y) \bigvee (\varkappa_i \wedge \pi_1)\,(y, z).$$

Proof. If $(x, x) \bigvee \varkappa_i\,(y, z)$ then there exist w.l.o.g. $\varkappa_0, \ldots, \varkappa_{n-1}$ and $(u_0, v_0), \ldots, (u_n, v_n) \in \mathfrak{D}$ with

$$(u_0, v_0) = (x, x), \qquad (u_n, v_n) = (y, z)$$

and

$$(u_i, v_i)\,\varkappa_i\,(u_{i+1}, v_{i+1}) \quad \text{for} \quad 0 \leqq i < n.$$

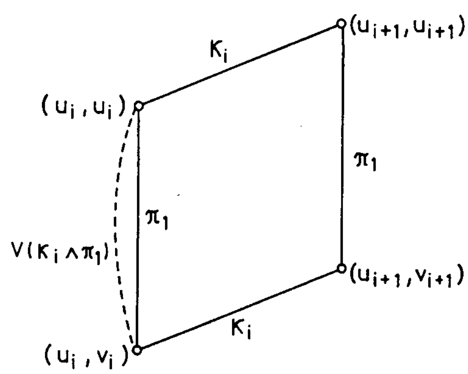
By induction we show that

$$(u_i, u_i) \bigvee (\varkappa_i \wedge \pi_1)\,(u_i, v_i).$$

Indeed this is trivial for $i = 0$. In passing from i to $i + 1$ we note that

$$(u_i, u_i)\,\varkappa_i\,(u_{i+1}, u_{i+1})$$

and, using the induction hypothesis we have the situation:



Thus the induction step is achieved with the Shifting lemma. Setting now $i = n$ the theorem is proved.

2.7. Corollary ([6]). $[\alpha, \bigvee \beta_i] = \bigvee [\alpha, \beta_i]$.

Proof. $\leqq$ is clear since $\beta_i \leqq \bigvee \beta_i$. Trivially $\Delta_\alpha^{\vee \beta_i} = \bigvee \Delta_\alpha^{\beta_i}$. Hence supposing $(x, y) \in [\alpha, \bigvee \beta_i]$, i.e. $(x, x) \bigvee \Delta_\alpha^{\beta_i}(x, y)$ we conclude with 2.6 the relation $(x, x) \bigvee (\Delta_\alpha^{\beta_i} \wedge \pi_1)\,(x, y)$ which clearly means $(x, y) \in \bigvee [\alpha, \beta_i]$. A second application of 2.6 yields a result of R. Freese and R. McKenzie:

2.8. Theorem ([1]). *Let* $\varphi\colon \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ *be an onto homomorphism and* α, β *congruences on* $\mathfrak{B}$. *Then* $\overleftarrow{\varphi}[\alpha, \beta] = [\overleftarrow{\varphi}\alpha, \overleftarrow{\varphi}\beta] \vee \ker \varphi$.

Proof. Using 2.4 we get that $(x, y) \in [\overleftarrow{\varphi}\alpha, \overleftarrow{\varphi}\beta]$ implies

$$(\varphi(x), \varphi(y)) \in [\overrightarrow{\varphi}\overleftarrow{\varphi}\alpha, \overrightarrow{\varphi}\overleftarrow{\varphi}\beta] = [\alpha, \beta]$$

because φ is onto. For the reverse inclusion suppose $(a, b) \in \overleftarrow{\varphi}[\alpha, \beta]$, i.e. $(x, y) \in [\alpha, \beta]$ with $x = \varphi(a)$ and $y = \varphi(b)$. The last relation can be written down as in 2.3. Since φ is onto there exist $(\bar{s}_i, \bar{t}_i) \in \overleftarrow{\varphi}\beta$ with $\varphi(\bar{s}_i) = s_i$ and $\varphi(\bar{t}_i) = t_i$ and there are similarly algebraic functions $\bar{\tau}_i$ on $\overleftarrow{\varphi}\alpha$ which arise from the given τ_i by substituting any constant (i.e. an element of α) by an arbitrary preimage under $\varphi \times \varphi$ (i.e. an element of $\overleftarrow{\varphi}\alpha$). Since $\varphi \times \varphi$ is a homomorphism we obtain:

$$\begin{aligned}
&\bar{\tau}_0(\bar{s}_0, \bar{s}_0) \ker \varphi \times \varphi\, (a, a)\,,\\
&\bar{\tau}_i(\bar{t}_i, \bar{t}_i) \ \ \ker \varphi \times \varphi\, \bar{\tau}_{i+1}(\bar{s}_{i+1}, \bar{s}_{i+1}) \quad \text{for} \quad 0 \leqq i < n\,,\\
&\bar{\tau}_n(\bar{t}_n, \bar{t}_n) \ker \varphi \times \varphi\, (a, b)\,.
\end{aligned}$$

Hence $(a, a)\, \Delta^{\overleftarrow{\varphi}\beta}_{\overleftarrow{\varphi}\alpha} \vee \ker \varphi \times \varphi\, (a, b)$. Application of 2.6 yields

$$(a, a)\, (\Delta^{\overleftarrow{\varphi}\beta}_{\overleftarrow{\varphi}\alpha} \wedge \pi_1) \vee (\ker \varphi \times \varphi \wedge \pi_1)\, (a, b)$$

which immediately gives the missing inclusion.

Another important property of the commutator is commutativity. To prove it we use for the first time the Cube lemma.

2.9. Theorem ([6]). $[\alpha, \beta] = [\beta, \alpha]$.

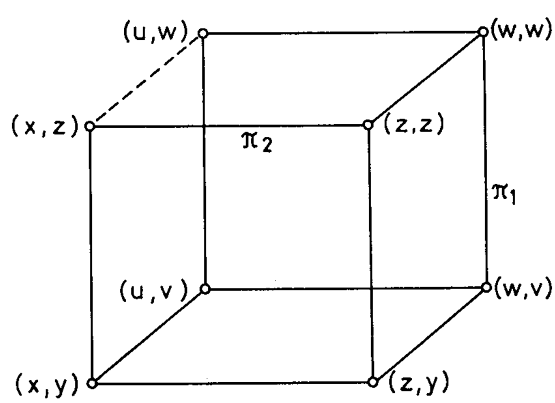
Proof. Our proof uses the Cube lemma to imitate Smith's proof in the permutable case. We define

$$\overline{\Delta}^{\beta}_{\alpha} := \{((x, y), (u, v)) \mid (x, u)\, \Delta^{\beta}_{\alpha}\, (y, v)\}\,.$$

Clearly $(x, y) \in [\alpha, \beta]$ implies $(x, x)\, \overline{\Delta}^{\beta}_{\alpha}\, (x, y)$, hence we are done if we can show that $\overline{\Delta}^{\beta}_{\alpha} = \Delta^{\alpha}_{\beta}$.

Obviously $\overline{\Delta}^{\beta}_{\alpha}$ is a binary relation on β containing $((x, x), (y, y))$ whenever $(x, y) \in \alpha$. Reflexivity and Symmetry are precisely properties 2.1 (iii) and 2.1 (ii).

For transitivity suppose $(x, u)\, \overline{\Delta}^{\beta}_{\alpha}\, (y, v)\, \overline{\Delta}^{\beta}_{\alpha}\, (z, w)$ which means $(x, y)\, \Delta^{\beta}_{\alpha}\, (u, v)$ and $(y, z)\, \Delta^{\beta}_{\alpha}\, (v, w)$. Further the relations $(z, y)\, \Delta^{\beta}_{\alpha}\, (w, v)$ and $(z, z)\, \Delta^{\beta}_{\alpha}\, (w, w)$ come from 2.1. We thus have the following situation



The Cube lemma thus yields $(x, z)\Delta_\alpha^\beta(u, w)$ i.e. $(x, u)\overline{\Delta}_\alpha^\beta(z, w)$. Compatibility of $\overline{\Delta}_\alpha^\beta$ is trivially seen, hence $\overline{\Delta}_\alpha^\beta$ is a congruence relation on β, containing $((x, x), (y, y))$ whenever $x \beta y$.

Hence $\overline{\Delta}_\alpha^\beta \geqq \Delta_\beta^\alpha$. We conclude $\Delta_\beta^\alpha = \overline{\overline{\Delta}}_\beta^\alpha \geqq \overline{\Delta}_\alpha^\beta \geqq \Delta_\beta^\alpha$ and therefore $\Delta_\beta^\alpha = \overline{\Delta}_\alpha^\beta$.

We note as a corollary another result from [6]:

2.10. Corollary ([6]). *Suppose $\varphi\colon \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is an onto homomorphism and α, β, γ are congruences on $\mathfrak{B}$. If there are congruences σ and τ on $\mathfrak{A}$ with $\sigma \wedge \tau \leqq \overleftarrow{\varphi}\gamma$, $\sigma \vee \overleftarrow{\varphi}\gamma \geqq \overleftarrow{\varphi}\alpha$, $\tau \vee \overleftarrow{\varphi}\gamma \geqq \overleftarrow{\varphi}\beta$ then $[\alpha, \beta] \leqq \gamma$.*

Proof. $[\sigma \vee \varphi\gamma, \tau \vee \overleftarrow{\varphi}\gamma] \leqq [\sigma, \tau] \vee \overleftarrow{\varphi}\gamma \leqq \overleftarrow{\varphi}\gamma$ applying 2.7, 2.9 and 2.2. Hence $[\overleftarrow{\varphi}\alpha, \overleftarrow{\varphi}\beta] \leqq \overleftarrow{\varphi}\gamma$. From 2.8 we get $\overleftarrow{\varphi}[\alpha, \beta] = [\overleftarrow{\varphi}\alpha, \overleftarrow{\varphi}\beta] \vee \ker \varphi \leqq \overleftarrow{\varphi}\gamma$, and after applying $\overrightarrow{\varphi}$ the result follows.

2.11. Corollary. *For congruences α, β, γ of the algebra $\mathfrak{A}$ we have $[\alpha, \beta] \leqq \gamma$ if and only if $[\overrightarrow{\varphi}\alpha, \overrightarrow{\varphi}\beta] = 0$ where φ is the canonical homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}/\gamma$.*

Proof. $[\overrightarrow{\varphi}\alpha, \overrightarrow{\varphi}\beta] = 0$ implies with 2.4 that $\overrightarrow{\varphi}[\alpha, \beta] = 0$ which is equivalent to $[\alpha, \beta] \leqq \ker \varphi \leqq \gamma$. On the other hand

$$\overleftarrow{\varphi}[\overrightarrow{\varphi}\alpha, \overrightarrow{\varphi}\beta] = [\overleftarrow{\varphi}\overrightarrow{\varphi}\alpha, \overleftarrow{\varphi}\overrightarrow{\varphi}\beta] \vee \ker \varphi = [\alpha \vee \ker \varphi, \beta \vee \ker \varphi] \vee \ker \varphi$$
$$\leqq [\alpha, \beta] \vee \ker \varphi.$$

Assuming $[\alpha, \beta] \leqq \gamma = \ker \varphi$ we get that $[\overrightarrow{\varphi}\alpha, \overrightarrow{\varphi}\beta] = 0$.

With 2.11 in mind we may be interested in

2.12. A syntactical description of $[\alpha, \beta] = 0$. *The statement "$[\alpha, \beta] = 0$" is equivalent to "for all term functions $p(x_1, \ldots, x_n)$ on $\mathfrak{A}$ and $(a_2, b_2), \ldots, (a_n, b_n) \in \alpha$ and $(x, y) \in \beta$ we have*

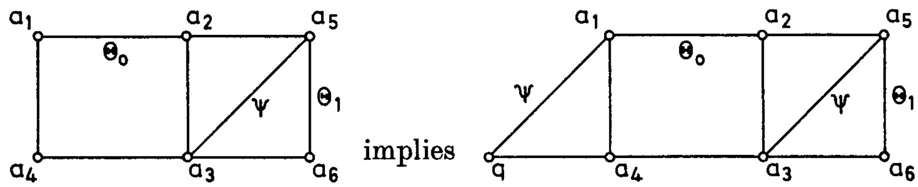
$$p(x, a_2, \ldots, a_n) = p(x, b_2, \ldots, b_n)$$

implies

$$p(y, a_2, \ldots, a_n) = p(y, b_2, \ldots, b_n)\text{''}.$$

Proof. According to Mal'cev [7], the right hand side states precisely, that $\delta_x^\beta := \{(y, y) \mid x \beta y\}$ is a class of some congruence on the algebra α for any arbitrary $x \in \mathfrak{A}$. If so it is certainly a class of Δ_α^β, hence $[\alpha, \beta] = 0$. On the other hand, if it were not a class of any congruence we might assume $(x, x)\Delta_\alpha^\beta(u, v)$, yielding $(u, u)\,\Delta_\alpha^\beta(u, v)$ with 2.6 and hence $(u, v) \in [\alpha, \beta]$.

3. Groups connected with the commutator. We recall from [3] that in every modular variety V there exists a 6-ary polynomial $q(x_1, \ldots, x_6)$ with the following property: Let Θ_0, Θ_1, Ψ be congruences on the algebra $\mathfrak{A}$ with $\Theta_0 \wedge \Theta_1 \leqq \Psi$. Let $a_1, \ldots, a_6$ be elements of $\mathfrak{A}$. Then



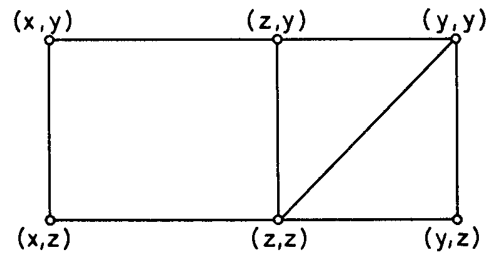
with $q := q(a_1, \ldots, a_6)$.

We define a new polynomial $p(x, y, z)$ by

$$p(x, y, z) := q(x, z, z, x, y, y).$$

Then $p(x, y, y) \equiv x$ is an equation valid in V (see [3]). Suppose now we have congruences α, β with $\alpha \geqq \beta$ and elements x, y, z with $x \, \alpha \, y \, \beta \, z$.

Then clearly $(y, y) \Delta_\alpha^\beta (z, z)$ and with $\psi := \Delta_\alpha^\beta$ we may apply the polynomial $q(x_1, \ldots, x_6)$ to the situation



and consequently find that

$$(\S) \qquad (x, y) \Delta_\alpha^\beta (p(x, y, z), z).$$

Setting $x = y$ we obtain

$$p(x, x, z) [\alpha, \beta] z.$$

Moreover since Δ_α^β is a congruence we find for $\vec{x} := (x_1, \ldots, x_n)$, $\vec{y} := (y_1, \ldots, y_n)$, $\vec{z} := (z_1, \ldots, z_n)$ with $x_i \alpha y_i \beta z_i$ and any n-ary operation f

$$(f(\vec{x}), f(\vec{y})) \Delta_\alpha^\beta (f(p(x_1, y_1, z_1), \ldots, p(x_n, y_n, z_n)), f(\vec{z})).$$

Using (§) again we get

$$(f(\vec{x}), f(\vec{y})) \Delta_\alpha^\beta (p(f(\vec{x}), f(\vec{y}), f(\vec{z})), f(\vec{z}))$$

and hence

$$(\S\S) \qquad f(p(x_1, y_1, z_1), \ldots, p(x_n, y_n, z_n)) [\alpha, \beta] p(f(\vec{x}), f(\vec{y}), f(\vec{z})).$$

This yields one direction of

3.1. An equational description of $[\alpha, \beta] = 0$ with $\alpha \geqq \beta$. *Suppose $\alpha \geqq \beta$ then $[\alpha, \beta] = 0$ if and only if for all $x_i \alpha y_i \beta z_i$ with $x_i, y_i, z_i \in \mathfrak{A}$ the equations $p(y_i, y_i, z_i) = z_i$ and*

$$f(p(x_1, y_1, z_1), \ldots, p(x_n, y_n, z_n)) = p(f(\vec{x}), f(\vec{y}), f(\vec{z}))$$

hold.

For the proof of the missing direction we define a congruence θ on α by

$$(x, y)\,\theta\,(u, z) :\Leftrightarrow x\alpha y\beta z \quad \text{and} \quad p(x, y, z) = u.$$

To show symmetry we suppose $p(x, y, z) = u$ and $x\alpha y\beta z$ and compute

$$\begin{aligned} p(u, z, y) &= p(p(x, y, z), p(y, y, z), p(y, y, y)) \\ &= p(p(x, y, y), p(y, y, y), p(z, z, y)) = p(x, y, y) = x. \end{aligned}$$

For transitivity suppose $x\alpha y\beta z$, $u\alpha z\beta s$, $p(x, y, z) = u$, $p(u, z, s) = r$ and compute:

$$\begin{aligned} p(x, y, s) &= p(p(x, y, y), p(y, y, y), p(z, z, s)) \\ &= p(p(x, y, z), p(y, y, z), p(y, y, s)) = p(u, z, s) = r. \end{aligned}$$

Using that $p(x, x, y) = y$ for $x\beta y$ we find $(x, x)\,\theta\,(y, y)$ for $x\beta y$ and hence $\theta \geqq \Delta_\alpha^\beta$.

Hence suppose $x[\alpha, \beta]y$, then $(x, x)\,\Delta_\alpha^\beta\,(x, y)$, therefore $(x, x)\,\theta\,(x, y)$ hence $p(x, x, y) = x$ which implies $x = y$. Thus $[\alpha, \beta] = 0$.

We pause for a simple application.

3.2. Definition (Nilpotency, Solvability). Starting with 1, the universal congruence on $\mathfrak{A}$ we define:

$$1^1 = 1, \qquad 1^{n+1} := [1^n, 1] \quad \text{and} \quad 1^{(1)} := 1, \qquad 1^{(n+1)} := [1^{(n)}, 1^{(n)}].$$

We say that $\mathfrak{A}$ is *nilpotent* (*solvable*) *of degree* $\leqq k$, *if and only if* $1^k = 0$ ($1^{(k)} = 0$). $\mathfrak{A}$ is called *nilpotent* (*solvable*) if for some natural number k $\mathfrak{A}$ is nilpotent (solvable) of degree $\leqq k$.

It is a simple exercise to convince oneself that within a given modular variety the class of algebras nilpotent (resp. solvable) of degree $\leqq k$ forms a subvariety V^k (resp. $V^{(k)}$).

Now 3.1 allows us to give an equational description of the varieties V^k and $V^{(k)}$. Namely, define sets of equations:

$$\mathscr{A}_1 := \mathscr{N}_1 := \{x = y\}.$$

$$\begin{aligned} \mathscr{A}_{i+1} := \{&f(p(\gamma_1, \sigma_1, \tau_1), \ldots, p(\gamma_n, \sigma_n, \tau_n)) \equiv p(f(\gamma_1, \ldots, \gamma_n), \\ &f(\sigma_1, \ldots, \sigma_n), f(\tau_1, \ldots, \tau_n)) \mid f \text{ is } n\text{-ary operation and} \\ &\mathscr{A}_i \vdash \gamma_k \equiv \sigma_k \equiv \tau_k \text{ for } 0 < k \leqq n\} \cup \\ &\{p(\sigma, \sigma, \tau) \equiv \tau \mid \mathscr{A}_i \vdash \sigma = \tau\}. \end{aligned}$$

$$\begin{aligned} \mathscr{N}_{i+1} := \{&f(p(x_1, \sigma_1, \tau_1), \ldots, p(x_n, \sigma_n, \tau_n)) \equiv p(f(x_1, \ldots, x_n), \\ &f(\sigma_1, \ldots, \tau_n), f(\sigma_1, \ldots, \tau_n)) \mid f \text{ is } n\text{-ary operation and} \\ &\mathscr{N}_i \vdash \sigma_k \equiv \tau_k \text{ for } 0 \leqq k \leqq n\} \cup \\ &\{p(\sigma, \sigma, \tau) \equiv \tau \mid \mathscr{N}_i \vdash \sigma \equiv \tau\}. \end{aligned}$$

Then the equations $\mathscr{A}_k$ (resp. $\mathscr{N}_k$) together with the equations of V describe the algebras in V which are solvable (resp. nilpotent) of degree $\leqq k$.

After this interlude we keep on supposing $[\alpha, \beta] = 0$ with $\alpha \geqq \beta$. Then from 3.1 we conclude that on every class of β $p(x, y, z)$ is a Mal'cev polynomial, commuting with itself. Hence on every class of β an abelian group $\mathcal{G}$ is defined (use 4.7(iii) of [4]) with $p(x, y, z) = x - y + z$. In fact, β is an affine congruence, see [5]. For $a \in \mathfrak{A}$ let us denote the algebra $([a]\beta, x - y + z)$ by $\mathcal{G}[a]$. Then we claim:

3.3. *For $(u, v) \in \alpha$, $\mathcal{G}[u]$ and $\mathcal{G}[v]$ are isomorphic.*

Proof. Define a map $\Psi_{u,v}$ by $\Psi_{u,v}(x) := p(v, u, x)$. Clearly $\Psi_{u,v}$ is a map from $\mathcal{G}[u]$ to $\mathcal{G}[v]$. Next we claim that $\Psi_{u,v} \circ \Psi_{v,u} = \mathrm{id}$ for $(u, v) \in \alpha$. Namely for $x \beta v$ we have $(v, u) \Delta_\alpha^\beta (x, p(u, v, x))$ since $(u, p(u, v, x)) \in \beta$ and hence

$$(v, u) \Delta_\alpha^\beta (p(v, u, p(u, v, x)), p(u, v, x)).$$

Thus we find that $p(v, u, p(u, v, x)) [\alpha, \beta] x$. Thus the mappings $\Psi_{u,v}$ are bijective. Moreover

$$\begin{aligned} \Psi_{u,v}(x - y + z) &= \Psi_{u,v}(p(x, y, z)) = p(v, u, p(x, y, z)) \\ &= p(p(v, v, v), p(u, u, u), p(x, y, z)) \\ &= p(p(v, u, x), p(v, u, y), p(v, u, z)) \\ &= \Psi_{u,v}(x) - \Psi_{u,v}(y) + \Psi_{u,v}(z). \end{aligned}$$

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