COALGEBRAS OF BOUNDED TYPE

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Abstract. Using results of Trnková, we first show that subcoalgebras are always closed under finite intersections. Assuming that the type functor F is bounded, we obtain a concrete representation of the terminal F-coalgebra. Several equivalent characterizations of boundedness are provided.

1. Introduction

Universal coalgebra as a theory of state based systems has seen a rapid development within recent years. For a long time the article [Rut96] by J. Rutten has been the definite reference. Several of the results there have needed special conditions on the type functor, most prominently, preservation of weak pullbacks.

Recently, more and more results have been proved that freed the general theory from particular assumptions on the type functor. The existence of terminal coalgebras could be proven under the rather weak hypothesis that the type functor is bounded. It was only through some fruitful discussions with Jiři Adámek that led us to realize the connection to a class of functors which has been thoroughly investigated in the sixties by a group of mathematicians in Prague. As a result, it turns out in our main theorem that a functor F is bounded if and only if there is a surjective natural transformation η from a functor $C \times (-)^M$ to F.

To our surprise, we came across a result of Vera Trnková, another member of the Prague group, which shows that every nontrivial $\mathcal{S}et$-endofunctor preserves nonempty pullbacks of monos. As a consequence, the collection of subcoalgebras of a fixed coalgebra is closed under finite intersections. Closure under arbitrary unions has been well known, so it turns out that the subcoalgebras of an arbitrary coalgebra form a topology.

This, and the main result of this paper show again, that the theory of coalgebras can not be obtained by straightforwardly dualizing the theory of universal algebra. One reason for this is that certain properties of the category $\mathcal{S}et$ play an important role, and these properties are not valid in $\mathcal{S}et^{op}$, the dual of $\mathcal{S}et$.

2. Preliminaries

2.1. **Elementary notions.** A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ associates with every object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$ and with every $\mathcal{C}$-morphism $f : A \to B$ a $\mathcal{D}$-morphism $F(f) : F(A) \to F(B)$ so that $F(id_A) = id_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$ whenever $f : A \to B$ and $g : B \to C$.

A typical example is the powerset functor $\mathcal{P} : \mathcal{S}et \to \mathcal{S}et$ which associates to every set X its powerset $\mathcal{P}(X) = \{U \mid U \subseteq X\}$ and with every map $f : X \to Y$ the map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ given by $\mathcal{P}(f)(U) := f[U] := \{f(u) \mid u \in U\}$.

We shall almost exclusively restrict ourselves to set-endofunctors, that is to functors $F : \mathcal{S}et \to \mathcal{S}et$ where $\mathcal{S}et$ denotes the category of sets. In the following we shall always assume that F is *nontrivial*, in that $F(X) \neq \emptyset$ whenever $X \neq \emptyset$.

If $X \neq \emptyset$ and $f : X \to Y$ is an injective map, then f has a left inverse, that is there exists a map $g : Y \to X$ with $g \circ f = id_X$. Hence $F(g) \circ F(f) = id_{F(X)}$, in particular, $F(f)$ is again injective. Similarly, if $f : X \to Y$ is surjective, then the axiom of choice yields a right inverse $g : Y \to X$ with $f \circ g = id_Y$. Consequently, any set-endofunctor preserves surjectivity, and injectivity of maps with nonempty domain.

A *natural transformation* η between set-endofunctors G and F consists of a map $\eta_X : G(X) \to F(X)$ for every set X, so that for every map $f : X \to Y$ the obvious diagram commutes:

$$\begin{array}{ccc} X & & G(X) \xrightarrow{\eta_X} F(X) \\ \downarrow f & & \downarrow G(f) \qquad \downarrow F(f) \\ Y & & G(Y) \xrightarrow{\eta_Y} F(Y) \end{array}$$

A natural transformation is called *surjective*, if η_X is a surjective map whenever $X \neq \emptyset$.

2.2. **Coalgebras.** Any nontrivial functor $F : \mathcal{S}et \to \mathcal{S}et$ will be called a *type functor*. A coalgebra $\mathcal{A} = (A, \alpha_A)$ of type F (also called F-coalgebra) consists of a set A and a map $\alpha_A : A \to F(A)$. Here A is called the *carrier set* and α_A is called the *structure map*.

By convention, we denote a coalgebra and its carrier set with the same letter and use different fonts for distinction. We also shall often drop the index to the structure map α when it is clear from the context.

2.2.1. *Homomorphisms.* A map $\varphi : A \to B$ between coalgebras $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$ is a *homomorphism*, if $\alpha_B \circ \varphi = F(\varphi) \circ \alpha_A$, that is if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ F(A) \arrow[r, "F(\varphi)"] & F(B) \end{tikzcd}$$

It is easy to check that the identity map id_A is always a homomorphism and that the composition $\psi \circ \varphi$ of two homomorphisms is a homomorphism again. Thus, F-coalgebras with their homomorphisms form a category which we shall denote by $\mathcal{S}et_F$.

An *isomorphism* is a homomorphism with a both sided inverse. It is well known [Rut96] that any bijective homomorphism is already an isomorphism. We call $\mathcal{A}$ and $\mathcal{B}$ *isomorphic*, and we write $\mathcal{A} \cong \mathcal{B}$, if there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

2.2.2. *Subcoalgebras.* If $\mathcal{A}$ is a coalgebra and $U \subseteq A$, we shall say that U is a *subcoalgebra* of $\mathcal{A}$, provided that U can be equipped with a coalgebra structure $\mathcal{U} = (U, \alpha_U)$ so that the set-theoretical inclusion map $\subseteq_U^A : U \hookrightarrow A$ is a homomorphism from $\mathcal{U}$ to $\mathcal{A}$.

The empty set $\emptyset$ is always a subcoalgebra, and if $\emptyset \neq U \subseteq A$, then $F(\subseteq_U^A)$ is injective, which entails that there can be at most one structure map $\alpha_U : U \to A$ making $\subseteq_U^A$ a homomorphism. Consequently, subcoalgebras are uniquely determined by their carrier sets.

2.2.3. *Colimits.* The category $\mathcal{S}et_F$ has arbitrary colimits, in fact, the forgetful functor from $\mathcal{S}et_F$ to $\mathcal{S}et$ creates colimits, see [Rut96]. Here we shall make use of the fact that every coalgebra $\mathcal{A}$ has a smallest homomorphic image. It can be obtained by forming the colimit (pushout) of all homomorphism with source $\mathcal{A}$. Alternatively, one may define a *congruence* to be the kernel of a homomorphism, then the smallest homomorphic image of $\mathcal{A}$ is obtained by factoring $\mathcal{A}$ through its largest congruence, see [GS99].

Sums are particular examples of colimits. If $(\mathcal{A}_i)_{i\in I}$ is a family of coalgebras, then the sum $\Sigma_{i\in I}\mathcal{A}_i$ in $\mathcal{S}et_F$ has as underlying set the disjoint union $\biguplus_{i\in I} A_i$ of the A_i and as canonical homomorphisms $e_i : \mathcal{A}_i \to \Sigma_{i\in I}\mathcal{A}_i$ the set theoretical inclusion maps from A_i into $\biguplus_{i\in I} A_i$.

2.3. **Two examples.** In computer science, coalgebras model state based systems. The carrier set represents the state space and the structure map often encodes several components - some describing *transitions* to successor states and some representing various types of *observations.*

2.3.1. *Automata.* For fixed sets C and M we consider the functor $F(X) = C \times X^M$. A map $f : X \to Y$ is assigned a map $F(f) : C \times X^M \to C \times Y^M$ by means of $F(f)(c, \tau) := (c, f \circ \tau)$.

An F-coalgebra $\mathcal{A}$ in this case consists of a set A and a map $\alpha : A \to C \times A^M$. Obviously, α can equivalently be written as a combination of two components which we shall denote by γ and δ:

$$\begin{aligned} \gamma &: A \to C \\ \delta &: A \times M \to A \end{aligned}$$

Thus, an F-coalgebra is just an automaton, where δ is the transition map, computing for every *state* $a \in A$ and every *input* $m \in M$ a new state $\delta(a, m)$, whereas γ calculates an output $\gamma(a)$. Indeed, a coalgebra homomorphism φ between two F-coalgebras $\mathcal{A}$ and $\mathcal{B}$ translates into the familiar notion of automata homomorphism:

$$\begin{aligned} \varphi(\gamma_A(a)) &= \gamma_B(\varphi(a)) \\ \varphi(\delta_A(a, m)) &= \delta_B(\varphi(a), m). \end{aligned}$$

One easily checks that a subset $U \subseteq A$ is a subcoalgebra iff it is a subautomaton, that is if

$$\forall u \in U. \forall m \in M. \delta(u, m) \in U.$$

2.3.2. *Nondeterministic automata.* In computer science, concurrency is often modelled by nondeterminism. In a nondeterministic automaton, the transition map is given the functionality

$$\delta : A \times M \to \mathcal{P}(A)$$

with $\mathcal{P}$ the powerset functor. Consequently, nondeterministic automata with input set M and output set C can be modelled as F-coalgebras where $F(X) = C \times \mathcal{P}(X)^M$.

2.4. **Terminal and cofree coalgebras.** Free algebras play a prominent role in universal algebra. In particular, the free algebra over the empty set of generators is *initial* in the category of algebras, in that it has precisely one homomorphism to each other algebra of the same type.

Dually, the *terminal* coalgebra, if it exists, incorporates all possible "behaviours" of coalgebras of the fixed type, see [Gum99]. For this reason, much effort has been devoted into showing the existence of terminal coalgebras for various functors.

Definition 2.1. *An F-coalgebra $\mathcal{Q}$ is called* terminal *(or* final*), if for every F-coalgebra $\mathcal{A}$ there is a unique homomorphism $\varphi : \mathcal{A} \to \mathcal{Q}$. Without the "uniqueness" condition, $\mathcal{Q}$ is called* weakly terminal.

From any weakly terminal coalgebra $\mathcal{W}$ the (unique) terminal coalgebra $\mathcal{Q}$ can be constructed by factoring $\mathcal{W}$ through its largest congruence.

2.4.1. *The terminal automaton.* We shall now construct the terminal coalgebra for the functor $F(X) = C \times X^M$. By M^* we denote the set of all finite words formed with elements from M. ε is the empty word and for $m \in M$ and $w \in M^*$ we let $m \cdot w$ denote the word with first letter m and rest w.

Now, the state set of the terminal automaton will be $Q := C^{M^*}$, the set of all mappings from M^* to C. Its transition structure is

$$\gamma_Q(\tau) := \tau(\varepsilon), \text{ and} \tag{1}$$

$$\delta_Q(\tau, m)(w) := \tau(m \cdot w). \tag{2}$$

Given any automaton $\mathcal{A}$, one checks that the unique homomorphism to $\mathcal{Q}$ is given by $\varphi : A \to C^{M^*}$ where $\varphi(a) : M^* \to C$ is recursively defined on M^* by

$$\begin{aligned} \varphi(a)(\varepsilon) &= \gamma_A(a), \text{ and} \\ \varphi(a)(m \cdot w) &= \varphi(\delta_A(a, m))(w). \end{aligned}$$

A good way to visualize an element $\tau \in Q$ may be as an infinite trees with nodes labeled by elements of C. Each node has $|M|$ sons and every word $w \in M^*$ denotes a path from the root to a particular node, or the "address" of that node. Thus $\gamma_Q(\tau)$ is the label of the root and $\delta^*(\tau, w)$ is the node with address w in the tree τ.

2.4.2. *Existence of terminal coalgebras.* In general, the existence of terminal coalgebras depends on the type functor F. In particular, it was observed by Lambek (see [Rut96] for a proof) that the structure map $\alpha_Q : Q \to F(Q)$ on any terminal F-coalgebra must be a bijection. It follows that terminal coalgebras cannot exist in $\mathcal{S}et_{\mathcal{P}}$. By the same token, there is no terminal nondeterministic automaton. We shall later consider a class of functors for which the existence of terminal coalgebras can be guaranteed.

2.4.3. *Cofree coalgebras.* Just as initial algebras are special instances of free algebras, the terminal coalgebra, if it exists, is the same as the cofree coalgebra over a one-element set. The general definition is:

Definition 2.2. *Let X be a set, $\mathcal{S}_X$ an F-coalgebra and $\epsilon_X : S_X \to X$ a map. $\mathcal{S}_X$ with ϵ_X is called* cofree *over X, if for any F-coalgebra $\mathcal{A}$ and any map $\varphi : A \to X$ there is precisely one F-homomorphism $\tilde{\varphi} : \mathcal{A} \to \mathcal{S}_X$ with $\epsilon_X \circ \tilde{\varphi} = \varphi$.*

$$\begin{tikzcd} A \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & S_X \arrow[d, "\epsilon_X"] \\ & X \end{tikzcd}$$

Cofree F-coalgebras over X can be seen as terminal coalgebras for the functor $X \times F(-)$ with constant set X:

If $(S_X, \alpha_X : S_X \to F(S_X))$ with $\epsilon_X : S_X \to X$ is a cofree F-coalgebra over X, then $(S_X, (\epsilon_X, \alpha_X) : S_X \to X \times F(S_X))$ is a terminal $X \times F(-)$-coalgebra.

On the other hand, if $(Q, \alpha_Q : Q \to X \times F(Q))$ is a terminal $X \times F(-)$-coalgebra, we obtain a cofree F-coalgebra on X by defining $S_X := Q$, $\alpha_X := \pi_2 \circ \alpha_Q$ and $\epsilon_X := \pi_1 \circ \alpha_Q$ where $\pi_1 : X \times F(Q) \to X$ and $\pi_2 : X \times F(Q) \to F(Q)$ are the projections.

2.4.4. *Cofree automata.* According to the above, the cofree automaton over X with output set C and input set M can be constructed as the set of all infinite trees τ where each node has $|M|$ successors, and carries both a "label" from C and a "color" from X.

$\epsilon_X(\tau) \in X$ is the color and $\gamma(\tau) \in C$ is the label of the root. $\delta(\tau, m)$ is the subtree of τ, whose root is the m-th son of τ.

We leave it to the reader to check that that this defines the automaton with input M and output C which is cofree over X.

2.5. **The functor arising from a natural transformation.** Suppose that $\eta : G \to F$ is a natural transformation between set-endofunctors G and F. Then η gives rise to a functor $H_\eta : \mathcal{S}et_G \to \mathcal{S}et_F$ which is defined on objects as

$$H_\eta(A, \alpha_A) := (A, \eta_A \circ \alpha_A)$$

and which is the identity on morphisms.

Lemma 2.3. *Let $\eta : G \to F$ be a natural transformation between nontrivial set-endofunctors G and F.*

(i) *Every subcoalgebra U of $\mathcal{A} \in \mathcal{S}et_G$ is an F-subcoalgebra of $H_\eta(\mathcal{A})$.*

If additionally η is surjective, then we have:

(ii) *H_η is surjective on objects, i.e. every coalgebra in $\mathcal{S}et_F$ is of the form $H_\eta(\mathcal{A})$ for some $\mathcal{A}$ in $\mathcal{S}et_G$.*

(iii) *If $\mathcal{Q}$ is weakly terminal in $\mathcal{S}et_G$ then $H_\eta(\mathcal{Q})$ is weakly terminal in $\mathcal{S}et_F$.*

(iv) *If $\mathcal{S}et_G$ has a terminal coalgebra $\mathcal{Q}$ then $\mathcal{S}et_F$ has a terminal coalgebra $\mathcal{Q}'$.*

Proof. (i) : If $\mathcal{U} = (U, \alpha_U)$ and $\mathcal{U} \leq \mathcal{A}$ then $\subseteq_U^A : \mathcal{U} \to \mathcal{A}$ is a G-homomorphism, so $H_\eta(\subseteq_U^A) : H_\eta(\mathcal{U}) \to H_\eta(\mathcal{A})$ is an F-homomorphism. But H_η is the identity on maps and on the carrier sets.
(ii) : Given $\mathcal{A} = (A, \alpha) \in \mathcal{S}et_F$, the case $A = \emptyset$ is trivial. Otherwise, let $(\eta_A)^- : F(A) \to G(A)$ be a right inverse of $\eta_A : G(A) \to F(A)$ and let $\mathcal{A}' := (A, (\eta_A)^- \circ \alpha)$. Then $H_\eta(\mathcal{A}') = \mathcal{A}$.
(iii) : Given an F-coalgebra $\mathcal{A}$, construct $\mathcal{A}'$ as before, then there exists a G-homomorphism $\varphi : \mathcal{A}' \to \mathcal{Q}$. Consequently, $\varphi = H_\eta(\varphi)$ is also an F-homomorphism from $\mathcal{A} = H_\eta(\mathcal{A}')$ to $H_\eta(\mathcal{Q})$.
(iv) : $\mathcal{Q}'$ is the smallest factor of $\mathcal{H}_\eta(\mathcal{Q})$. It is obtained by factoring with the largest congruence relation. □

3. Intersections of Subcoalgebras

It is well known (for a proof see [Gum99]) that the union of subcoalgebras of a fixed F-coalgebra $\mathcal{C}$ is again a subcoalgebra of $\mathcal{C}$. V. Trnková proves in [Trn69] that every set-endofunctor preserves pullbacks given by nonempty intersections of two sets. From this one may infer that the intersection of two subcoalgebras of a given coalgebra $\mathcal{A}$ is again a subcoalgebra. We shall give a direct proof of the latter result:

Theorem 3.1. *Let F be an arbitrary $\mathcal{S}et$-endofunctor and $\mathcal{A}$ an F-coalgebra. If U and V are subcoalgebras of $\mathcal{A}$ then so is $U \cap V$.*

Proof. $\emptyset$ is always a subcoalgebra, so we may assume $U \cap V \neq \emptyset$. We choose $w_0 \in U \cap V$ and consider the maps $p : U \to U \cap V$ and $q : A \to V$ defined as

$$p(u) := \begin{cases} u, & \text{if } u \in U \cap V; \\ w_0 & \text{otherwise,} \end{cases} \quad \text{and} \quad q(a) := \begin{cases} a, & \text{if } a \in V; \\ w_0 & \text{otherwise.} \end{cases}$$

One observes the equalities

$$q \circ \subseteq^A_V \; = \; id_V \tag{3}$$

and

$$\subseteq^V_{U \cap V} \; \circ \, p \; = \; q \circ \subseteq^A_U . \tag{4}$$

We now define a coalgebra structure β on $U \cap V$ by

$$\beta := F(p) \circ \alpha_U \circ \subseteq^U_{U \cap V} .$$

Here, and in all the following diagrams we denote set inclusions as well as their F-images by hooked arrows:

$$\begin{tikzcd}
& U \arrow[rr, hook] \arrow[dl, dashed, "p"] \arrow[dd, "\alpha_U"] & & A \arrow[dd, "\alpha_A"] \arrow[dl, dashed, "q"] \\
U \cap V \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, dashed, "\beta"'] & & V \arrow[ur, hook] \arrow[dd, "\alpha_V"] & \\
& F(U) \arrow[rr, hook] \arrow[dl, dashed, "F(p)"] & & F(A) \arrow[dl, dashed, "F(q)"] \\
F(U \cap V) \arrow[ur, hook] \arrow[rr, hook] & & F(V) \arrow[ur, hook] &
\end{tikzcd}$$

We now calculate:

$$\begin{aligned}
F(\subseteq^A_{U\cap V}) \circ \beta &= F(\subseteq^A_{U\cap V}) \circ F(p) \circ \alpha_U \circ \subseteq^U_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(\subseteq^V_{U\cap V}) \circ F(p) \circ \alpha_U \circ \subseteq^U_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(q) \circ F(\subseteq^A_U) \circ \alpha_U \circ \subseteq^U_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(q) \circ \alpha_A \circ \subseteq^A_U \circ \subseteq^U_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(q) \circ \alpha_A \circ \subseteq^A_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(q) \circ \alpha_A \circ \subseteq^A_V \circ \subseteq^V_{U\cap V} \\
&= F(\subseteq^A_V) \circ F(q) \circ F(\subseteq^A_V) \circ \alpha_V \circ \subseteq^V_{U\cap V} \\
&= F(\subseteq^A_V) \circ \alpha_V \circ \subseteq^V_{U\cap V} \\
&= \alpha_A \circ \subseteq^A_V \circ \subseteq^V_{U\cap V} \\
&= \alpha_A \circ \subseteq^A_{U\cap V} .
\end{aligned}$$

□

This corollary cannot be improved in general. In fact, in [Gum98] it is shown that the filter functor $\mathcal{F}$ asociating with every set X the set $\mathcal{F}(X)$ of all filters on X preserves finite intersection, but not infinite ones, so as a result, $\mathcal{F}$-subcoalgebras are not closed under arbitrary intersections.

As a consequence of the above, we can carry over to arbitrary type functors a result, which we had previous only been able to prove under additional assumptions in [GS98].

Lemma 3.2. *(i): If $\mathcal{U} \leq \Sigma_{i\in I}\mathcal{A}_i$ is a subcoalgebra of a sum of coalgebras, then $\mathcal{U} \cong \Sigma_{i\in I}(\mathcal{U} \cap \mathcal{A}_i)$.*
(ii): If $\mathcal{U}_i \leq \mathcal{A}_i$ for all $i \in I$, then $\Sigma_{i\in I}\mathcal{U}_i \leq \Sigma_{i\in I}\mathcal{A}_i$.

Proof. (i): Let $\mathcal{U} \leq \Sigma_{i\in I}\mathcal{A}_i$, then by the above, we have the subcoalgebras $U \cap A_i$ of $\mathcal{A}_i$. The universal property of the sum $\Sigma_{i\in I}(\mathcal{U} \cap \mathcal{A}_i)$ gives us a unique homomorphism ρ to $\mathcal{U}$ with $\rho \circ f_i = \subseteq^U_{U\cap A_i}$. As remarked in section 2.2.3, the forgetful functor creates colimits, thus the carrier set of $\Sigma_{i\in I}(\mathcal{U} \cap \mathcal{A}_i)$ is the disjoint union $\biguplus_{i\in I}(U \cap A_i)$ and the map ρ is at the same time the unique set map from $\biguplus_{i\in I}(U \cap A_i)$ to U satisfying $\rho \circ f_i = \subseteq^U_{U\cap A_i}$. In particular, ρ is bijective, so by [Rut96] it is an isomorphism.

$$\begin{tikzcd}
U \arrow[rr, hook] & & \Sigma_{i\in I}A_i \\
& \Sigma_{i\in I}(U \cap A_i) \arrow[ul, "\rho"'] \arrow[ur, hook] & \\
U \cap A_i \arrow[uu, hook] \arrow[ur, "f_i"'] \arrow[rr, hook] & & A_i \arrow[uu, "e_i"']
\end{tikzcd}$$

(ii): The universal map $\rho : \Sigma_{i\in I}\mathcal{U}_i \to \Sigma_{i\in I}\mathcal{A}_i$ is the same as the universal map between $\biguplus_{i\in I} U_i$ and $\biguplus_{i\in I} A_i$, i.e. the natural inclusion. □

This lemma is particularly of interest when the closure of classes of coalgebras under class operators needs to be determined. For a class $\mathcal{K}$ of F-coalgebras let $\mathcal{S}(\mathcal{K})$, resp. $\Sigma(\mathcal{K})$, be the class of all subcoalgebras, resp. sums, of members of $\mathcal{K}$, then we have:

Corollary 3.3. *For any class $\mathcal{K}$ of coalgebras*

$$\mathcal{S}(\Sigma(\mathcal{K})) = \Sigma(\mathcal{S}(\mathcal{K})).$$

4. Bounded, loosely bounded, and small functors

The notion of a *bounded functor* has been defined in the context of coalgebras by J. Rutten. He calls a functor $F : \mathcal{S}et \to \mathcal{S}et$ *bounded*, if there is a cardinal bound κ for the size of one-generated coalgebras. In this case, the one-generated coalgebras form a set of generators, from which the existence of cofree coalgebras can be inferred. Ruttens definition, however, implicitly assumes that F weakly preserves arbitrary intersections of infinitely many sets in order for the notion of one-generated subcoalgebras to make sense.

In [GS99] we have therefore reformulated the definition of boundedness to make it apply to arbitrary $\mathcal{S}et$-endofunctors. In the case where one-generated coalgebras exist, it obviously agrees with Rutten's definition:

Definition 4.1. *A functor $F : \mathcal{S}et \to \mathcal{S}et$ is* bounded *by some set M, if for every F-coalgebra $\mathcal{A}$ and every $a \in A$ there exists a subcoalgebra $\mathcal{U}$ of $\mathcal{A}$ with $a \in U$ and $|U| \leq |M|$.*

In [GS99] we gave an elementary proof, showing that cofree F-coalgebras exist whenever F is bounded. In this section we shall obtain a rather detailed characterization of bounded functors, theorem 4.7 below, from which an even more concrete construction of cofree coalgebras can be derived.

We start with some elementary properties of bounded functors.

Lemma 4.2. *The functor $G(X) := C \times X^M$ is $|M^*|$-bounded.*

Proof. From the description of $C \times X^M$-coalgebras as automata, one checks directly, that for every automaton A and for every $a \in A$ the smallest subautomaton $\langle a \rangle$ containing $a \in A$ is given by

$$\{\delta^*(a, w) \mid w \in M^*\},$$

where $\delta^* : A \times M^* \to A$ is recursively defined as usual:

$$\begin{aligned} \delta^*(a, \varepsilon) &= a \\ \delta^*(a, m \cdot w) &= \delta^*(\delta(a, m), w). \end{aligned}$$

Consequently, $G(X) = C \times X^M$ is bounded by M^*. □

As a direct consequence of lemma 2.3(ii) we also get:

Lemma 4.3. *If G is bounded, and $\eta : G \to F$ is a surjective natural transformation, then F is bounded.*

4.1. **Loosely bounded functors.** F is bounded by M iff every F-coalgebra $\mathcal{A}$ is the union of subcoalgebras of size at most $|M|$. Thus the following simpler condition is a direct consequence of boundedness:

Definition 4.4. *A set-endofunctor F is called* loosely bounded *by some set M, if every nontrivial F-coalgebra contains a nontrivial subcoalgebra of size at most $|M|$.*

Somewhat surprisingly, it will turn out as a consequence of theorem 4.7 that loosely bounded functors must already be bounded. Thus this definition will provide an easier criterion for checking boundedness.

4.2. **Small functors.** The notion of *boundedness*, which concerns any set-endofunctor F, refers to coalgebraic notions. It would be convenient, if it could be equivalently formulated without reference to coalgebras. We shall show that this is indeed possible. We use the notion of *small functor* that has been intensively studied by a group of mathematicians in Prague in the sixties, most prominently by J. Adámek, P. Goralčík, V. Koubek, and V. Trnková. Small functors are just the "accessible" $\mathcal{S}et$-endofunctors [Adá].

Definition 4.5. *Given a set M, a functor $F : \mathcal{S}et \to \mathcal{S}et$ is called M-*small *if $F(X) = \bigcup\{F(\varphi)[F(M)] \mid \varphi : M \to X\}$ for every set $X \neq \emptyset$.*

Thus, the definition requires every element of $F(X)$ to be in the range of $F(\varphi)$ for an appropriate map $\varphi : M \to X$. A prototypical example of a bounded functor is given by the familiar $F(X) = C \times X^M$:

Lemma 4.6. *The functor $F(X) = C \times X^M$ is M-small.*

Proof. Given an arbitrary $(c, \mu) \in C \times X^M$, then $\mu : M \to X$, so we simply choose $(c, id_M) \in C \times M^M$ and obtain

$$F(\mu)(c, id_M) = (c, \mu \circ id_M) = (c, \mu).$$

□

4.3. **The characterization of bounded functors.** The following theorem brings together the notions of *smallness* and *boundedness*. As a consequence, we find that a functor F is already bounded by M, if every F-coalgebra has a nonempty subcoalgebra of size at most $|M|$. The equivalence of *(iii)* and *(iv)* has been remarked in [Trn69].

Theorem 4.7. *Let $F : \mathcal{S}et \to \mathcal{S}et$ be a nontrivial set-endofunctor. Then the following are equivalent:*

- (i) *F is bounded.*
- (ii) *F is loosely bounded.*
- (iii) *F is small.*
- (iv) *There are sets C and M and a surjective natural transformation from $C \times (-)^M$ to F.*

Proof. $(i) \to (ii)$ is trivial.
$(ii) \to (iii)$: Given a set $X \neq \emptyset$ and $t \in F(X)$, we define a coalgebra structure on X by the constant map $\alpha_X : X \to F(X)$ which takes every $x \in X$ to the element $t \in F(X)$. By assumption, F is loosely bounded by some set M, so (X, α_X) has a nonempty subcoalgebra $U \leq X$ with $|U| \leq |M|$. We can therefore choose a surjective map $\psi : M \to U$. It follows that $F(\psi) : F(M) \to F(U)$ is surjective too. Select an arbitrary $u \in U$, then we can find some $s \in F(M)$ with $F(\psi)(s) = \alpha_U(u)$. Now with $\varphi := \subseteq_U^X \circ \psi$ we will have the map $\varphi : M \to A$, as required by the definition of smallness.

$$\begin{tikzcd}
M \arrow[r, two heads, "\psi"'] \arrow[rr, bend left, "\varphi"] & U \arrow[r, hook] \arrow[d, "\alpha_U"] & X \arrow[d, "\alpha_X"] \\
F(M) \arrow[r, two heads, "F(\psi)"] \arrow[rr, bend right, "F(\varphi)"'] & F(U) \arrow[r, hook] & F(X)
\end{tikzcd}$$

We can now calculate:

$$\begin{aligned}
F(\varphi)(s) &= F(\subseteq_U^X) \circ F(\psi)(s) \\
&= F(\subseteq_U^X)(\alpha_U(u)) \\
&= (F(\subseteq_U^X) \circ \alpha_U)(u) \\
&= (\alpha_X \circ \subseteq_U^X)(u) \\
&= \alpha_X(u) \\
&= t.
\end{aligned}$$

$(iii) \to (iv)$: If F is M-small, take $C := F(M)$, and let $G(X) := C \times X^M$. Recall that for a map $f : X \to Y$ we have $G(f)(c, \mu) = (c, f \circ \mu)$. For any set X, we introduce the map $\eta_X : C \times X^M \to F(X)$ defined by

$$\eta_X(c, \mu) := F(\mu)(c).$$

We check that $\eta : G \to F$ is a natural transformation: Let a map $f : X \to Y$ be given and any pair $(c, \mu) \in C \times X^M$, then

$$\begin{aligned}
(F(f) \circ \eta_X)(c, \mu) &= F(f)(F(\mu)(c)) \\
&= (F(f) \circ F(\mu))(c) \\
&= F(f \circ \mu)(c) \\
&= \eta_Y(c, f \circ \mu) \\
&= \eta_Y(G(f)(c, \mu)) \\
&= (\eta_Y \circ G(f))(c, \mu).
\end{aligned}$$

The fact that F is M-small translates immediately into η_X being surjective for every $X \neq \emptyset$.
$(iv) \to (i)$ combines lemmata 4.2 and 4.3. $\square$

From the proof we observe that for the implications $(i) \to (ii) \to (iii) \to (iv)$ we can stay with the same bound $|M|$, whereas in the step $(iv) \to (i)$ we need to pass from $|M|$ to $|M^*|$, as a consequence of lemma 4.2. Consequently we obtain as a corollary:

Corollary 4.8. *If M is an infinite set, then the following are equivalent:*

(i) *F is bounded by M.*
(ii) *F is loosely bounded by M.*
(iii) *F is M-small.*
(iv) *There is a set C and a surjective natural transformation from $C \times (-)^M$ to F.*

As a consequence of theorem 4.7, coalgebras with bounded type functors can be naturally derived from automata. From this, we easily infer the closure of bounded functors under sums, products, and compositions:

Corollary 4.9. *If F and F' are bounded, then so are $F + F'$, $F \times F'$, and $F \circ F'$.*

Proof. Sum, product and composition of $F = C \times (-)^M$ with $F' = C' \times (-)^{M'}$ are obviously bounded. It remains to verify that from surjective natural transformations $\eta : G \to F$ and $\eta' : G' \to F'$ one obtains surjective natural transformations $\eta + \eta' : G + G' \to F + F'$, $\eta \times \eta' : G \times G' \to F \times F'$, and $\eta'' : F \circ F' \to G \circ G'$, where the latter is defined on any set X as

$$\eta''_X := G(\eta'_X) \circ \eta_{F'(X)}.$$

□

5. The existence of terminal and cofree coalgebras

It is well known that terminal coalgebras exist for bounded type functors. Boundedness of F guarantees the existence of a set of generators, so an application of the special adjoint functor theorem (see [Bor94]) would guarantee the existence of the terminal coalgebra. In [GS99] we have given a direct construction.

As a consequence of our main theorem 4.7, however, we even know how the terminal coalgebra arises from the terminal automaton. If $\mathcal{Q}$ is the terminal automaton, as described in subsection 2.4.1, then the terminal F-coalgebra is the smallest factor of $H_\eta(\mathcal{Q})$, where η is the surjective natural transformation from $G(-) = C \times (-)^M$ to F.

Cofree coalgebras can be constructed in a similar manner. We have already remarked how the cofree F-coalgebra over X arises from the terminal coalgebra of type $X \times F(-)$. If F is bounded, then so is $X \times F(-)$, for a surjective natural transformation $\eta : C \times (-)^M \to F(-)$

can be extended to a surjective natural transformation

$$id_X \times \eta : X \times C \times (-)^M \to X \times F(-).$$

5.1. **An application: Terminal $\mathcal{P}_\omega$-coalgebras.** To see a concrete example, consider the finite powerset functor $\mathcal{P}_\omega$ which associates with every set X the set of all finite subsets of X. Given a map $f : X \to Y$ let $\mathcal{P}_\omega(f)(U) := f[U]$ for every finite subset $U \subseteq X$. With $C := \mathcal{P}_\omega(\omega)$ and $M := \omega$, we construct a natural transformation $\eta : C \times (-)^M \to \mathcal{P}_\omega(-)$. by defining for every set X:

$$\eta_X(U, \mu) := \mu[U].$$

One easily checks that η is natural and surjective, hence $\mathcal{P}_\omega$ is bounded by ω.

By lemma 2.3, the terminal $\mathcal{P}_\omega$-coalgebra is the smallest quotient of $H_\eta(\mathcal{Q})$ where $\mathcal{Q}$ is the automaton with state set $\mathcal{P}_\omega(\omega)^{\omega^*}$.

Q can be visualized as the set of all countably branching infinite trees τ, where each node is labeled by some finite subset of ω. For $n \in \omega$, let $\tau_n := \lambda w.\tau(n \cdot w)$ be the n-th son of τ.

Thus, $\mathcal{Q} = (\mathcal{P}_\omega(\omega)^{\omega^*}, \alpha_Q)$, where $\alpha_Q(\tau) = (\tau(\varepsilon), \lambda m.\delta_Q(\tau, m)) = (\tau(\varepsilon), \lambda m.\tau_m)$. Now $\mathcal{H}_\eta(\mathcal{Q}) = (Q, \eta_Q \circ \alpha_Q)$ where

$$(\eta_Q \circ \alpha_Q)(\tau) = \{\tau_m \mid m \in \tau(\varepsilon)\}.$$

5.2. **The terminal image finite transition system.** $\mathcal{P}_\omega$-coalgebras are also known as *image finite transition systems*. A structure map $\alpha : A \to \mathcal{P}_\omega(A)$ corresponds uniquely to a binary relation $R_\alpha \subseteq A \times A$ where for every $a \in A$, the set $\{b \in A \mid aRb\}$ is finite. Such a binary relation can be interpreted as a transition relation, in which case it is customary write

$$a \to b$$

instead of aRb. For the above coalgebra $\mathcal{H}_\eta(\mathcal{Q})$ this reads as

$$\tau \to \sigma \iff \exists n \in \tau(\varepsilon).\ \sigma = \tau_n.$$

This $\mathcal{P}_\omega$-coalgebra is weakly terminal. In order to obtain the terminal coalgebra, one has to factor by the largest congruence relation. It is well known that the functor $\mathcal{P}_\omega$ *weakly preserves pullbacks*, so the largest congruence relation is the same as the largest *bisimulation* (see [Rut96] for these notions). In our context, this is the largest relation $\sim$ on Q, satisfying:

$\tau \sim \sigma$ implies

(i) $\forall m \in \tau(\varepsilon).\exists n \in \sigma(\varepsilon).\tau_m \sim \sigma_n$, and
(ii) $\forall n \in \sigma(\varepsilon).\exists m \in \sigma(\varepsilon).\tau_n \sim \sigma_m$.

Thereby we have obtained an easy characterization of the terminal $\mathcal{P}_\omega(-)$ coalgebra. It may be instructive to compare this description with those given by M. Barr [Bar93] or J. Worrel [Wor99].

6. Conclusion

For arbitrary coalgebras we have observed that subcoalgebras are closed under finite intersections, hence the formation of sums and subcoalgebras commute.

For bounded functors F, we have given a rather detailed description of F-coalgebras using a surjective natural transformation from a particular automaton. This description appears to be useful for characterizing cofree F-coalgebras. In fact all accounts known to us in the literature of coalgebraic types for which cofree objects exist over arbitrarily large color sets have a bounded type functor.

Recently, however, Jiři Adámek [Adá], has shown to us the existence of an unbounded functor that does have cofree coalgebras over every color set X.

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