Distributivity of Categories of Coalgebras

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Abstract. We prove that for any $\mathcal{S}et$-endofunctor F the category $\mathcal{S}et_F$ of F-coalgebras is *distributive* if F preserves preimages, i.e. pullbacks along an injective map, and that the converse is also true whenever $\mathcal{S}et_F$ has finite products.

1 Introduction

In the category $\mathcal{S}et$ of sets the equation

$$A \times (B + C) = A \times B + A \times C$$

holds, i.e., products distribute over disjoint unions. In general, we call a category $\mathcal{C}$ with finite sums *distributive*, if for all objects $A, B, C \in \mathcal{C}$ for which $A \times B$ and $A \times C$ exist, also $A \times (B + C)$ exists, and the canonical morphism

$$A \times B + A \times C \to A \times (B + C)$$

is an isomorphism. Distributive categories were studied e.g. by Cockett [Coc93], and by Carboni, Lack, and Walters [CLW93] - the latter even champions distributive categories as the appropriate setting for discussing datatypes ([Wal89,Wal91,Wal92]).

Coalgebras of various types, as general mathematical models of state based systems (see [Rut00b] for an introduction) have found applications in diverse fields, including functional programming ([Gib99], [GH00]), automata theory ([Rut98]), semantics and verification of object oriented programs ([HHJT98]), concurrency theory ([RT94]), final semantics ([TR98,Wor00]), hidden algebra ([Wor98]), analysis ([Rut00a]), and foundations of mathematics ([BM96]).

In this note we ask, under which conditions the category $\mathcal{S}et_F$ of coalgebras of a type functor F is distributive. Assuming the existence of finite products in $\mathcal{S}et_F$, this turns out to be equivalent to the type functor F preserving preimages with non-empty domains (theorem 1). It has been shown in [GS00a] that this very condition is equivalent to the property that homomorphic preimages of F-subcoalgebras are always F-subcoalgebras.

Even if finite products fail to exist in $\mathcal{S}et_F$, we will be able to conclude that F preserving preimages with non-empty domain is equivalent to a categorical property of $\mathcal{S}et_F$ closely related to distributivity, *extensiveness* - indeed any extensive and finitely complete category is distributive.

A key observation is that preservation of non-empty preimages by F is equivalent to the property that in $\mathcal{S}et_F$ each homomorphism into a sum induces a split of its domain (proposition 1).

2 Basic Notions

2.1 *F*-coalgebras and homomorphisms

Let $F : \mathcal{S}et \to \mathcal{S}et$ be a $\mathcal{S}et$-endofunctor. An F-*coalgebra* is a pair $\mathcal{A} = (A, \alpha_A)$, consisting of a set A and a map $\alpha_A : A \to F(A)$. A is called the *carrier set* and α_A is called the *structure map* of $\mathcal{A}$.

If $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$ are F-coalgebras, then a map $\varphi : A \to B$ is called a *homomorphism*, if $\alpha_B \circ \varphi = F(\varphi) \circ \alpha_A$, that is, if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ F(A) \arrow[r, "F(\varphi)"] & F(B) \end{tikzcd}$$

F-coalgebras and their homomorphisms form a category $\mathcal{S}et_F$. It is well known that all colimits in $\mathcal{S}et_F$ exist, and they are formed just as in $\mathcal{S}et$. In particular, the sum $\Sigma_{i \in I} \mathcal{A}_i$ of a family of F-coalgebras $\mathcal{A}_i = (A_i, \alpha_i)$ has as carrier the disjoint union $\biguplus_{i \in I} A_i$ and the coalgebra structure is the unique map $\alpha : \biguplus_{i \in I} A_i \to F(\biguplus_{i \in I} A_i)$ with $\alpha \circ e_i = F(e_i) \circ \alpha_i$ for all $i \in I$, where each e_i is the canonical embedding of A_i into the disjoint union $\biguplus_{i \in I} A_i$.

2.2 Subcoalgebras

A subset $U \subseteq A$ is called a *subcoalgebra* of $\mathcal{A} = (A, \alpha_A)$, provided there exists a coalgebra structure $\alpha_U : U \to F(U)$ so that the inclusion map $\subseteq_U^A : U \to A$ is a homomorphism from $\mathcal{U} = (U, \alpha_U)$ to $\mathcal{A}$. The structure map on any subcoalgebra is uniquely determined, so we will use the term "subcoalgebra" interchangeably for the coalgebra $\mathcal{U}$ and for its carrier set U.

The subcoalgebras of an F-coalgebra are easily seen to be closed under arbitrary unions, which implies that they form a complete lattice, where the join operation is given by the set-theoretical union, and for any set $U \subseteq A$ there is a largest subcoalgebra $[U]$ contained in U, the subcoalgebra *cogenerated* by U.

Less obviously, the subcoalgebras of a given coalgebra are also closed unter finite intersection (see [GS]): The reason for this is that every $\mathcal{S}et$-endofunctor F preserves *non-empty finite intersections* (as has been proved by Trnková, see [Trn69]). This means that F preserves the pullback of a finite family of injective mappings $(j_i : U_i \hookrightarrow U)_{i \in I}$ provided that the domain of this pullback, which is just $\bigcap_{i \in I} j_i[U_i]$, is not empty. Trnková proved in addition that F can be turned into a functor F^r preserving non-empty and empty finite intersections just by modifying F on the empty set and the empty mappings. This modification does not change the F-coalgebras, since obviously $\mathcal{S}et_F = \mathcal{S}et_{F^r}$, so we will always assume in the following that F is a $\mathcal{S}et$-functor which preserves all finite intersections.

In addition, we may assume $FA \neq \emptyset$ for any nonempty set A, for otherwise we would have $FB = \emptyset$ for any set B, making F the trivial functor.

2.3 Limits in $\mathcal{S}et_F$

While colimits in $\mathcal{S}et_F$ are formed just like in $\mathcal{S}et$, the situation is more complicated for limits. If F preserves a certain type of limit, $\mathcal{S}et_F$ has this same type of limit, and it is formed as in $\mathcal{S}et$. However, $\mathcal{S}et_F$ can have a limit that is not preserved by F. Indeed, under rather weak conditions on F the category $\mathcal{S}et_F$ is complete (see [GS00b]), but it should be noted that the base set of a limit in $\mathcal{S}et_F$ usually differs from the corresponding limit in $\mathcal{S}et$. In short, the forgetful functor from $\mathcal{S}et_F$ to $\mathcal{S}et$ preserves colimits, but not limits. As an example, $\mathcal{S}et_F$ has equalizer for any type functor F ([GS00b]): If $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ are two F-homomorphisms, their equalizer $\mathrm{eq}(\varphi, \psi)$ in $\mathcal{S}et_F$ is given by the largest subcoalgebra contained in $\{a \in A \mid \varphi a = \psi a\}$, i.e. by the subcoalgebra cogenerated by the equalizer of the maps φ and ψ in $\mathcal{S}et$. In the next section we will see that $\mathcal{S}et_F$ also has *preimages* for any functor F.

3 Preimages in $\mathcal{S}et$ and $\mathcal{S}et_F$

If $f : A \to B$ is a map and $V \subseteq B$, the *preimage* $f^-(V) := \{a \in A \mid fa \in V\}$ *of* V *under* f is given by the pullback in $\mathcal{S}et$ of f along the inclusion map $\subseteq^B_V : V \hookrightarrow B$.

$$\begin{tikzcd} f^-(V) \arrow[r, hook] \arrow[d, "\hat{f}"'] & A \arrow[d, "f"] \\ V \arrow[r, hook] & B. \end{tikzcd}$$

Here $\hat{f}$ is the domain-codomain-restriction of f. If $\mathcal{A}, \mathcal{B}$ are coalgebras and $\mathcal{V}$ is a subcoalgebra of $\mathcal{B}$, then $f^-(V)$ need not be a subcoalgebra of $\mathcal{A}$, however we have:

Lemma 1. *Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be an* F*-homomorphism,* $\mathcal{V} \leq \mathcal{B}$ *a subcoalgebra. Then the pullback of the inclusion morphism* $\leq : \mathcal{V} \hookrightarrow \mathcal{B}$ *along* φ *in* $\mathcal{S}et_F$*, exists and is given by* $[\varphi^-(V)]$*, the subcoalgebra of* $\mathcal{A}$ *cogenerated by the set* $\varphi^-(V)$*.*

$$\begin{tikzcd} {[\varphi^-(V)]} \arrow[r, hook, "\leq"] \arrow[d, "\hat{\varphi}"'] & \mathcal{A} \arrow[d, "\varphi"] \\ \mathcal{V} \arrow[r, hook, "\leq"'] & \mathcal{B} \end{tikzcd}$$

Proof. The diagram obviously commutes. Given a coalgebra $\mathcal{C}$ and two homomorphisms $\psi_1 : \mathcal{C} \to \mathcal{A}$, $\psi_2 : \mathcal{C} \to \mathcal{V}$ with $\varphi \circ \psi_1 = \leq \circ \psi_2$, then $\psi_1(C) \subseteq \varphi^-(V)$. Since homomorphic images of subcoalgebras are subcoalgebras of the image, ψ_1

factors through $[\varphi^-(V)]$ in $\mathcal{S}et_F$ by means of a homomorphism $\theta : \mathcal{C} \to [\varphi^-(V)]$.

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\psi_1"] \arrow[ddr, bend right, "\psi_2"'] & & \\
& {[\varphi^-(V)]} \arrow[r, hook, "\leq"] \arrow[d, "\hat{\varphi}"'] & \mathcal{A} \arrow[d, "\varphi"] \\
& \mathcal{V} \arrow[r, hook, "\leq"'] & \mathcal{B}
\end{tikzcd}$$

It follows $\leq \circ\, \hat{\varphi} \circ \theta = \leq \circ\, \psi_2$, therefore $\hat{\varphi} \circ \theta = \psi_2$, so θ is a mediating morphism, and it is obviously unique.

This lemma raises the question, under which conditions $\varphi^-(V)$ itself is a subcoalgebra of $\mathcal{A}$, or, equivalently, $\varphi^-(V) = [\varphi^-(V)]$. From [GS00a], we know that this is equivalent to F *preserving non-empty preimages*, i.e., pullbacks along injective maps with non-empty domain. Since we may assume that F preserves finite intersections, it can be easily seen that F preserves *all* preimages as soon as it preserves all non-empty preimages (for details see [Sch01]). We conclude:

Lemma 2. *The following are equivalent:*

(1) F preserves preimages.
(2) F preserves non-empty preimages.
(3) Given a map $f : A \to B$ and $V \subseteq B$ with $\emptyset \neq f^-(V) \neq A$, then for each $x \in FA$, $y \in FV$ with $(Ff)x = F(\subseteq^B_V)y$ there is a (necessarily unique) $z \in F(f^-(V))$ with $F(\subseteq^A_{f^-(V)})z = x$.
(4) Homomorphic preimages of F-subcoalgebras are always F-subcoalgebras.
(5) $[\varphi^-(V)] = \varphi^-(V)$ for each F-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and each $\mathcal{V} \leq \mathcal{B}$.

4 Domain Splitting

Consider a map $f : A \to B + C$. We can investigate its properties by distinguishing the cases $fa \in B$, resp., $fa \in C$. In other terms, f induces a splitting of its domain via $A = A_B + A_C$, where $A_B = \{a \in A \mid fa \in B\} = f^-(B)$, and $A_C = \{a \in A \mid fa \in C\} = f^-(C)$. If $\varphi : \mathcal{A} \to \mathcal{B} + \mathcal{C}$ is an F-homomorphism and F preserves preimages, by lemma 2 we have the same domain splitting $\mathcal{A} = \varphi^-(B) + \varphi^-(C)$ in $\mathcal{S}et_F$. It turns out that the existence of such splittings is indeed equivalent to the preservation of preimages by F.

Proposition 1. *The following are equivalent:*

(1) F preserves preimages.
(2) Every F-homomorphism $\varphi : \mathcal{A} \to \mathcal{B} + \mathcal{C}$ induces a splitting of its domain, that is $\mathcal{A} = \mathcal{A}_B + \mathcal{A}_C$ for some subcoalgebras $\mathcal{A}_B, \mathcal{A}_C \leq \mathcal{A}$ with $\varphi(A_B) \subseteq B$, $\varphi(A_C) \subseteq C$. In this case necessarily $A_B = \varphi^-(B)$ and $A_C = \varphi^-(C)$.

Proof. (1)⇒(2) has been discussed above, so we prove (2)⇒(1) by checking the third condition of Lemma 2.

Given $f : A \to B$ and $V \subseteq B$ with $\emptyset \neq f^{-}(V) \neq A$ and elements $x \in FA$, $y \in FV$ with $(Ff)x = F(\subseteq_V^B)y$, we set $A_0 := f^{-}(V)$ and $A_1 := A \setminus A_0$.

$$\begin{tikzcd}
A_0 \arrow[r, hook] \arrow[d, "f_{|A_0}^{|V}"'] & A \arrow[d, "f"] & A_1 \arrow[l, hook'] \arrow[d, "f_{|A_1}^{|W}"] \\
V \arrow[r, hook] & B = V + W & W, \arrow[l, hook']
\end{tikzcd}$$

Here, W is the complement of V in B and $f_{|A_0}^{|V}$, resp. $f_{|A_1}^{|W}$, are domain-codomain-restrictions of f, and $f^{-}(W) = A_1$, since preimages commute with complements.

Fix an element $k \in F(A_1)$ and introduce coalgebra structures α_A on A and α_B on B by defining, for arbitrary $a \in A$ and $b \in B$:

$$\alpha_A a := \begin{cases} x & , \text{if } a \in A_0 \\ F(\subseteq_{A_1}^A)k & , \text{if } a \in A_1, \end{cases} \qquad \text{and} \qquad \alpha_B b := \begin{cases} (Ff)x & , \text{if } b \in V \\ F(f_{|A_1})k & , \text{if } b \in W. \end{cases}$$

These structure maps turn f into an F-homomorphism $(A, \alpha_A) \to (B, \alpha_B)$, since

$$(Ff)(\alpha_A a) = \begin{cases} (Ff)x = \alpha_B(fa) & , \text{if } a \in A_0 \\ (Ff)(F(\subseteq_{A_1}^A)k) = F(f_{|A_1})k = \alpha_B(fa) & , \text{if } a \in A_1. \end{cases}$$

V, resp. W, with the constant structure maps with result y, resp. $F(f_{|A_1}^{|W})k$, are easily checked to be subcoalgebras of (B, α_B), so $\mathcal{B} = \mathcal{V} + \mathcal{W}$ in $\mathcal{S}et_F$. This allows us to conclude that A_0 is a subcoalgebra of $\mathcal{A}$, so there is a coalgebra structure $\rho : A_0 \to F(A_0)$ turning $\subseteq_{A_0}^A$ into a homomorphism. Now, $z := \rho(u)$ for an arbitrary $u \in A_0$ is the required element, since

$$F(\subseteq_{A_0}^A)z = F(\subseteq_{A_0}^A)(\rho u) = \alpha_A(\subseteq_{A_0}^A u) = x.$$

The proof generalizes to show that preservation of preimages by F is equivalent to the fact that every F-homomorphism $\varphi : \mathcal{A} \to \sum_{i \in I} \mathcal{B}_i$ into a (possibly infinite) sum induces a corresponding splitting of its domain.

5 Preimage preservation implies distributivity

In this section we will show that $\mathcal{S}et_F$ is distributive provided that F preserves preimages. In the next section we shall then formulate a converse to this result.

Definition 1. *A category $\mathcal{C}$ with finite sums is* distributive*, if binary products distribute over sums, i.e., for all $A, B, C \in \mathcal{C}$ we have: If the products $A \times B$ and $A \times C$ exist, then the product $A \times (B + C)$ exists and the canonical morphism*

$$A \times B + A \times C \to A \times (B + C)$$

is an isomorphism.

Of course, one can generalize this definition to infinite sums and infinite products, and all our proofs extend to this more general case.

Proposition 2. *If F preserves preimages, then the category $\mathcal{S}et_F$ is distributive.*

Proof. Let F preserve preimages. Let $\mathcal{A} \in \mathcal{S}et_F$ and a family $(\mathcal{B}_i)_{i\in I}$ of F-coalgebras be given, so that $\mathcal{A} \times \mathcal{B}_i$ exists for each $i \in I$. Let $p_i : \mathcal{A} \times \mathcal{B}_i \to \mathcal{A}$ and $q_i : \mathcal{A} \times \mathcal{B}_i \to \mathcal{B}_i$ be the canonical projections of the products and $e_i : \mathcal{B}_i \to \sum_{j\in I} \mathcal{B}_j$ the canonical injections. We claim that $\sum_{i\in I}(\mathcal{A} \times \mathcal{B}_i)$ together with the projections

$$[p_i]_{i\in I} : \sum_{i\in I}(\mathcal{A} \times \mathcal{B}_i) \to \mathcal{A} \quad \text{and} \quad \sum_{i\in I} q_i : \sum_{i\in I}(\mathcal{A} \times \mathcal{B}_i) \to \sum_{i\in I} \mathcal{B}_i$$

is the product of $\mathcal{A}$ with $\sum_{i\in I} \mathcal{B}_i$ in $\mathcal{S}et_F$.

Let $(\mathcal{Q}, \varphi : \mathcal{Q} \to \mathcal{A}, \psi : \mathcal{Q} \to \sum_{i\in I} \mathcal{B}_i)$ be a competitor. By proposition 1 we obtain a decomposition $\mathcal{Q} = \sum_{i\in I} \mathcal{Q}_i$ with $Q_i = \psi^{-}(B_i)$. Therefore, we have for each $i \in I$ pairs of homomorphisms

$$\varphi_{|Q_i} : \mathcal{Q}_i \to \mathcal{A}, \;\; \psi_{|Q_i} : \mathcal{Q}_i \to \mathcal{B}_i,$$

inducing unique mediating morphisms $(\varphi_{|Q_i}, \psi_{|Q_i}) : \mathcal{Q}_i \to \mathcal{A} \times \mathcal{B}_i$, so that

$$\sum_{i\in I}(\varphi_{|Q_i}, \psi_{|Q_i}) : \mathcal{Q} \to \sum_{i\in I}(\mathcal{A} \times \mathcal{B}_i)$$

is a mediating morphism for $(\mathcal{Q}, \varphi, \psi)$.

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B}_i \arrow[rr, "p_i"] \arrow[ddr, "q_i"'] & & \mathcal{A} & \\
& \mathcal{Q}_i \arrow[ul, dashed] \arrow[r, hook] \arrow[d, "\psi_{|Q_i}"] & \mathcal{Q} \arrow[u, "\varphi"] \arrow[r, dashed] \arrow[d, "\psi"] & \sum \mathcal{A} \times \mathcal{B}_i \arrow[ul, "{[p_i]_{i\in I}}"'] \arrow[dl, "\sum q_i"] \\
& \mathcal{B}_i \arrow[r, hook, "e_i"'] & \sum \mathcal{B}_i &
\end{tikzcd}$$

If $\rho : \mathcal{Q} \to \sum_{i\in I}(\mathcal{A} \times \mathcal{B}_i)$ is another mediating morphism, then for each $i \in I$ we have

$$(\sum_{i\in I} q_i) \circ \rho_{|Q_i} = e_i \circ \psi_{|Q_i},$$

and $\rho_{|Q_i}$ factors through $\mathcal{A}\times\mathcal{B}_i$. In fact $\rho_{|Q_i}$ is a mediating morphism for $\varphi_{|Q_i}$ and $\psi_{|Q_i}$. This implies $\rho_{|Q_i} = (\varphi_{|Q_i}, \psi_{|Q_i})$ by uniqueness, so $\rho = \sum_{i\in I}(\varphi_{|Q_i}, \psi_{|Q_i})$.

With slightly more notational overhead we can check that in $\mathcal{S}et_F$ infinite products distribute over infinite sums when F preserves preimages.

Notice that there are different notions of distributive category in the literature ([CLW93,Coc93]). The differences consist in “how many” limits the category in question is supposed to have. The definition we have given is the one which requires only those limits that are absolutely necessary to define the notion.

6 Distributivity + finite products implies preimage preservation

This section is devoted to finding a converse to proposition 2. If $\mathcal{S}et_F$ is distributive and has finite products, we shall show that F preserves preimages. For this we first observe that equalizers commute with sums in $\mathcal{S}et_F$. In proposition 3 we will then see that the preservation of preimages by F is equivalent to the fact that in $\mathcal{S}et_F$ *pullbacks commute with binary sums.* By expressing a pullback in the canonical way as the equalizer of a product we then come to the desired conclusion (proposition 4).

6.1 Equalizers

It is easy to see that in $\mathcal{S}et_F$ equalizers commute with sums, i.e., if $(\varphi_i, \psi_i : \mathcal{A}_i \to \mathcal{B})_{i\in I}$ is a family of pairs of homomorphisms, the equalizer of $\sum_{i\in I} \varphi_i : \sum_{i\in I} \mathcal{A}_i \to \mathcal{B}$ with $\sum_{i\in I} \psi_i : \sum_{i\in I} \mathcal{A}_i \to \mathcal{B}$ is given by the sum of the equalizers of the φ_i with the ψ_i. To see this, compute

$$\mathrm{eq}(\sum_{i\in I} \varphi_i, \sum_{i\in I} \psi_i) = [\{a \in \biguplus_{i\in I} A_i \mid (\sum_{i\in I} \varphi_i)a = (\sum_{i\in I} \psi_i)\}]$$
$$= [\biguplus_{i\in I} \{a \in A_i \mid \varphi_i a = \psi_i a\}] = \sum_{i\in I} [\{a \in A_i \mid \varphi_i a = \psi_i a\}] = \sum_{i\in I} \mathrm{eq}(\varphi_i, \psi_i).$$

6.2 Pullbacks and sums

Definition 2. *In a category* $\mathcal{C}$ *with binary sums, we say that* pullbacks commute with binary sums, *if for all morphisms* $f : A \to C$, $g_1 : B_1 \to C$, $g_2 : B_2 \to C$ *we have: If the pullbacks* $\mathrm{pb}(f, g_1)$ *and* $\mathrm{pb}(f, g_2)$ *exist, the pullback* $\mathrm{pb}(f, [g_1, g_2])$ *exists, too, and*

$$\mathrm{pb}(f, g_1 + g_2) = \mathrm{pb}(f, g_1) + \mathrm{pb}(f, g_2).$$

It is again obvious how to extend the definition to the infinite case.

Proposition 3. *The following are equivalent:*

1. *F preserves preimages.*
2. *In $\mathcal{S}et_F$ pullbacks commute with infinite sums.*
3. *In $\mathcal{S}et_F$ pullbacks commute with binary sums.*

Proof. (1)⇒(2) is proved similarly to proposition 2, and (2)⇒(3) is trivial.

(3)⇒(1): We check the condition of proposition 1. Let $\varphi : \mathcal{A} \to \mathcal{B}_1 + \mathcal{B}_2$ be a homomorphism. For $i = 1, 2$ we form the pullback

$$\begin{tikzcd}
{[\varphi^{-}(B_i)]} \arrow[r, hook] \arrow[d, "\hat{\varphi}"'] & \mathcal{A} \arrow[d, "\varphi"] \\
\mathcal{B}_i \arrow[r, hook] & \mathcal{B}_1 + \mathcal{B}_2
\end{tikzcd}$$

in $\mathcal{S}et_F$. Then by assumption the following diagram is a pullback:

$$\begin{tikzcd}
{[\varphi^-(B_1)] + [\varphi^-(B_2)]} \arrow[r, hook] \arrow[d, "\hat{\varphi}"'] & \mathcal{A} \arrow[d, "\varphi"] \\
\mathcal{B}_1 + \mathcal{B}_2 \arrow[r, hook, "\mathrm{id}_{\mathcal{B}_1+\mathcal{B}_2}"'] & \mathcal{B}_1 + \mathcal{B}_2
\end{tikzcd}$$

On the other hand the domain of the pullback of φ along $\mathrm{id}_{\mathcal{B}_1+\mathcal{B}_2}$ must be $\mathcal{A}$ itself, hence $[\varphi^-(B_1)] + [\varphi^-(B_2)] = \mathcal{A}$, which implies (1) by proposition 1.

Since the pullback of $\varphi : \mathcal{A} \to \mathcal{C}$, $\psi : \mathcal{B} \to \mathcal{C}$ is nothing but the equalizer of $\varphi \circ \pi_1, \psi \circ \pi_2 : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, where π_1, π_2 are the projections of $\mathcal{A} \times \mathcal{B}$, we obtain:

Proposition 4. *If $\mathcal{S}et_F$ has finite products and is distributive, F preserves preimages.*

Summarizing propositions 1, 2, 3, and 4, we conclude:

Theorem 1. *The following are equivalent:*

- *F preserves preimages.*
- *Preimages of subcoalgebras under homomorphisms are subcoalgebras.*
- *In $\mathcal{S}et_F$ any homomorphism into a sum induces a splitting of its domain.*
- *In $\mathcal{S}et_F$ pullbacks commute with sums.*

Each of these conditions implies that $\mathcal{S}et_F$ is distributive. If $\mathcal{S}et_F$ has finite products, then the converse is also true.

7 Towards a generalization: Extensive Categories

In [CLW93]) the notion of an *extensive category* was introduced to capture categories in which sums exist and are well-behaved.

Definition 3 ([CLW93]). *Let $\mathcal{C}$ be a category with finite sums and pullbacks along injections into finite sums. $\mathcal{C}$ is* extensive*, if for all commutative diagrams*

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "f"'] & Z \arrow[d, "h"] & Y \arrow[l, "i_Y"'] \arrow[d, "g"] \\
A \arrow[r, "i_A"'] & A + B & B \arrow[l, "i_B"]
\end{tikzcd}$$

in $\mathcal{C}$, with the lower row being a sum, we have: Both squares are pullbacks iff (Z, i_X, i_Y) is a sum.

Proposition 5. *$\mathcal{S}et_F$ is extensive iff F preserves preimages.*

Proof. If $\mathcal{S}et_F$ is extensive, proposition 1 shows that F preserves preimages.

To prove the converse, let F preserve preimages. Given a commutative diagram as shown above in $\mathcal{S}et_F$ with the lower row a sum and the squares pullbacks, the upper row is a sum by proposition 1. Suppose now the two rows are sums. We have to show that the left square is a pullback. It suffices to show that this square is a preimage in $\mathcal{S}et$ since F preserves preimages. So let $z \in Z$ and $a \in A$ with $i_A a = hz$ be given. Since $Z = i_X(X) + i_Y(Y)$, either $z \in i_X(X)$ or $z \in i_Y(Y)$. If we had $z = i_Y(y)$ for some $y \in Y$, we could conclude $hz = i_B(gy) \in i_B(B)$ which is impossible since $hz \in i_A(A)$ by assumption and $i_A(A)$ is disjoint from $i_B(B)$. So we have $z \in i_X(X)$, showing that the left square is a pullback. A symmetric proof shows that also the right square is a pullback, thus $\mathcal{S}et_F$ is extensive.

The heart of the second part of the proof is: $\mathcal{S}et$ itself is extensive, and since F preserves preimages, sums and preimages are formed in $\mathcal{S}et_F$ as in $\mathcal{S}et$, so extensiveness carries over from $\mathcal{S}et$ to $\mathcal{S}et_F$.

In general, we can define F-coalgebras of an endofunctor $F : \mathcal{C} \to \mathcal{C}$ for any base category $\mathcal{C}$, as pairs $\mathcal{A} = (A, \alpha_A)$, consisting of an object $A \in \mathcal{C}$ and an arrow $\alpha_A : A \to FA$ in $\mathcal{C}$. Homomorphisms are defined in the same way as for $\mathcal{C} = \mathcal{S}et$. We ask now: Under which conditions does extensiveness of $\mathcal{C}$ imply extensiveness of $\mathcal{C}_F$? We make the following assumptions on $\mathcal{C}$, resp. F:

- $\mathcal{C}$ is extensive and has infinite sums.
- $\mathcal{C}$ is finitely complete with terminal object 1.
- $\mathcal{C}$ has epi-(regular mono)-factorisations.
- F preserves regular monos and takes non-initial objects to non-initial objects.

Lemma 3. *In $\mathcal{C}$ any canonical injection $e_A : A \to A + B$ of a sum is a regular mono.*

Proof. The following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[d] & A + B \arrow[d] \\ 1 \arrow[r, "e_1"'] & 1 + 1 \end{tikzcd}$$

Here $e_1 : 1 \to 1+1$ is the first canonical injection of the sum. Since regular monos are stable under pullbacks, it suffices to prove that e_1 is a regular mono. Observe that e_1 is the equalizer of $\mathrm{id} : 1 + 1 \to 1 + 1$ with $e_1 \circ !_{1+1} : 1 + 1 \to 1 \to 1 + 1$, where $!_{1+1} : 1 + 1 \to 1$ denotes the unique morphism into the terminal object.

7.1 Subcoalgebras

We define a *subcoalgebra in $\mathcal{C}_F$* to be a regular subobject. One may check that this agrees with the previous definition from section 2.2 when $\mathcal{C} = \mathcal{S}et$ (s. [GS00a] for a proof). It is easy to see (for details compare [Hug01]):

Lemma 4. *The forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ preserves regular monos and creates epis, regular monos, epi-(regular mono)-factorisations, and exact sequences of the following form, where $\hookrightarrow$ denotes a regular mono:*

$$\bullet \hookrightarrow \bullet \rightrightarrows \bullet$$

U creates every colimit and every limit that is preserved by F.

It turns out that the definition of a cogenerated subcoalgebra, $[U]$, is indeed a categorical one. Let $\mathbf{Sub}_F(\mathcal{A})$ denote the partial order of subcoalgebras of $\mathcal{A} \in \mathcal{C}_F$, $\mathbf{Sub}(A)$ the partial order of regular subobjects of A in the category $\mathcal{C}$. The functor $U_\alpha : \mathbf{Sub}_F(\mathcal{A}) \to \mathbf{Sub}(A)$, mapping any subcoalgebra of $\mathcal{A}$ to its base object, has a right adjoint $[-]_\alpha : \mathbf{Sub}(A) \to \mathbf{Sub}_F(\mathcal{A})$, which, in the case of $\mathcal{C} = \mathcal{S}et$, coincides with $[-]$.

7.2 Preimages

If $f : A \to B$ is a morphism in $\mathcal{C}$, we can pull back subobjects of B along f, obtaining a map $f^* : \mathbf{Sub}(B) \to \mathbf{Sub}(A)$. $f^*(V)$ is called the *preimage of V along f*. We say that F *preserves preimages* if it preserves pullbacks along regular monos. This means that we have $F(f^*V) = (Ff)^*(FV)$ for every $f : A \to B$ and any $V \in \mathbf{Sub}(B)$. F is said to *preserve non-empty preimages* if this equation holds, except for V being the initial object.

With these notions it is easy to see that the implications (1)$\Rightarrow$ (2)$\Rightarrow$(4) $\iff$ (5) from lemma 2 and (1)$\Rightarrow$(2) from proposition 1 are still true for $\mathcal{C}$ in place of $\mathcal{S}et$. Our proofs of the reverse implications made essential use of the fact that $\mathcal{S}et$ is a well-pointed topos; we do not know whether such a condition is indeed needed. Examining our proofs, we obtain (for details see [Hug01]):

Proposition 6. *Let $F : \mathcal{C} \to \mathcal{C}$ preserve regular monos and non-empty preimages. If $\mathcal{C}$ is extensive, finitely complete, and has epi-regular mono factorizations, $\mathcal{C}_F$ is extensive and distributive.*

References

[BM96] J. Barwise and L. Moss, *Vicious circles*, CSLI Lecture Notes, 1996.

[CLW93] A. Carboni, S. Lack, and R.F.C. Walters, *Introduction to extensive and distributive categories*, J. of Pure and Applied Algebra (1993), no. 84, 145–158.

[Coc93] J. R. B. Cockett, *Introduction to distributive categories*, Mathematical Structures in Computer Science **3** (1993), no. 3, 277–307.

[GH00] Jeremy Gibbons and Graham Hutton, *Proof methods for corecursive programs*, submitted, 2000.

[Gib99] J. Gibbons, *Lecture notes on algebraic and coalgebraic methods for calculating functional programs*, from Estonian Winter School on Computer Science, 1999.

[GS] H.P. Gumm and T. Schröder, *Coalgebras of bounded type*, Submitted.

[GS00a] H.P. Gumm and T. Schröder, *Coalgebraic structure from weak limit preserving functors*, Electronic Notes in Theoretical Computer Science (2000), no. 33, 113–133.

[GS00b] H.P. Gumm and T. Schröder, *Products of coalgebras*, to appear in Algebra Universalis, 2000.

[HHJT98] U. Hensel, M. Huisman, B. Jacobs, and H. Tews, *Reasoning about classes in object-oriented languages: Logical models and tools*, European Symposium on Programming, LNCS, no. 1381, Springer, 1998, pp. 105–121.

[Hug01] Jesse Hughes, *A study of categories of algebras and coalgebras*, Ph.D. thesis, Dept. of Philosophy, Carnegie Mellon University, 2001.

[RT94] J. Rutten and D. Turi, *Initial algebra and final coalgebra semantics for concurrency*, Proc. of the REX workshop A Decade of Concurrency – Reflections and Perspectives (J. de Bakker et al., eds.), LNCS, vol. 803, Springer-Verlag,1994, pp. 530–582.

[Rut98] J.J.M.M. Rutten, *Automata and coinduction (an exercise in coalgebra)*, CONCUR '98 (D. Sangiorigi et al., eds.), LNCS, no. 1466, 1998, pp. 194–218.

[Rut00a] J.J.M.M. Rutten, *Behavioural differential equations: a coinductive calculus of streams, automata, and power series*, Tech. report, Centrum voor Wiskunde en Informatica, 2000.

[Rut00b] J.J.M.M. Rutten, *Universal coalgebra: a theory of systems*, Theoretical Computer Science (2000), no. 249, 3–80.

[Sch01] Tobias Schröder, *Coalgebren und Funktoren*, Ph.D. thesis, FB Mathematik und Informatik, Philipps-Universitt Marburg, 2001.

[TR98] Daniele Turi and Jan Rutten, *On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces*, Mathematical Structures in Computer Science **8** (1998), no. 5, 481–540.

[Trn69] V. Trnková, *Some properties of set functors*, Comm. Math. Univ. Carolinae (1969), no. 10,2, 323–352.

[Wal89] R.F.C. Walters, *Data-types in distributive categories*, Bulletion of the Australian Maths Society (1989), no. 40, 79–82.

[Wal91] R.F.C. Walters, *Categories and computer science*, Cambridge University Press, 1991.

[Wal92] R.F.C. Walters, *An imperative language based on distributive categories*, Mathematical Structures in Computer Science (1992), no. 2, 249–256.

[Wor98] J. Worrell, *Toposes of coalgebras and hidden algebras*, 1998, pp. 215–233.

[Wor00] J. Worrell, *On coalgebras and final semantics*, Ph.D. thesis, Oxford, 2000.