EQUATIONAL AND IMPLICATIONAL CLASSES OF CO-ALGEBRAS.

(EXTENDED ABSTRACT)

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ABSTRACT. If $T : \mathbf{Set} \to \mathbf{Set}$ is a functor which is bounded and preserves weak pullbacks then a class of T-coalgebras is a covariety, i.e closed under $\mathcal{H}$ (homomorphic images), $\mathcal{S}$ (sub-coalgebras) and Σ (sums), if and only if it can be defined by a set of "coequations". Similarly, classes closed under $\mathcal{H}$ and Σ can be characterized by implications of coequations. These results are analogous to the theorems of G.Birkhoff and of A.I.Mal'cev in classical universal algebra.

1. INTRODUCTION

The recently developed theory of coalgebras under a functor T provides a highly attractive framework for describing the semantics and the logic of various types of transition systems. In contrast to the algebraic semantics of abstract data types where data objects are constructed recursively and equality is proven by induction, coalgebras support definitions by co-recursion and define equivalence by co-induction. This view is appropriate in many contexts, prominently when modelling objects and classes in object-oriented languages ([Rei95, Jac96]) or infinite data objects such as processes and streams.

1.1. **Transitions and transition systems.** A *transition* Θ is nothing but a binary relation on a set S, i.e. $\Theta \subseteq S \times S$. Θ is called *image finite*, if for every $s \in S$, the set $s\Theta = \{t \in S \mid s\Theta t\}$ is finite. Θ is called *deterministic* if it is the graph of a function $\theta : S \to S$, i.e. $\Theta = \{(s, \theta(s)) \mid s \in S\}$.

A *transition system* is a family $T = (\Theta_a)_{a \in A}$ of transitions on S. Related to this notion is that of an automaton where additionally one may have a set $F \subseteq S$ of *accepting states* or an *output function* $\gamma : S \to B$.

In order to emphasize the dynamical aspect of transitions or transition systems, we describe them by a map α from S to some structured set. Unary relations will be modelled by a map into $Bool = \{true, false\}$ and binary relations $R \subseteq S \times X$ by a map from S into the powerset $\mathcal{P}(X)$. With $\mathcal{P}_{fin}(S)$ we denote the lattice of finite subsets of S.

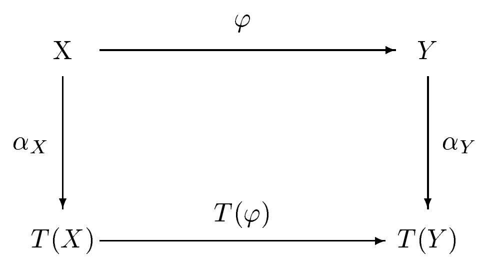
In particular, a map α of type :

$S \to S$	is a deterministic transition,
$S \to \mathcal{P}(S)$	is a nondeterministic transition (relation),
$S \to \mathcal{P}_{fin}(S)$	is an image finite nondeterministic transition,
$S \to S^A$	is a deterministic transition system,
$S \to \mathcal{P}_{fin}(S)^A$	models a nondeterministic transition system in which all transitions are image finite,
$S \to B \times S^A$	is an automata with output, and
$S \to \mathcal{P}_{fin}(S)^A \times Bool$	models an automaton with bounded nondeterminism and an acceptance condition.

2. Coalgebras

In all of the examples we are given a functor $T : \mathbf{Set} \to \mathbf{Set}$ and a map $\alpha : S \to T(S)$. Any pair (X, α_X) where $\alpha_X : X \to T(X)$ will be called a *coalgebra of type* T.

A *homomorphism* between coalgebras (X, α_X) and (Y, α_Y) is a map $\varphi : X \to Y$ for which the following diagram commutes:



Thus the class of all coalgebras of a fixed type T becomes a category $\mathbf{Set}_T$. From this a number of standard coalgebraic constructions can be derived. In particular, (X, α_X) is a subcoalgebra of (Y, α_Y) if $X \subseteq Y$ and the natural embedding "$\subseteq$" is a homomorphism. In this case, the coalgebra structure on X is unique, so we also refer to the set X as a subcoalgebra of (Y, α_Y). X is called *fully invariant* in Y, if $\varphi(X) \subseteq X$ for every endomorphism $\varphi : Y \to Y$. A coalgebra is called *simple*, if it does not have any proper homomorphic image.

2.1. **Preservation of weak generalized pullbacks.** Under some mild conditions on the functor T, the class of all coalgebras of type T will be very well behaved and expose a structure theory which is largely dual to the classical theory of universal algebra. Specifically, we must assume that T preserves *weak generalized pullbacks*, which is to say that T transforms the limit of a collection $(\varphi_i)_{i \in I}$ of maps having a common codomain into a weak limit of the family $(T(\varphi_i))_{i \in I}$. This concept is introduced in [Rut96] and criteria for verifying it are studied in [Rut98] and [Gum98]. All of the functors mentioned above do share this property.

If T preserves weak generalized pullbacks then the following are just some facts known about the class $\mathbf{Set}_T$ (c.f. [Rut96]):

- In $\mathbf{Set}_T$, monomorphisms are injective, epimorphisms are surjective and bijective morphisms are isomorphisms.
- The subcoalgebras of a fixed coalgebra (S, α_S) are closed under unions and intersections, in particular, for any subset $X \subseteq S$ there is a largest subcolagebra $[X]$ contained in X and a smallest subcoalgebra $\langle X \rangle$ containing X. The latter is called the subcoalgebra *generated by* X. If X is a one-element set,

$X = \{x\}$, then we write $\langle x\rangle$ instead of $\langle\{x\}\rangle$ and call such a a subcoalgebra *one-generated.*

- Images and preimages of subcoalgebras under homomorphisms are subcoalgebras.

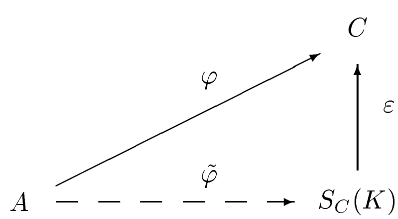
2.2. **Covarieties and Coquasivarieties.** We will particularly be interested in certain subclasses of $\mathbf{Set}_T$ which are called *covarieties.* Here a covariety is a class of T-coalgebras closed under the operators

$\mathcal{H}$ homomorphic images,
$\mathcal{S}$ subcoalgebras, and
Σ sums.

Classes closed under $\mathcal{H}$ and under Σ are called *coquasivarieties.*

If a coalgebra B has subcoalgebras $A_i, i \in I$ so that B is the union of the A_i, then B is called a *conjunct sum* of the A_i. If K is a class of T-coalgebras, denote by $\Sigma_c(K)$ the class of all conjunct sums and by $\mathcal{S}_1(K)$ the class of all one-generated subcoalgebras of members of K. It was shown in [GS98] that the covariety generated by a class K can be obtained as $\Sigma_c\mathcal{H}\mathcal{S}_1(K)$, and that in general the operators $\mathcal{H}$ and $\mathcal{S}$ commute, i.e. that $\mathcal{H}\mathcal{S}(K) = \mathcal{S}\mathcal{H}(K)$.

2.3. **Bounded functors and cofree coalgebras.** Let C be a set. We refer to the elements of C as "colors" and and to every set map from a coalgebra A to C as a "coloring". A coalgebra $S_C(K)$ together with a coloring $\varepsilon : S_C(K) \to C$ is called *cofree over* C, with respect to a class K, if for every coalgebra A in K and any coloring $\varphi : A \to C$ there exists exactly one homomorphism $\tilde{\varphi} : A \to S_C(K)$ such that $\varphi = \varepsilon \circ \tilde{\varphi}$. We write S_C for $S_C(\mathbf{Set}_T)$.



Cofree coalgebras exist, provided that there is a bound on the cardinality of one-generated T-coalgebras. In this case, the functor T is called *bounded.* All of the functors mentioned above, except for $\mathcal{P}(-)$, are bounded.

If T is bounded then $S_C(K)$ can be constructed as a conjunct sum of one-generated subcoalgebras in K, in particular, if K is a *covariety* then $S_C(K)$ is in K.

However, there is another useful way of looking at cofree coalgebras: A "colored T-coalgebra", i.e. a coalgebra A together with a coloring $\varphi : A \to C$ may be viewed as a coalgebra for the functor $C \times T(-)$. A cofree coalgebra $S_C(K)$ with its coloring ε is then nothing but a a final object in the category of $C \times T(-)$-coalgebras. The elements of $S_C(K)$ therefore correspond uniquely to colored one-generated subcoalgebras of coalgebras in K which, considered as $C \times T(-)$-coalgebras, are simple.

3. Co-Equations

A *coequation* is defined as an element of a cofree coalgebra $S_C(K)$. More precisely, each $e \in S_C(K)$ is called a *coequation with colors in* C.

Since for any subset $C' \subseteq C$ we get a a canonical embedding $\tilde{\subseteq} : S_{C'}(K) \to S_C(K)$, there will be a smallest set $C_e \subset C$ such that $e \in S_{C_e}(K)$. This will be called the set colors *occurring* in e.

Given any coequation e with colors in C and a coalgebra A, we shall say that e *holds in* A, in symbols

$$A \models e$$

if for every coloring $\varphi : A \to C$ we have $e \notin \tilde{\varphi}(A)$.

In order to check e locally on the elements of A, note that e can be identified with the simple colored subcoalgebra it generates. Therefore, to check whether e holds at a in A, we simply have to check that for no coloring of $\langle a \rangle$ we obtain $\langle e \rangle$ as a color preserving homomorphic image. We shall use this observation later when studying an example.

For a set E of coequations we define $\mathbf{Mod}(E)$, the *coequational class* of E as [1]

$$\mathbf{Mod}(E) = \{A \in \mathbf{Set}_T \mid \forall e \in E.\ A \models e\}.$$

Conversely, let K be a class of T-coalgebras, and let C be a bound for T. We define

$$\mathbf{Ceq}(K) = \{e \in S_C \mid \forall A \in K.\ A \models e\}.$$

The following lemma is easy to check:

Lemma 3.1. *Let E be a set of coequations, then* $\mathbf{Mod}(E)$ *is closed under $\mathcal{H}$, $\mathcal{S}$ and Σ, i.e. a covariety.*

But the converse turns out to be true too. This is the coequational version of Birkhoff's theorem:

Theorem 3.1. *Covarieties are the same as coequational classes, specifically, for any class K of coalgebras,*

$$\mathbf{Mod}(\mathbf{Ceq}(K)) = \mathcal{H}\mathcal{S}\Sigma(K).$$

On the equational side, we can define a consequence relation, also denoted by "$\models$" as

$$E \models f :\Leftrightarrow \forall A \in \mathbf{Set}_T.\ (\forall e \in E.\ A \models e) \Rightarrow A \models f$$

Using results from [GS98] we can infer an internal description for this consequence relation. To this end define $\ll f \gg$ as the smallest fully invariant subcoalgebra of $S_C(K)$ generated by f. Then we have:

Theorem 3.2. $E \models f$ *if and only if* $\ll f \gg \cap E \neq \emptyset$

[1] Of course this is an abuse of set notation

4. Co-Implications

If E is a set of coequations and f a single coequation then the expression $E \Rightarrow t$ is called a *co-implication.* Let C be the set of colors occurring in E or in f. We say that $E \Rightarrow f$ *holds* in some coalgebra A if for any coloring $\varphi : A \to C$ we have

$$E \cap \tilde{\varphi}(A) = \emptyset \text{ implies } f \notin \tilde{\varphi}(A).$$

Given a set Q of co-implications, we can define $\mathbf{Mod}(Q)$ as the class of all coalgebras satisfying all co-implications in Q. Similarly, let $\mathbf{Cimp}(K)$ be the set of all co-implications satisfied in all members of K.

We can now show that the classes definable by co-implications are precisely the co-quasivarieties:

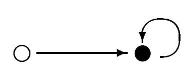
Theorem 4.1. *Let K be any class of T-coalgebras, then*

$$\mathcal{H}\Sigma(K) = \mathbf{Mod}(\mathbf{Cimp}(K)).$$

5. An Example

We elaborate a simple example. Let $\mathcal{I}$ be the identity functor on **Set**. An $\mathcal{I}$-coalgebra is a map $\alpha : S \to S$. Consider the subclass K of $\mathbf{Set}_{\mathcal{I}}$ consisting of all (S, α_S) such that $\forall s \in S.\ \exists n \in \mathbb{N}.\ \alpha^n(s) = s$. It is easy to check that K is a covariety, i.e. it is closed under $\mathcal{H}$, $\mathcal{S}$, and Σ.

K can be described by the following two-color coequation:



The figure represents a simple colored one-generated $\mathcal{I}$-coalgebra which cannot be obtained as a color preserving homomorphic image of any colored coalgebra in K. Conversely, if $A \notin K$, there exists $a \in A$ such that $a \notin B_a$ where B_a is defined as $\{\alpha^n(a) \mid 0 < n \in \mathbb{N}\}$. For a coloring φ, painting every element in B_a black and painting a white, we shall obtain the coequation as $\tilde{\varphi}(a)$.

References

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