Covarieties and Complete Covarieties

H. Peter Gumm, Tobias Schröder

Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
Marburg, Germany
{gumm,tschroed}@mathematik.uni-marburg.de

Abstract

We present two ways to define covarieties and complete covarieties, i.e. covarieties that are closed under total bisimulation: by closure operators and by subcoalgebras of coalgebras.

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1 Introduction

Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor. An *F-coalgebra* is a set A together with a map $\alpha_A : A \to F(A)$. α_A is often referred to as the "transition structure" on A.

An *F-homomorphism* between two F-coalgebras (A, α_A), (B, α_B) is a map $f : A \to B$ with $F(f) \circ \alpha_A = \alpha_B \circ f$. The class of all F-coalgebras together with F-homomorphisms forms a category which is denoted by $\mathbf{Set_F}$.

In [6] J.J.M.M. Rutten has shown how coalgebras can be used to model various kinds of transition systems. He develops the basic theory of coalgebras, analogous to the fundamental theory of universal algebra. We shall assume familiarity with this article. Further examples and applications of coalgebras can be found in [4] and [3].

Here we are trying to extend the basic theory by investigating and characterizing covarieties, i.e. classes of coalgebras that are closed under homomorphic images, sums, and subcoalgebras.

In particular, we are interested in the following question: Given two coalgebras A and B, how can we determine whether they generate the same covariety ?

It turns out that it is enough to consider only homomorphic images of one-generated subcoalgebras of A and B (Corollary 2.9).

A pair consisting of a coalgebra A and a subcoalgebra B is shown to determine a class $\mathcal{Q}(A, B)$, which is a *quasi-covariety*, that is a class of coalgebras closed under homomorphic images and under sums. If A has the "extension property", then $\mathcal{Q}(A, B)$ is closed under subcoalgebras, i.e. a covariety. Assuming that cofree coalgebras exist, we show that A has the extension property iff it is a retract of a cofree coalgebra, hence every covariety arises in the above way.

Finally we propose the notion of *complete covarieties*, i.e. covarieties that are closed under bisimulation. For these classes we are able to get results like this: If A and B are coalgebras, then A and B generate the same complete covariety iff they fulfil the same formulae of an appropiate language.

The category $\mathbf{Set_F}$ of all F-coalgebras has a number of useful properties. In particular, epimorphisms are surjective and, more general, the forgetful functor from $\mathbf{Set_F}$ to $\mathbf{Set}$ creates every colimit and it creates every limit which is preserved by F (see [1]).

An important observation of [6] is that in most applications the functor F preserves "weak pullbacks". With this assumption a number of further properties can be utilized in $\mathbf{Set_F}$. For instance, monos are injective, images and preimages of subcoalgebras are subcoalgebras, and the intersection of finitely many subcoalgebras is a subcoalgebra. In order that the intersection of arbitrarily many subcoalgebras is a subcoalgebra, we need to assume that F preserves "weak generalized pullbacks" (see [2]), that is weak limits of arbitrary families $(\varphi_i : A_i \to C)_{i \in I}$ of maps with common codomain. [1]

We therefore will assume in the rest of the paper that $F : \mathbf{Set} \to \mathbf{Set}$ is a functor that preserves weak generalized pullbacks. As a consequence, for any coalgebra A and any subset $X \subseteq A$, the *coalgebra generated by* X *in* A exists. It is the intersection of all subcoalgebras of A containing X and will be denoted by $\langle X \rangle$. [2]

[1] A considerable amount of confusion has been created in much of the previous literature where authors have assumed preservation of weak pullbacks when in fact their arguments required preservation of weak generalized pullbacks. In [2] it is shown that the former requirement is not enough to even guarantee existence of 1-generated subcoalgebras

[2] In fact it would be enough for this paper to require that the functor $F : \mathbf{Set} \to \mathbf{Set}$ preserves weak pullbacks and that $\langle X \rangle$ always exists for any F-coalgebra (A, α_A) and any $X \subseteq A$ However, we do not know of any instance where such an F would not automatically preserve weak generalized pullbacks.

2 Covarieties

2.1 Conjunct representations of coalgebras

Conjunct representations of coalgebras are dual to subdirect representations of algebras. Conjunctly irreducible coalgebras will be the building blocks of which all coalgebras can be constructed by way of a conjunct repr esentation.

Definition 2.1 *A* conjunct representation *of a coalgebra* $A \in \mathbf{Set_F}$ *is a family* $(\phi_i : A_i \to A)_{i \in I \in \mathbf{Set}}$ *of homomorphisms where*

(i) *all* ϕ_i *are injective and*
(ii) $\bigcup_{i \in I} \phi_i(A_i) = A$.

Remark 2.1 *Let* $(\phi_i : A_i \to A)_{i \in I}$ *be a conjunct representation of* A *and let* $e_i : A_i \hookrightarrow \Sigma_{i \in I} A_i$ *be the canonical injections. Then there is a surjective homomorphism* $\phi : \Sigma_{i \in I} A_i \to A$, *such that all* $\phi \circ e_i$ *are injective. Therefore,* A *is called a* conjunct sum *of the* A_i.

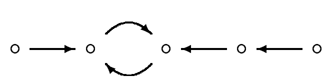
Definition 2.2 *A coalgebra* A *is called* conjunctly irreducible *if for each conjunct representation* $(\phi_i : A_i \to A)_{i \in I}$ *at least one* ϕ_i *is an isomorphism.*

Given a coalgebra A, then for every element $a \in A$, we have a natural embedding of $\langle a \rangle$, the coalgebra generated by the one-element set $\{a\}$, into A, providing us with a trivial representation of A as a conjunct sum. Thus we see immediately:

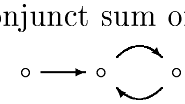
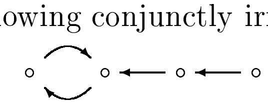
Proposition 2.1 *A coalgebra* $A \in \mathbf{Set_F}$ *is conjunctly irreducible iff it is one-generated, i.e.* $A = \langle a \rangle$ *for some* $a \in A$.

Corollary 2.2 *Each coalgebra is a conjunct sum of conjunctly irreducible subcoalgebras.*

As an example, consider coalgebras under the identity functor $I(S) = S$. These are the simplest cases of deterministic systems. Let A be the following five-element coalgebra whose transition structure is indicated by arrows:



Then A is a conjunct sum of the following conjunctly irreducible summands:

 and 

We define operators that are dual to $\mathcal{H}$, $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{P}_S$ in universal algebra.

Definition 2.3 *Let $\mathcal{K} \subseteq \mathbf{Set_F}$ be a class of F-coalgebras. We denote by*

- *$\mathcal{H}(\mathcal{K})$ the class of all homomorphic images,*
- *$\mathcal{S}(\mathcal{K})$ the class of all subcoalgebras,*
- *$\Sigma(\mathcal{K})$ the class of all sums,*
- *$\Sigma_C(\mathcal{K})$ the class of all conjunct sums*

of coalgebras in $\mathcal{K}$. By

- *$\mathcal{S}_1(\mathcal{K})$*

we denote the class of all one-generated subcoalgebras of coalgebras in $\mathcal{K}$. We write $B \leq A$, if B is a subcoalgebra of A.

One easily checks:

Lemma 2.3 *$\mathcal{H}$, $\mathcal{S}$, Σ, and Σ_C are closure operators.*

Definition 2.4 *A* covariety *is a class $\mathcal{K} \subseteq \mathbf{Set_F}$ that is closed under $\mathcal{H}$, $\mathcal{S}$ and Σ.*

In analogy to the situation in universal algebra one obtains:

Proposition 2.4 *Let $\mathcal{K} \subseteq \mathbf{Set_F}$ a class. Then*

(i) $\mathcal{SH}(\mathcal{K}) \subseteq \mathcal{HS}(\mathcal{K})$,
(ii) $\Sigma\mathcal{S}(\mathcal{K}) \subseteq \mathcal{S}\Sigma(\mathcal{K})$,
(iii) $\Sigma\mathcal{H}(\mathcal{K}) \subseteq \mathcal{H}\Sigma(\mathcal{K})$.

In particular, for each class $\mathcal{K} \subseteq \mathbf{Set_F}$,

$$\mathcal{HS}\Sigma(\mathcal{K})$$

is the least covariety that contains $\mathcal{K}$.

In universal algebra the operators $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{P}$ do not commute. In the coalgebraic context, however, we get further commutations, as shown in the following two propositions :

Proposition 2.5 *$\mathcal{S}$ and Σ commute.*

Proof. Proposition 2.4 yields $\Sigma\mathcal{S}(\mathcal{K}) \subseteq \mathcal{S}\Sigma(\mathcal{K})$, so it remains to show that $\mathcal{S}\Sigma(\mathcal{K}) \subseteq \Sigma\mathcal{S}(\mathcal{K})$.

Let ϕ be an embedding of A into $\Sigma_i B_i$, $B_i \in \mathcal{K}$, and let e_i be the canonical injection from B_i into $\Sigma_i B_i$. Then the following diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] & \Sigma_i B_i \\ & B_i \arrow[u, "e_i"'] \end{tikzcd}$$

can be completed commutatively. To see this, form for each i the pullback of ϕ and e_i in the category of sets. This results in the set

$$R_i = \{(a, b_i) \mid \phi(a) = e_i(b_i), a \in A, b_i \in B_i\}$$

with canonical projections π_i^1 and π_i^2. They are injective because ϕ and e_i are.

Now, according to [6], each R_i is a bisimulation, so it can be equipped with a transition structure, turning R_i into a coalgebra in such a way that π_i^1 and π_i^2 are homomorphisms.

$$\begin{tikzcd} A \arrow[r, "\phi"] & \Sigma_i B_i \\ R_i \arrow[u, "\pi_i^1"] \arrow[r, "\pi_i^2"'] & B_i \arrow[u, "e_i"'] \end{tikzcd}$$

We now consider $\Sigma_i R_i$ and claim that this coalgebra is isomorphic to A. Indeed, let ε be the homomorphism defined by the π_i^1, then the following diagram commutes for each i:

$$
\begin{tikzcd}
A \arrow[rr, "\phi"] & & \Sigma_i B_i \\
& \Sigma_i R_i \arrow[ul, "\varepsilon"'] \arrow[ur, "\phi \circ \varepsilon"'] & \\
R_i \arrow[uu, "\pi_i^1"] \arrow[ur, "f_i"'] \arrow[rr, "\pi_i^2"'] & & B_i \arrow[uu, "\mathrm{e}_i"']
\end{tikzcd}
$$

For every $a \in A$ there exists an index i and some $b \in B_i$ with $\phi(a) = e_i(b)$. Thus $(a, b) \in R_i$, so $a = \pi_i^1(a, b) = \varepsilon(f_i(a, b))$, hence ε is onto.

Next assume that there are $x, x' \in \Sigma_i R_i$ with $\phi \circ \varepsilon(x) = \phi \circ \varepsilon(x')$, then there are indices j, k and elements $r \in R_j$ and $r' \in R_k$ with $f_j(r) = x$ and $f_k(r') = x'$. Therefore $e_j \circ \pi_j^2(r) = e_k \circ \pi_k^2(r')$. It follows that $j = k$, and $r = r'$, since e_j and π_j^2 are injective. Therefore ε is injective.

Proposition 2.6 *$\mathcal{H}$ and $\mathcal{S}$ commute.*

Proof. $\mathcal{SH}(\mathcal{K}) \subseteq \mathcal{HS}(\mathcal{K})$ by proposition 2.4. For $A \in \mathcal{HS}(\mathcal{K})$ there exists $C \in \mathcal{K}$, a monomorphism $e : B \to C$, and a surjective homomorphism $\phi : B \to A$. Pushouts exist in $\mathbf{Set_F}$, so let D be the pushout of e and ϕ.

$$
\begin{tikzcd}
\mathrm{B} \arrow[r, "e"] \arrow[d, "\phi"'] & C \arrow[d, "p_2"] \\
A \arrow[r, "p_1"'] & D
\end{tikzcd}
$$

Pushouts of epis are always epi, so p_2 is epi. The forgetful functor $U : \mathbf{Set_F} \to \mathbf{Set}$ creates colimits [6] and in $\mathbf{Set}$ pushouts of injective maps are injective, hence p_1 is injective. This shows that A is isomorphic to a subcoalgebra of a homomorphic image of $C \in \mathcal{K}$, i.e. $A \in \mathcal{SH}(\mathcal{K})$.

2.3 The covariety generated by a coalgebra

In order to see whether two coalgebras A and B generate the same covariety, we need only study their one-generated coalgebras. This is already suggested by Proposition 2.1.

For the operator $\mathcal{S}_1$ the following equalities are immediate:

- $\mathcal{S}_1\mathcal{H}(\mathcal{K}) = \mathcal{H}\mathcal{S}_1(\mathcal{K})$
- $\mathcal{S}_1\Sigma(\mathcal{K}) = \mathcal{S}_1(\mathcal{K})$

Corollary 2.7 *Every one-generated coalgebra in $\mathcal{H}\mathcal{S}\Sigma(\mathcal{K})$ is already an element of $\mathcal{H}\mathcal{S}_1(\mathcal{K})$.*

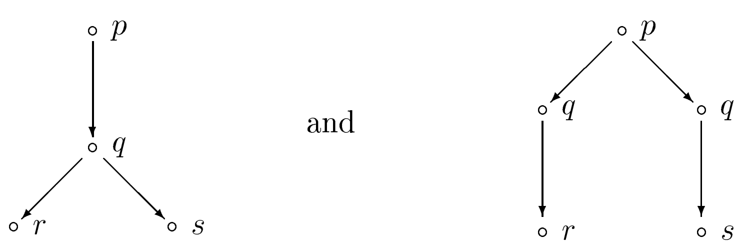
This yields a useful description of the covariety generated by a class $\mathcal{K}$ of coalgebras.

Proposition 2.8 *The least covariety that contains $\mathcal{K}$ is $\Sigma_C\mathcal{H}\mathcal{S}(\mathcal{K})$, more precisely $\Sigma_C\mathcal{H}\mathcal{S}_1(\mathcal{K})$.*

Corollary 2.9 *Let $\mathcal{K}_1, \mathcal{K}_2 \subseteq$ $\mathbf{Set_F}$ be classes of covarieties. The covariety generated by $\mathcal{K}_1$ is contained in the covariety generated by $\mathcal{K}_2$ if and only if*

$$\mathcal{S}_1(\mathcal{K}_1) \subseteq \mathcal{H}\mathcal{S}_1(\mathcal{K}_2).$$

This criterion is easy to check. For instance, we see immediately that the following two Kripke structures



generate different covarieties.

3 The definition of covarieties by homomorphisms

From an arbitrary coalgebra A and any subcoalgebra $B \leq A$ we are going to define a *quasi-covariety*, that is a class of coalgebras closed under homomorphic images and sums.

Definition 3.1 *Let $A, B \in \mathbf{Set_F}$ be coalgebras, $B \leq A$. The class $\mathcal{Q}(A,B)$ is defined as the class of all coalgebras $C \in \mathbf{Set_F}$ with the property that each homomorphism $\phi : C \to A$ factors through B, i.e. that*

$$\phi(C) \subseteq B.$$

$\mathcal{Q}(A,B)$ is not necessarily a covariety but we have:

Proposition 3.1 *$\mathcal{Q}(A,B)$ is a quasi-covariety.*

Proof. Let $C \in \mathcal{Q}(A,B)$, $\psi : C \twoheadrightarrow C'$ a surjective homomorphism, $\phi : C' \to A$ a homomorphism. Since $C \in \mathcal{Q}(A,B)$, there exists a homomorphism $\widetilde{\phi \circ \psi}$ with $\phi \circ \psi = \widetilde{\phi \circ \psi} \circ \leq$.

We need a morphism $\tilde{\phi}$ "splitting" the following diagram:

$$\begin{tikzcd}
C' \arrow[r, "\phi"] \arrow[dr, "\tilde{\phi}"] & A \\
C \arrow[u, "\psi"] \arrow[r, "\widetilde{\phi \circ \psi}"'] & B \arrow[u, "\leq"']
\end{tikzcd}$$

Obviously, $\operatorname{Ker}\psi \subseteq \operatorname{Ker}\widetilde{\phi \circ \psi}$, so there is exactly one homomorphism $\tilde{\phi}$, making the bottom left triangle commute. Since the outer rectangle commutes we also have

$$\phi \circ \psi \ = \leq \circ\, \widetilde{\phi \circ \psi} \ = \leq \circ\, \tilde{\phi} \circ \psi.$$

ψ is epi, so

$$\phi \ = \leq \circ\, \tilde{\phi},$$

which means that the upper right triangle commutes. Thus $C' \in \mathcal{Q}(A,B)$.

Let now $A_i \in \mathcal{Q}(A,B)$ for $i \in I$. We need to show that $\Sigma_{i\in I} A_i \in \mathcal{Q}(A,B)$. For an arbitrary morphism $\phi : \Sigma_{i\in I} A_i \to A$, let e_i, for each $i \in I$, be the canonical injection of A_i into the sum. Since $A_i \in \mathcal{Q}(A,B)$, $\phi \circ e_i$ factors through B via some $\widetilde{\phi \circ e_i}$.

The universal property of the sum yields now exactly one homomorphism $\tilde{\phi} : \Sigma_i A_i \to B$ so that for each i the bottom left triangle in the following diagram commutes:

$$\begin{tikzcd}
\Sigma_i A_i \arrow[r, "\phi"] \arrow[dr, "\tilde{\phi}"] & A \\
A_i \arrow[u, "\mathrm{e}_i"] \arrow[r, "\widetilde{\phi \circ \mathrm{e}_i}"'] & B \arrow[u, "\leq"']
\end{tikzcd}$$

For each $i \in I$ we have

$$\phi \circ e_i \ = \leq \circ \ \widetilde{\phi \circ e_i} \ = \leq \circ \ \tilde{\phi} \circ e_i,$$

so

$$\phi \ = \leq \circ \ \tilde{\phi}$$

follows from the universal property of the sum.

3.2 Invariance and the Extension Property

Given a coalgebra A, there may be different subcoalgebras B, B', giving rise to the same quasi-covariety. Amongst those we can always choose one which is “invariant”, in a sense to be defined below. Next, we discuss a property of A that guarantees that $\mathcal{Q}(A, B)$ is closed under subcoalgebras, i.e. a covariety. In that case, different invariant subcoalgebras of A produce different covarieties.

Definition 3.2 *Let A be a coalgebra, $B \leq A$. B is called* invariant *in A, if for each homomorphism $\phi : A \to A$ we have:*

$$\phi(B) \subseteq B.$$

Proposition 3.2 *If $B \in \mathcal{Q}(A, B)$ holds, then B is invariant in A.*

Proof. Assume $\phi : A \to A$. Let $\phi_{|B} : B \to A$ be the restriction of ϕ to B. As $B \in \mathcal{Q}(A, B)$, $\phi_{|B}$ must factor through B, which is to say $\phi(B) \subseteq B$.

Proposition 3.3 *For each $B \leq A$ there exists an invariant subcoalgebra $B^o \leq B$ with*

$$\mathcal{Q}(A, B) = \mathcal{Q}(A, B^o).$$

Proof. We can choose

$$B^o = \bigcup\{f(C) | C \in \mathcal{Q}(A, B), f \in \mathrm{Hom}(C, A)\}.$$

Evidently,

$$\mathcal{Q}(A, B) = \mathcal{Q}(A, B^o).$$

Let $\phi : A \to A$ be an endomorphism, $b \in B^o$. Then there is a $C \in \mathcal{Q}(A, B)$ and a homomorphism $f : C \to A$ with $b \in f(C)$. Since also $C \in \mathcal{Q}(A, B^o)$ the map $\phi \circ f$ factors through B^o, i.e. $\phi(f(C)) \subseteq B^o$. Hence $\phi(b) \in B^o$.

Definition 3.3 *A coalgebra A has the* extension property*, if for all coalgebras C we have: If $C \leq C'$ then any homomorphism $f : C \to A$ can be extended to a homomorphism $\hat{f} : C' \to A$.*

Example 3.1 *Every final and every cofree coalgebra (see definition 3.5) has the extension property.*

Proposition 3.4 *Let A have the extension property, $B \leq A$. Then $\mathcal{Q}(A, B)$ is a covariety.*

Proof. According to proposition 3.1 it remains to prove that $\mathcal{Q}(A, B)$ is closed under subcoalgebras: Let $C \in \mathcal{Q}(A, B)$, $D \leq C$, and $\phi : D \to A$ a homomorphism. Then there is an $\hat{\phi} : C \to A$ extending ϕ to C. $\hat{\phi}$ factors through B via a homomorphism ψ since $C \in \mathcal{Q}(A, B)$. Now we can set $\tilde{\phi} \ := \ \psi \circ \leq$. Then

$$\phi \ = \ \hat{\phi} \circ \leq = \leq \circ\, \psi \circ \leq = \leq \circ\, \tilde{\phi}$$

The following diagram illustrates the situation:

$$\begin{tikzcd}
C \arrow[r, "\hat{\phi}"] \arrow[dr, "\psi"] & A \\
D \arrow[u, "\leq"] \arrow[r, "\tilde{\phi}"'] & B \arrow[u, "\leq"']
\end{tikzcd}$$

Proposition 3.5 *Let A be a coalgebra with extension property and $B \leq A$. If B is invariant in A then $B \in \mathcal{Q}(A, B)$.*

Proof. Let $\phi : B \to A$ be a homomorphism. The extension property for A allows us to extend ϕ to a homomorphism $\hat{\phi} : A \to A$. B being invariant in A yields $\phi(B) = \hat{\phi}_{|B}(B) \subseteq B$.

Proposition 3.6 *Let A have the extension property. If $B \subset B'$ and B' is invariant in A then*

$$\mathcal{Q}(A, B) \subset \mathcal{Q}(A, B').$$

Proof. $\subseteq$ is obvious. Clearly, if B' were in $\mathcal{Q}(A, B)$, then the inclusion morphism $\leq: B' \to A$ would factor through B, yielding $B' = B$. Thus $B' \notin \mathcal{Q}(A, B)$, yet $B' \in \mathcal{Q}(A, B')$ according to proposition 3.5.

Summarizing, we have:

Theorem 3.7 *Let A be a coalgebra, $B \leq A$. The class $\mathcal{Q}(A, B)$ is closed under homomorphic images and sums. If A has the extension property then $\mathcal{Q}(A, B)$ is a covariety. For fixed A the covarieties $\mathcal{Q}(A, B)$ correspond exactly to the invariant subcoalgebras of A.*

3.3 Cofree coalgebras and bounded functors

In this section we will see that each covariety has the form $\mathcal{Q}(A, B)$ if the functor F has an additional property. This is a restatement of a result of Rutten ([6]).

Definition 3.4 *The functor F is called* bounded*, if there is set C such that the cardinality of each one-generated subcoalgebra in $\mathbf{Set_F}$ is bounded by the cardinality of C. In this case we call F* bounded by *C.*

Definition 3.5 *Let C be a set. An F-Coalgebra $(A, \alpha_A) \in \mathbf{Set_F}$ is called* cofree over *C if there is a map $\varepsilon_C : A \to C$ such that for every F-Coalgebra $(B, \alpha_B) \in \mathbf{Set_F}$ and every map $\phi : B \to C$ there is exactly one homomorphism $\tilde{\phi} : B \to A$ with $\varepsilon_C \circ \tilde{\phi} = \phi$. ε_C is often called the "color map".*

This means that $(A, (\alpha_A, \varepsilon_C))$ is a final $F \times C$-coalgebra. If F is bounded then for each set C there exists a cofree coalgebra over C.

We now restate in our language Rutten's theorem:

Proposition 3.8 ([6]) *Let F be bounded by a set C. Then every F-covariety has the form $\mathcal{Q}(S_C, B)$ where S_C is a cofree coalgebra over C and B a subcoalgebra of S_C.*

Definition 3.6 *A pair of morphisms $\iota : A \to B$ and $\pi : B \to A$ is called a* retraction *if $\pi \circ \iota = \mathrm{id}_A$. In this case A is called a* retract *of B.*

We can now characterize coalgebras with the extension property:

Proposition 3.9 *Let F be a bounded functor and A a coalgebra in $\mathbf{Set_F}$. Then A has the extension property if and only if it is a retract of some cofree coalgebra.*

Proof. Suppose that A has the extension property. Denote by $|A|$ the base set of A and consider the cofree coalgebra S over the set $|A|$ with color map $\varepsilon_{|A|} : S \to |A|$. The map $\mathrm{id}_A : A \to |A|$ yields a unique homomorphism $\iota : A \to S$ with $\varepsilon_{|A|} \circ \iota = \mathrm{id}_A$, so ι is injective. Thus the subcoalgebra $C = \iota(A)$ of F is isomorphic to A and we can write $\iota = \leq \circ \phi$ where $\phi : A \to C$ agrees with ι on all elements of A. Since A has the extension property, the inverse $\phi^{-1} : C \to A$ can be extended to a homomorphism $\pi : S \to A$ with $\pi \circ \leq = \phi^{-1}$, hence $\pi \circ \iota = \pi \circ \leq \circ \phi = \phi^{-1} \circ \phi = \mathrm{id}_A$.

$$\begin{tikzcd} S \arrow[rrr, bend left, "\pi"] \arrow[drr, "\varepsilon_{|A|}"'] & \geq & C \arrow[r, shift left, "\phi^{-1}"] & A \arrow[l, shift left, "\phi"] \arrow[dl, "id_A"] \\ & & {|A|} & \end{tikzcd}$$

Assume now that S is cofree over the color set C and that A is a retract of S with retraction pair $\iota : A \to S$ and $\pi : S \to A$, satisfying $\pi \circ \iota = \mathrm{id}_A$. Let $\phi : D \to A$ be any homomorphism and $D \leq D'$. We must extend ϕ to a homomorphism $\phi' : D' \to A$. Define a map $\delta : D \to C$ as $\varepsilon_C \circ \iota \circ \phi$, and extend it to a map $\delta' : D' \to C$ so that $\delta = \delta' \circ \leq$. Let $\psi : D' \to S$ be the unique homomorphism with $\varepsilon_C \circ \psi = \delta'$. Then we calculate

$$\varepsilon_C \circ \psi \circ \leq = \delta' \circ \leq = \delta = \varepsilon_C \circ \iota \circ \phi.$$

It is easy to see that ε_C can always be left-cancelled, hence

$$\psi \circ \leq = \iota \circ \phi,$$

so we finally set $\phi' = \pi \circ \psi$, and calculate

$$\phi' \circ \leq = \pi \circ \psi \circ \leq = \pi \circ \iota \circ \phi = \mathrm{id}_A \circ \phi = \phi.$$

4 Complete Covarieties

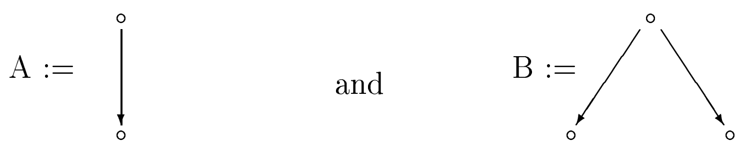
4.1 Total Bisimulations

When dealing with transition systems one usually does not distinguish between systems that are bisimilar. A bisimulation R between coalgebras A and B is defined as a relation $R \subseteq A \times B$ on which a coalgebra structure can be defined so that the projections $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are homomorphisms. If additionally π_1 and π_2 are surjective then we shall call R a *total bisimulation.*

Notice that a homomorphism $\phi : A \to B$, viewed as a subset of $A \times B$ is a bisimulation; this is total iff ϕ is surjective. We sometimes write $(\mathrm{Gr}\,\phi)$, resp. $(\mathrm{Gr}\,\phi)^{-1}$ for the relation given by ϕ resp. for the inverse of this relation.

We shall now consider classes of F-coalgebras which are not only closed under $\mathcal{H}$, $\mathcal{S}$, and Σ but beyond this under total bisimulations. Such a class is called a *complete covariety.*

Each complete covariety is of course a covariety. The reversal is not true as can be seen in the following example of $\mathcal{P}$-coalgebras:



By corollary 2.9, $B \notin \mathcal{HS}\Sigma(\{A\})$, even though there is a total bisimulation between A and B.

Definition 4.1 *For a class $\mathcal{K} \subseteq \mathbf{Set_F}$ of coalgebras we define $\mathcal{B}(\mathcal{K})$ as the class of all coalgebras for which there is a total bisimulation with some coalgebra in $\mathcal{K}$.*

We have already mentioned, that a surjective homomorphism between coalgebras A and B is a total bisimulation. Conversely, if R is a total bisimulation between A and B, then R is a coalgebra together with surjective homomorphisms onto both A and B. It follows that B is a homomorphic image of a homomorphic preimage of A. This proves the following reduction:

Proposition 4.1 *A class $\mathcal{K}$ is closed under total bisimulations if and only if it is closed under homomorphic images and under homomorphic preimages.*

Basic properties of the operator $\mathcal{B}$ are summed up in the following proposition whose proof is straightforward:

Proposition 4.2 *Let $\mathcal{K} \subseteq \mathbf{Set_F}$ be a class, then*

(i) $\mathcal{B}$ *is a closure operator,*
(ii) $\mathcal{H}(\mathcal{K}) \subseteq \mathcal{B}(\mathcal{K})$,
(iii) $\mathcal{S}\mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}\mathcal{S}(\mathcal{K})$,
(iv) $\Sigma\mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}\Sigma(\mathcal{K})$.

Corollary 4.3 *Let $\mathcal{K} \subseteq \mathbf{Set_F}$ be a class. The smallest complete covariety containing $\mathcal{K}$ is*

$$\mathcal{B}\mathcal{S}\Sigma(\mathcal{K}).$$

This description can be refined as in the case of covarieties:

Proposition 4.4 *For all classes $\mathcal{K} \subseteq \mathbf{Set_F}$ we have*

$$\mathcal{S}_1\mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}\mathcal{S}_1(\mathcal{K}).$$

Thus we obtain a description of the complete covarieties analogous to proposition 2.8:

Proposition 4.5 *Let $\mathcal{K} \subseteq \mathbf{Set}_F$ be a class. The complete covariety generated by $\mathcal{K}$ is*

$$\Sigma_C\mathcal{B}\mathcal{S}_1(\mathcal{K}).$$

Proof. By corollary 4.3, $\mathcal{B}\mathcal{S}\Sigma(\mathcal{K})$ is the smallest complete covariety containing $\mathcal{K}$. According to corollary 2.8 this is contained in

$$\Sigma_C\mathcal{H}\mathcal{S}_1(\mathcal{B}\mathcal{S}\Sigma(\mathcal{K})).$$

On the other hand:

$$\begin{aligned}\Sigma_C\mathcal{H}\mathcal{S}_1(\mathcal{B}\mathcal{S}\Sigma(\mathcal{K})) &\subseteq \Sigma_C\mathcal{H}\mathcal{B}\mathcal{S}_1\mathcal{S}\Sigma(\mathcal{K})\\ &= \Sigma_C\mathcal{H}\mathcal{B}\mathcal{S}_1\Sigma(\mathcal{K})\\ &= \Sigma_C\mathcal{B}\mathcal{S}_1\Sigma(\mathcal{K})\\ &= \Sigma_C\mathcal{B}\mathcal{S}_1(\mathcal{K}).\end{aligned}$$

Like any variety, complete covarieties can also be written in the form $\mathcal{Q}(A, B)$, provided that F is bounded. We shall now study for which choices of A and B we have that $\mathcal{Q}(A, B)$ is a complete covariety.

Let $\bar{\mathcal{F}}$ be the final F-coalgebra. For each F-coalgebra A let s_A be the unique homomorphism $A \to \bar{\mathcal{F}}$.

Proposition 4.6 *Let $\mathcal{K} \subseteq \mathbf{Set_F}$ be a class. $\mathcal{K}$ is a complete covariety if and only if there is a $U \leq \bar{\mathcal{F}}$ with $\mathcal{K} = \mathcal{Q}(\bar{\mathcal{F}}, U)$.*

Proof. Let $\mathcal{K} = \mathcal{Q}(\bar{\mathcal{F}}, U)$ for a $U \leq \bar{\mathcal{F}}$. U is invariant in $\bar{\mathcal{F}}$, since $\mathrm{id}_{\bar{\mathcal{F}}}$ is the only $\bar{\mathcal{F}}$-endomorphism.

$\mathcal{K}$ is a covariety since $\bar{\mathcal{F}}$ has the extension property. We need to show $\mathcal{B}(\mathcal{K}) \subseteq \mathcal{K}$: Let $B \in \mathcal{B}(\mathcal{K})$, then there exists an $A \in \mathcal{K}$ and a total bisimulation between B and A. In particular, the largest bisimulation $\sim$ between B and A is total. Therefore, for every $b \in B$ there exists an $a \in A$ with $a \sim b$. But because $\bar{\mathcal{F}}$ is final this means

$$s_B(b) = s_A(a) \in U,$$

therefore $B \in \mathcal{Q}(\bar{\mathcal{F}}, U)$.

To prove the other direction we set

$$U := \bigcup_{A \in \mathcal{K}} s_A(A).$$

Clearly, $U \leq \bar{\mathcal{F}}$, so we claim

$$\mathcal{K} = \mathcal{Q}(\bar{\mathcal{F}}, U).$$

$\subseteq$ is obvious. Let $A \in \mathcal{Q}(\bar{\mathcal{F}}, U)$; then $s_A(A) \leq U$ holds. But we have $U \in \mathcal{K}$ since for each $u \in U$ there is a $b \in B \in \mathcal{K}$ with $\langle u \rangle \sim \langle b \rangle$. This shows $U \in \mathcal{BS}\Sigma(\mathcal{K}) = \mathcal{K}$ and therefore also $A \in \mathcal{K}$.

5 Coalgebraic Logic

Definition 5.1 *A class $\mathcal{L}$ (of formulae) is called an F-*language *if for each $A \in \mathbf{Set_F}$ we have a satisfaction relation*

$$\models_A \subseteq A \times \mathcal{L}.$$

For $a \models_A \phi$ we say that ϕ holds in a. *We write $A \models \phi$ if $a \models_A \phi$ for all $a \in A$. If $\mathcal{K} \subseteq \mathbf{Set_F}$ is a class, we write $\mathcal{K} \models \phi$ if $A \models \phi$ for all $A \in \mathcal{K}$.*

We call $\models$ (or $\mathcal{L}$) homomorphism-stable *if for all $A, B \in \mathbf{Set_F}$, all $a \in A$, and all homomorphisms $f : A \to B$:*

$$a \models_A \phi \iff f(a) \models_B \phi.$$

Proposition 5.1 *Let $\models$ be homomorphism-stable, and let R be a bisimulation between A and B from $\mathbf{Set_F}$. For all $a \in A$, $b \in B$ we then have:*

$$aRb \Rightarrow (a \models_A \phi \iff b \models_B \phi).$$

In particular, if $U \leq A$ then for all $u \in U$:

$$u \models_U \phi \iff u \models_A \phi.$$

and for families $(A_i)_{i \in I} \subseteq \mathbf{Set_F}$ we have for all $i \in I$ and all $a \in A_i$:

$$a \models_{\Sigma_{i \in I} A_i} \phi \iff a \models_{A_i} \phi.$$

Proof. Because R is a bisimulation the two projections $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are homormorphisms. Thus, if $(a,b) \in R$ then

$$(a \models_A \phi \iff (a,b) \models_R \phi \iff b \models_B \phi).$$

Languages that are nothing but homomorphism-stable are not quite interesting because we can obtain trivial examples such as $\models_A = \emptyset$ for all $A \in \mathbf{Set_F}$. We need a second property:

Definition 5.2 *An F-language $\mathcal{L}$ is called* characterizing *if for each $a \in A \in \mathbf{Set_F}$ there is a formula $\Phi(A,a) \in \mathcal{L}$ such that $a \models_A \Phi(A,a)$ holds and for each $b \in B \in \mathbf{Set_F}$:*

$$b \models_B \Phi(A,a) \Rightarrow b \sim a.$$

If $\mathcal{L}$ is homomorphism-stable, then in order to prove that $\mathcal{L}$ is characterizing it suffices to prove that we can characterize the elements of the final F-coalgebra $\bar{\mathcal{F}}$:

Proposition 5.2 *Let $\mathcal{L}$ be homomorphism-stable. For each $a \in \bar{\mathcal{F}}$ let there be a $\Phi(\bar{\mathcal{F}},a) \in \mathcal{L}$ with $a \models_{\bar{\mathcal{F}}} \Phi(A,a)$ and*

$$\forall b \in \bar{\mathcal{F}}.(b \models \Phi(\bar{\mathcal{F}},a) \Rightarrow b = a)$$

($\bar{\mathcal{F}}$ is simple, so $b = a$ is equivalent to $b \sim a$.) Then $\mathcal{L}$ is characterizing, and for $A \in \mathbf{Set_F}$ we can choose

$$\Phi(A,a) := \Phi(\bar{\mathcal{F}}, s_A(a)).$$

Proof. Let $b \in B \in \mathbf{Set_F}$. From

$$b \models \Phi(A,a) = \Phi(\bar{\mathcal{F}}, s_A(a))$$

we infer

$$s_B(b) = s_A(a)$$

because $\mathcal{L}$ is homomorphism-stable, and it follows $a \sim b$ by the bisimulation $(\mathrm{Gr}\, s_A) \circ (\mathrm{Gr}\, s_B)^{-1}$.

A simple – although not very interesting – example for a language that is homomorphism-stable and characterizing can be achieved by

$$\mathcal{L} := \bar{\mathcal{F}},$$

$$a \models_A \phi :\iff s_A(a) = \phi$$

for all F-coalgebras (A, α_A) and all $a \in A$.

6 Characterization of complete covarieties

Let $\mathcal{L}$ be an F-language. If $\mathcal{L}$ is characterizing (by means of the formula $\Phi(A, a)$) and homomorphism-stable, we can construct a language $\mathcal{L}' \supseteq \mathcal{L}$ as follows: $\mathcal{L}'$ contains all formulae

$$\bigvee_{i \in I} \phi_i, \; I \in \mathbf{Set}, \phi_i \in \mathcal{L}$$

with the evident semantics. $\mathcal{L}'$ is characterizing and homomorphism-stable because $\mathcal{L}$ is. With this language $\mathcal{L}'$ we can define characterizing formulae for coalgebras $A \in \mathbf{Set_F}$: We set

$$\Phi(A) := \bigvee_{a \in A} \Phi(A, a)$$

Evidently $A \models \Phi(A)$ holds, and we have

Proposition 6.1 *For each $A, B \in \mathbf{Set_F}$ we have:*

$$(\forall \phi \in \mathcal{L}' : A \models \phi \Rightarrow B \models \phi) \iff B \models \Phi(A) \iff B \in \mathcal{BS}\Sigma(A).$$

More generally we have for a class $\mathcal{K} \subseteq \mathbf{Set_F}$:

$$(\forall \phi \in \mathcal{L}' : ((\forall A \in \mathcal{K} : A \models \phi) \Rightarrow B \models \phi)) \iff B \in \mathcal{BS}\Sigma(\mathcal{K}).$$

L. Moss has constructed in [5] characterizing and homomorphism-stable languages for a large class of functors.

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