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Ideals in universal algebras

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Dedicated to Alfred Tarski on his 80th birthday

0. Introduction

In many familiar classes of algebraic structures kernels of congruence relations are uniquely specified by the inverse images $\varphi^{-1}(0)=\{x \mid \varphi(x)=0\}$ of a specified constant 0. On the one hand, $\varphi^{-1}(0)$ is nothing else but the 0-class of the kernel congruence of φ, on the other hand $\varphi^{-1}(0)$ can be axiomatized intrinsically, namely $\varphi^{-1}(0)$ is an ideal (in rings, Boolean algebras, or more generally in Heyting algebras), a normal subgroup, resp. normal subloop (in groups, resp. loops) or a filter (in Implication algebras or Boolean algebras again, where 0 is replaced by the unit). In this paper we investigate common features of all the above structures by using a general notion of "ideal", which makes sense in all universal algebras having a constant 0 and which specializes to the familar concepts of ideal, normal subgroup or filter in each of the algebras quoted above. In all universal algebras the 0-classes of congruence relations are easily seen to be ideals, but we shall require that conversely each ideal is the 0-class of a unique congruence relation. Such algebras, or rather classes of algebras with this property will be called "classes with ideal determined congruences" or shortly *ideal determined.*

In Part 1, after presenting the precise definitions, we shall show that the ideal determined varieties are characterized by a Mal'cev condition, which turns out to be a combination of Fichtner's condition for 0-regularity together with a ternary term $r(x, y, z)$ which is a weakened form of Mal'cev's permutability term. From a result of Hagemann it follows that ideal-determined varieties have modular congruence lattices, so the theory of commutators becomes readily available. In

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Part 2 we study the commutator $[I, J]$ of two ideals. Introducing the concept of commutator terms we can describe $[I, J]$ as the set of all elements which result from applying commutator terms to I and J.

Acknowledgement

The notion of ideals for multioperator groups was invented by Higgins [7], it was used by Magari [9] and defined for general algebras in Ursini [12].

1. Definitions and the Mal'cev type characterization

Let K be a class of universal algebras for a fixed type Δ. We require throughout that all algebras do have a constant 0 which is either in Δ or at least equationally defined. 0 and K remain fixed throughout. If we agree to abbreviate the n-tuple $(x_1, \ldots, x_n)$ by $\vec{x}$ we may define:

1.1 DEFINITION. (i) A term $p(\vec{x}, \vec{y})$ is called an *ideal term in* $\vec{y}$ if $p(\vec{x}, \vec{0}) \approx 0$ is an identity in $\mathcal{K}$.

(ii) A nonempty subset I of an algebra $\mathcal{A} \in \mathcal{K}$ is an *ideal* if for every ideal term $p(\vec{x}, \vec{y})$ in $\vec{y}$, $\vec{a} \in A^n$, $\vec{i} \in I^m$ we have $p(\vec{a}, \vec{i}) \in I$.

These definitions are from Ursini [12]. We note immediately that the composition of ideal terms yields ideal terms again, so to check whether a subset of an algebra is an ideal, it is enough to verify Definition 1.1 (ii) with $p(\vec{x}, \vec{y})$ ranging over enough ideal terms from which all others can be obtained by composition.

In the case of rings it is sufficient to use 0, $y_1 - y_2$, $x_1 \cdot y_1$, $y_1 \cdot x_1$. In the case of groups we have to consider 1, $y_1 \cdot y_2^{-1}$, $x_1 y_1 x_1^{-1}$. The following Lemma is obvious:

1.2 LEMMA. *The intersection of ideals is an ideal and for any set $S \subseteq A$, the ideal generated by S in A consists of all $p(\vec{a}, \vec{s})$ where $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a} \in A^n$, $\vec{s} \in S^n$. In particular, the set of all ideals of the algebra $\mathcal{A}$ is an algebraic lattice.*

Each ideal has to contain 0, so typical examples of ideals are congruence classes containing 0; indeed it is trivial to check for each congruence Θ, that $[0]\Theta := \{x \in A \mid x\Theta 0\}$ is an ideal. The converse fails to hold generally, so we define.

1.3 DEFINITION. The class $\mathcal{K}$ is *ideal determined*, if every ideal I is the 0-class of a unique congruence relation I^δ. In this case $-^\delta$ establishes an isomorphism between congruence and ideal lattices. We remark that uniqueness in the above definition is indispensable if trivial examples like e.g. pointed sets are to be avoided.

If $\mathcal{K}$ is an ideal determined class, sums of ideals I and J are easy to describe, namely setting $[I]J^{\delta} := \{s \in A \mid (i, s) \in J^{\delta}$ for some $i \in I\}$, we find:

1.4 LEMMA. *$I+J=[I]J^{\delta}=[J]I^{\delta}$, where $I+J$ is the supremum of the ideals I and J.*

Proof. $I+J$ is the ideal generated by $I\cup J$, so for $s\in I+J$ there is an ideal term $p(\vec{x}, \vec{y})$ such that $s=p(\vec{a}, \vec{k})$ where $\vec{a}\in A^n$ and $\vec{k}\in(I\cup J)^m$. We may write $s=p(\vec{a}, i_1, \ldots, i_r, j_{r+1}, \ldots, j_m)$ where the i's are in I and the j's are in J. With $b=p(\vec{a}, i_1, \ldots, i_r, 0, \ldots, 0)$ we have $b\in I$ and $(b, s)\in J^{\delta}$. The other inclusion is trivial.

1.4 COROLLARY. *Let θ and ψ be congruences, then $(0, x)\in \Theta \vee \psi$ if and only if $(0, x)\in \Theta \circ \psi$*

Hence, and since congruence classes are uniquely determined by their 0-class, B. Jónsson's [8] proof that permutability of congruences implies the arguesian law carries through without any change, yielding:

1.5 COROLLARY: *If $\mathcal{K}$ is an ideal determined class, then the congruence (ideal) lattices of algebras in $\mathcal{K}$ are arguesian, and hence also modular.*

We are now going to give a Mal'cev type characterization for *varieties* $\mathcal{K}$ with ideal determined congruences.

The notion of ideal determined congruences requires that congruences are uniquely determined by their 0-classes. A class $\mathcal{U}$ is called 0-regular if congruences on algebras in $\mathcal{U}$ are uniquely determined by their 0-classes. Varieties which are 0-regular have been characterized:

1.6 (Fichtner [1]). If $\mathcal{K}$ is a variety then $\mathcal{K}$ is 0-regular if and only if there exists a natural number n, binary terms $d_1, \ldots, d_n$ and quaternary terms $q_1, \ldots, q_n$ such that the equations

$$d_i(x, x)=0 \quad \text{for} \quad 1\le i\le n$$

$$x=q_1(x, y, 0, d_1(x, y))$$

$$q_i(x, y, d_i(x, y), 0)=q_{i+1}(x, y, 0, d_{i+1}(x, y)) \quad \text{for} \quad 1\le i<n$$

$$q_n(x, y, d_n(x, y), 0)=y$$

Another way to phrase 1.6 is:

1.7 COROLLARY. *A variety* $\mathcal{K}$ *is* 0*-regular iff there exist binary terms* $d_1, \ldots, d_n$ *such that the equations*

$$d_i(x, x) = 0 \quad \text{for} \quad 1 \le i \le n$$

and the implication

$$d_1(x, y) = d_2(x, y) = \cdots = d_n(x, y) = 0 \Rightarrow x = y$$

hold in $\mathcal{K}$.

Given a variety of universal algebras, then to check whether it is ideal determined seems to require the effort of finding all ideal terms. However, the following theorem, which is the main result of part 1. makes the task of identifying ideal determined classes very easy:

1.8 THEOREM. *A variety* $\mathcal{K}$ *with a constant* 0 *is ideal determined, if and only if it is* 0*-regular and there exists a ternary term* $r(x, y, z)$ *such that*

$$r(x, x, y) = y$$
$$r(0, x, x) = 0$$

are equations in $\mathcal{K}$.

P. Köhler helped us to find the following corollary:

1.9 COROLLARY. *A variety* $\mathcal{K}$ *with constant* 0 *is ideal determined iff it is* 0*-regular and there exists a binary term* $s(x, y)$ *such that*

$$s(x, x) = 0$$
$$s(0, x) = x$$

are equations in $\mathcal{K}$.

This follows immediately from 1.8 if one defines $s(x, y) := r(0, x, y)$ and conversely: $r(x, y, z) := s(s(x, y), z)$. Yet another corollary arises if we take 0-regularity and add the property of 1.4 which we will denote by *permutability at* 0 for short:

1.10 COROLLARY. *A variety $\mathcal{K}$ with constant* 0 *is ideal determined iff it is* 0-*regular and permutable at* 0.

It is easy to see that a variety is permutable at 0 if and only if a binary term $s(x, y)$ which satisfies the equations of 1.9 exists. For the *proof of* 1.8 we assume that $\mathcal{K}$ is ideal determined and look at $F_{\mathcal{K}}\{x, y\}$, the free 2-generated algebra in $\mathcal{K}$ with generators x and y. Let $\Theta_{(x,y)}$ be the smallest congruence relation collapsing x and y. Since $(0, y) \in \Theta_{(0,x)} \circ \Theta_{(x,y)}$ we get from 1.4 that $(0, y) \in \Theta_{(x,y)} \circ \Theta_{(0,x)}$ so the usual Mal'cev type argument yields a binary term $s(x, y)$ such that the equations $s(x, x) = 0$ and $s(0, y) = y$ are satisfied in the variety $\mathcal{K}$. Hence $r(x, y, z) := s(s(x, y), z)$ is the required term satisfying $r(x, x, y) = y$ and $r(0, x, x) = 0$.

For the converse we have to show that each ideal is the 0-class of some congruence relation. Let I be an ideal of $\mathcal{A}$. Then by Mal'cev's description of congruence classes [10], we only have to verify: If τ is an algebraic function mapping one element of I into I, then it maps every element of I into I. So let τ be an algebraic function, $i, j \in I$ and suppose $\tau(i) \in I$. Then there is a term $t(x_1, \ldots, x_n, x)$ and elements $a_1, \ldots, a_n \in A$ with $\tau(x) = t(a_1, \ldots, a_n, x)$. Consider the term

$$s(\vec{x}, y_1, y_2, y_3) := r(y_1, t(x_1, \ldots, x_n, y_2), t(x_1, \ldots, x_n, y_3))$$

Then $s(\vec{x}, y_1, y_2, y_3)$ is an ideal term in (y_1, y_2, y_3). Hence, substituting a_i for x_i and the ideal elements $\tau(i)$, i and j for y_1, y_2 and y_3, we get that

$$s(\vec{a}, \tau(i), i, j) = r(\tau(i), \tau(i), \tau(j)) = \tau(j) \quad \text{must be in } J.$$

J. Hagemann has shown in [6], that 0-regular varieties are modular and *n-permutable*. n-permutability means, that the join $\Theta \vee \psi$ of two congruences is already given by their n-fold relational product, where n is fixed throughout the variety. Thus, ideal determined varieties are n-permutable. Corollary 1.4 seems to suggest moreover, that they might already be permutable, since they are permutable at 0. This, however, is not the case as the following example shows:

1.11 EXAMPLE. An *implication algebra* is a groupoid satisfying the equations

$(xy)x = x$ (1)

$(xy)y = (yx)x$ (2)

$x(yz) = y(xz)$ (3).

It follows that $xx = yy$, so $1 := xx$ is an equationally defined constant, satisfying: $1x = x$. The calculations are:

$$\underline{xx} = [(xy)x]x = [x(xy)](xy) = x[[x(xy)]y] = x[[((xy)x)(xy)]y] =$$
$$= x[(xy)y] = \underline{(xy)(xy)}. \text{ In particular } \underline{xx} = [x(yy)][x(yy)] =$$
$$= [y(xy)][y(xy)] = \underline{yy}.$$

Thus $(xx)y = (yy)y = \underline{y}$. We also note that from $ab = 1 = ba$ we infer $a = 1a = (ba)a = (ab)b = 1b = \underline{b}$. So with $d_1(x, y) := xy$, $d_2(x, y) := yx$ and $r(x, y, z) := (xy)z$ we may apply our theorem to find that implication algebras comprise an ideal determined variety. The three-element example from Mitschke [1] shows that implication algebras do not have permutable congruence relations. Although implication algebras are 0-regular, their congruences are *not* determined by *each* of their classes, so they are not regular. Implication algebras do have 3-permutable congruences though, and so do all of the examples we mentioned so far of ideal determined varieties. We do not know, whether this must always be true in each ideal determined variety.

2. The commutator of ideals

In this chapter we shall assume that $\mathscr{K}$ is a variety which is ideal determined. Since we know that $\mathscr{K}$ is congruence modular, we do have the theory of commutators available (see Hagemann, Herrmann [6], Freese, McKenzie [2] and Gumm [4], [3]). In short, the commutator is a multiplication of congruences α and β denoted by $[\alpha, \beta]$ which reduces to the concept of the classical commutator of normal subgroups in groups, to the ideal generated by $IJ + JI$ in rings, if I and J are the ideals corresponding to α and β, and to the intersection of congruences in lattices, or more generally in distributive varieties. In our framework of ideal determined varieties we should be able to describe the commutator $[I, J]$ of two ideals by only referring to I and J without passing first to the congruence relations I^δ and J^δ corresponding to I and to J. Thus we simply write $[I, J]$ for the ideal which is the 0-class of $[I^\delta, J^\delta]$.

First we have to explain the transfer from I to I^δ. If $d_1, \dots, d_n$ are the terms from 1.6, we have:

2.1 LEMMA. (i) $(a, b) \in I^\delta$ *iff* $\forall_{0<i\le n} d_i(a, b) \in I$. (ii) I^δ *is generated as a congruence relation by* $D = \{(0, d_i(a, b)) \mid (a, b) \in I^\delta\}$.

Proof. (i) follows immediately from Fichtner's result 1.6. Similarly, let Θ be the congruence relation generated by D, then clearly Θ contains $I \times I$, so (ii) follows from (i).

The following characterization of the commutator $[\alpha, \beta]$ of two congruences α and β will serve as our definition, see [3].

First think of α as a subalgebra of $\mathcal{A} \times \mathcal{A}$ which happens to be an equivalence relation. On this algebra α let Δ_α^β be the congruence generated by all pairs $((a, a), (b, b))$ where $(a, b) \in \beta$. Then $[\alpha, \beta] = \{(c, d) \mid \exists_{a \in A}(a, c)\Delta_\alpha^\beta(a, d)\}$. It was shown in [3], that this is equivalent to

$$[\alpha, \beta] = \{(c, d) \mid (c, c)\Delta_\alpha^\beta(c, d)\} = \{(c, d) \mid \exists_a (a, a)\Delta_\alpha^\beta(c, d)\}.$$

Hence $[\alpha, \beta] = 0_{\mathcal{A}}$ (where $0_{\mathcal{A}}$ denotes the trivial congruence relation "$=$" in $\mathcal{A}$), if and only if the sets $\delta_a^\beta = \{(c, c) \mid c\beta a\}$ are classes of some congruence relation, which then has to be Δ_α^β. According to a theorem of Mal'cev [10], the δ_a^β are congruence classes if and only if for every algebraic function $\tau(x)$ and $u, v \in \delta_a^\beta$ we have: $\tau(u) \in \delta_b^\beta$ implies $\tau(v) \in \delta_b^\beta$. Keeping in mind that the underlying algebra is α so $\tau(x)$ is actually given by $(t(a_1, \ldots, a_n x_1), t(b_1, \ldots, b_n, x_2))$ where t is a term and $(a_i, b_i) \in \alpha$ we get (see [4]): $[\alpha, \beta] = 0_{\mathcal{A}}$ iff for every term $t(x_1, \ldots, x_n, y)$, elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ with $a_i \alpha b_i$ and elements c, d with $c\beta d$ we have the implication:

$$t(a_1, \ldots, a_n, c) = t(b_1, \ldots, b_n, c) \textit{ implies}$$
$$t(a_1, \ldots, a_n, d) = t(b_1, \ldots, b_n, d).$$

This is called the "Term condition" by several authors. Since the commutator operation is respected by homomorhisms, it is indeed enough to know, when $[\alpha, \beta] = 0_{\mathcal{A}}$. The following three propositions describe that case, the first one assumes only 0-regularity.

2.2 PROPOSITION. *Let $\mathcal{V}$ be a 0-regular variety, and α, β congruences on the algebra $\mathcal{A}$ from $\mathcal{V}$. Let I and J be the ideals $[0]\alpha$ and $[0]\beta$. Then the following are equivalent.*

(i) $[\alpha, \beta] = 0_{\mathcal{A}}$ $([I, J] = \{0\})$
(ii) $\Delta^J := \{(x, x) \mid x \in J\}$ *is a class of a congruence relation on the algebra* α.
(iii) *If $t(\vec{x}, y)$ is a term, $\vec{a}, \vec{b} \in A^n$ with $a_i \alpha b_i$ and if $c, d \in J$, then $t(\vec{a}, c) = t(\vec{b}, c)$ implies $t(\vec{a}, d) = t(\vec{b}, d)$.*
(iv) *If $t(\vec{x}, y)$ is a term, $\vec{a}, \vec{b} \in A^n$ with $a_i \alpha b_i$ and if $c \in J$ then $t(a, 0) = t(b, 0)$ iff $t(\vec{a}, c) = t(\vec{b}, c)$.*

2.3 PROPOSITION. *Let $\mathcal{K}$ be an ideal determined variety and I, J ideals of $\mathcal{A} \in \mathcal{K}$. Then the following are equivalent:*

(i) $[I, J] = \{0\}$.
(ii) *If $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $a_i I^\delta b_i$ and $\vec{c} \in J^m$, then $t(\vec{a}, \vec{c}) = t(\vec{b}, \vec{c})$.*
(iii) *If $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $a_i I^\delta b_i$ and $\vec{c} \in J^m$, then $t(\vec{a}, \vec{c}) = 0$ implies $t(\vec{b}, \vec{c}) = 0$.*

2.4 PROPOSITION. *In an ideal determined variety*

$$[I, J] = \{t(\vec{a}, \vec{c}) \mid t(\vec{x}, \vec{y}) \text{ is an ideal term in } \vec{y}, \vec{c} \in I^m \text{ and}$$
$$t(\vec{b}, \vec{c}) = 0 \text{ for some } \vec{b} \text{ with } a_i I^\delta b_i\}.$$

Proof of 2.2. (i) implies (ii), since $\Delta^J = \delta_0^\beta$ with $\beta = J^\delta$. If (ii) holds, then clearly, since Δ^J generates Δ_α^β, from $(c, d) \in [\alpha, \beta]$ we infer $d_i(c, d)[\alpha, \beta]0$ for all i. Hence $(0, 0)\Delta_\alpha^\beta(0, d_i(c, d))$ for all $0 < i \leq n$, hence $(0, d_i(c, d)) \in \Delta^J$, thus $d_i(c, d) = 0$ for $0 < i \leq n$. Hence $c = d$ and (i) holds. (iii) and (iv) are obviously equivalent and follow from (i), since they are special cases of the term condition. For (iii) $\to$ (ii), one modifies (iii) by saying $t(\vec{a}, c) = t(\vec{b}, c) \in J$ implies $t(\vec{a}, d) = t(\vec{b}, d) \in J$, which is allowed, since $cJ^\delta d$, then this statement is the syntactical formulation of (ii). To prove 2.3, note that (ii) simply states, that Δ^J is an ideal in the algebra I^δ. (iii) is a corollary of Proposition 2.4, whose proof is as follows: $x[\alpha, \beta]0$ *iff* $(0, 0)\Delta_\alpha^\beta(x, 0)$ iff $(x, 0)$ is in the ideal generated by Δ^J in the algebra I^δ, iff $t((a_1, b_1), \ldots, (a_n, b_n), (c_1, c_1), \ldots, (c_m, c_m)) = (x, 0)$ for some ideal term $t(\vec{x}, \vec{y})$ in $\vec{y}$.

If we recall the example of rings R, the commutator of two ideals I and J is generated by $IJ + JI$, so it consists of all elements

$$\sum_k a_k i_k j_k b_k + \sum \bar{a}_k \bar{j}_k \bar{i}_k \bar{b}_k, \qquad a_k, \bar{a}_k, b_k, \bar{b}_k \in R, \qquad i_k, \bar{i}_k \in I \quad \text{and} \quad j_k, \bar{j}_k \in J.$$

Hence it is the result of applying the terms $m(\vec{x}, \vec{y}, \vec{z}) = \sum x_k y_k z_k u_k + \sum \bar{x}_k \bar{z}_k \bar{y}_k \bar{u}_k$ with the x_k, $\bar{x}_k$, u_k, $\bar{u}_k$ replaced with elements from $\mathcal{R}$, the y_k, $\bar{y}_k$ and z_k, $\bar{z}_k$ replaced with elements from I, resp. J. Such terms therefore may be called commutator terms. Commutator terms in that sense may also be described for groups. The property that seems to matter is in the following definition.

2.5 DEFINITION. A term $m(\vec{x}, \vec{y}, \vec{z})$ is a *commutator term in $\vec{y}$ and $\vec{z}$*, if it is an ideal term in $\vec{y}$ and an ideal term in $\vec{z}$. The following theorem shows that indeed the commutator terms describe commutators.

2.6 THEOREM. *Let $\mathcal{K}$ be an ideal determined variety, and I, J ideals in $\mathcal{A} \in \mathcal{K}$. Then*

$$[I, J] = \{m(\vec{a}, \vec{i}, \vec{j}) \mid \vec{a} \in A^n, \vec{i} \in I^m, \vec{j} \in J^r \text{ where } m(\vec{x}, \vec{y}, \vec{z}) \text{ is a commutator term in } \vec{y} \text{ and } \vec{z}\}.$$

Proof. Let us abbreviate the right side with $\langle I, J\rangle$, then we have to show that $\langle I, J\rangle = [I, J]$. Clearly $\langle I, J\rangle \subseteq [I, J]$, since for every commutator term $m(\vec{x}, \vec{y}, \vec{z})$, $\vec{a} \in A^n$, $\vec{i} \in I^m$, $\vec{j} \in J^r$ we have: $m(\vec{a}, 0, \vec{j}) = 0$ and, since $m(\vec{x}, \vec{y}, \vec{z})$ is an ideal term in $\vec{z}$, we conclude with 2.4, that $m(\vec{a}, \vec{i}, \vec{j}) \in [I, J]$. For the reverse inclusion note first, that the definition of $\langle I, J\rangle$ is obviously respected by the homomorphism theorem, i.e. $\varphi\langle I, J\rangle = \langle \varphi I, \varphi J\rangle$, so we may restrict ourselves to the case $\langle I, J\rangle = 0$ and prove that then $[I, J] = 0$. Hence suppose as in 2.3 (iii), that $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a}, \vec{b} \in A^n$ with $a_i I^\delta b_i$ and $j \in J^m$ and $p(\vec{a}, \vec{j}) = 0$. Now let $d_{i_1}, \ldots, d_{i_n}$ and d_j be any of the terms for 0-regularity. Then $d_j(p(\vec{y}, \vec{z}), p(\vec{0}, \vec{z}))$ is a commutator term in $\vec{y}$ and $\vec{z}$ and $d_j(p(d_{i_1}(a_1, b_1), \ldots, d_{i_n}(a_n, b_n), \vec{j}), p(\vec{0}, \vec{j})) = 0$. Since this is true for all j, we have:

(*) $p(d_{i_1}(a_1, b_1), \ldots, d_{i_n}(a_n, b_n), \vec{j}) = p(\vec{0}, \vec{j})$, hence $p(\vec{e}, \vec{j}) = p(\vec{f}, \vec{j})$ where $e_i = d_r(a_i, b_i)$ and $f_i = d_s(a_i, b_i)$ for any arbitrary r and s.

Next we show:

(**) If $\tau_1, \ldots, \tau_n$ are arbitrary unary algebraic functions, $\vec{e}, \vec{i} \in I^n$, $\vec{j} \in J^m$, then $p(\vec{i}, \vec{j}) = p(\vec{e}, \vec{j})$ implies $p(\overrightarrow{\tau(i)}, \vec{j}) = p(\overrightarrow{\tau(e)}, \vec{j})$.

Indeed, since the $\tau_k(x)$ come from terms $t_k(\vec{x}_k, x)$ with $\tau_k(x) = t_k(a_{k1}, \ldots, a_{kr}, x)$ we look at

$$d_j(p(t_1(\vec{x}_1, v_1), \ldots, t_n(\vec{x}_n, v_n), \vec{z}), p(t_1(\vec{x}_1, u_1), \ldots, t_n(\vec{x}_n, u_n), \vec{z})).$$

Again, this is a commutator term in $(v_1, \ldots, v_n, u_1, \ldots, u_n)$ and in $\vec{z}$. As before we conclude:

$$p(\tau_1(i_1), \ldots, \tau_n(i_n), \vec{j}) = p(\tau_1(e_1), \ldots, \tau_n(e_n), \vec{j}).$$

Since the pairs (e_i, f_i) generate I^δ we conclude from (*) and (**) that $p(\vec{a}, \vec{j}) = p(\vec{b}, \vec{j}) = 0$. Finally, it seems worthwhile to note:

2.7 PROPOSITION. *The term $r(x, y, z)$ has the property that $r(x, y, z) = x - y + z$ in every affine algebra.*

Proof. Using the "term condition", we find:

$$d_i(r(0, y, 0), r(0, y, 0)) = d_i(r(y, y, 0), r(0, y, y)),$$

hence

$$d_i(r(0, x, x), r(x, x, 0))[1, 1]d_i(r(y, x, x), r(x, x, y)),$$

i.e.

$$0 = d_i(0, 0)[1, 1]d_i(r(y, x, x), y) \quad \text{for all } i,$$

so $y[1, 1]r(y, x, x)$. Thus $r(x, y, z)$ is a Mal'cev term in every affine algebra. The rest follows from the general theory of commutators.

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