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Is there a Mal'cev theory for single algebras?

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This article solves a problem stated by A. F. Pixley at the conference on universal algebra in Oberwolfach, 1973. To obtain our solution we prove and apply a theorem (2.2) on idempotent compatible functions, which might be of some independent interest.

1.1 DEFINITIONS. A *congruence equality* $e = g$ is an expression where e and g are terms in variables and binary function symbols $\wedge$(meet), $\vee$(join), and $\circ$(relational product). $e = g$ holds for a concrete sublattice **L** of the partition lattice $\Pi(S)$ on a set S iff $e = g$ is true for every interpretation of the variables in $e = g$ by partitions from **L**. $e = g$ holds in an algebra **A** iff it holds in $\mathfrak{C}(A)$, the lattice of all congruences on **A**. $e = g$ holds in a variety **V** iff $e = g$ holds in all algebras from **V**. For a more precise definition see Wille [8].

1.2 EXAMPLES. (α) $x \circ y = y \circ x$ (permutability) (β) $(x \circ y) \wedge (z \circ x) = x \circ (y \wedge z)$ (arithmeticity). (β) implies permutability and distributivity so it is immediate that (β) implies every other congruence equality which is not equivalent to $x = y$.

A. I. Mal'cev (resp. A. F. Pixley) has shown in [3] (resp. [4]) that in a variety (α) (resp. (β)) holds if and only if there exists a term p (resp. m) in the language of **V** such that

(α'): $p(x, y, y) = p(y, y, x) = x$

(resp.

(β'): $m(x, y, y) = m(y, y, x) = m(x, y, x) = x$)

are equations valid in **V**.

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Wille [8] and independently Pixley [5] have shown that every set of congruence equalities can be characterized in a similar way by a weak Mal'cev condition. For the definition of weak Mal'cev conditions and their properties see Taylor [7].

For the sake of brevity let us call a ternary function a "Mal'cev-function" resp. a "Pixley-function" if it satisfies the above equations (α'), resp. (β').

Surprisingly there is a "local version" of the characterization of (β), which was proved by A. F. Pixley in [6]:

1.3 THEOREM (Pixley). *Let* **A** *be an algebra with* $\mathfrak{C}(\mathbf{A})$ *finite. Then* **A** *satisfies* (β) *if and only if there is a Pixley-function on* **A** *which is compatible with all congruences on* **A**.

Remark. This was extended recently by I. Korec [2] to algebras **A** with $|\mathbf{A}| \leq \aleph_0$ dropping the restriction that $\mathfrak{C}(\mathbf{A})$ be finite.

In fact there is a stronger theorem proved in Pixley's and in Korec's paper namely:

1.3′ THEOREM (Pixley, Korec). *Let* **L** *be a concrete sublattice of* $\pi(S)$ *with either* $|\mathbf{L}|$ *finite or* $|S|$ *countable then* (β) *holds in* **L** *if and only if there is a Pixley-function on* S, *compatible with all members of* **L**.

The resulting problem, stated by A. F. Pixley at the conference on universal algebra in Oberwolfach, 1973 is the following:

PROBLEM. Is there a Mal'cev theory for single algebras?

The following definition will be used to give the above problem a precise formulation:

1.4 DEFINITION. Let $e = g$ be a congruence equality and let M be the corresponding (weak) Mal'cev condition. $e = g$ is said to be *locally Mal'cev characterizable* if for any finite algebra **A**, $e = g$ holds in $\mathfrak{C}(\mathbf{A})$ if and only if there exist compatible functions on **A**, corresponding to the function symbols in M and satisfying the equations given by M.

Before we go further let us note that Mal'cev conditions for congruence equalities are idempotent, i.e. all compatible functions locally characterizing congruence equalities have to be idempotent.

§2. The algebra $\mathbf{A}_6$

We define a unary algebra, $\mathbf{A}_6$, on a six-element set in the following way:

$$\mathbf{A}_6 := (A, f, g) \quad \text{where} \quad A := \{0, 1, \ldots, 5\},\ f, g: A \to A$$

with

$$f(x) := x + 2 \pmod 6$$

and

$$g(0) = g(5) = 1,\ g(1) = g(4) = 0,\ g(2) = g(3) = 5.$$

Then it is easy to check that $\mathbf{A}_6$ has exactly two nontrivial congruences, θ_1 and θ_2 which are given by the partitions:

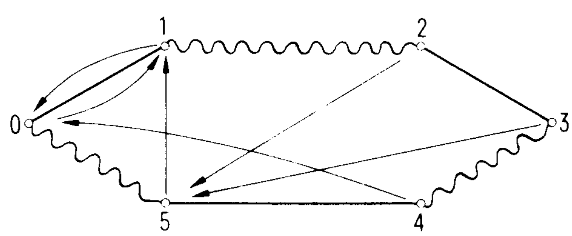
$$\theta_1 = \{\{x, x+1\} \mid x \equiv 0 (\mathrm{mod}\ 2)\}$$

$$\theta_2 = \{\{x, x+1\} \mid x \equiv 1 (\mathrm{mod}\ 2)\} \quad (+\text{always will denote addition mod } 6)$$

Moreover we have: $\theta_1 \wedge \theta_2 = \omega$
and

$$\theta_1 \circ \theta_2 \circ \theta_1 = \theta_2 \circ \theta_1 \circ \theta_2 = \theta_1 \vee \theta_2 = \iota.$$

This can be easily checked looking at the diagram below where elements of A congruent mod θ_1, resp. θ_2, are connected by straight, resp. waved lines. The arrows indicate the operation g.



Let us from now on always write x—y for $x\theta_1 y$ and x~~y for $x\theta_2 y$.

Now we investigate admissible operations on $\mathbf{A}_6$, i.e. mappings $h: A^n \to A$ which are compatible with θ_1 and θ_2. In view of our final goal and in view of the remark at the end of chapter 1, we may restrict our attention to all admissible functions which are idempotent.

For an arbitrary algebra $\mathbf{B} = (B, F)$ we define:

$$IA_n(\mathbf{B}) := \{h : B^n \to B \mid h \text{ is idempotent and compatible with all congruences on } \mathbf{B} \text{ and } h \text{ is not a projection}\}.$$

2.1 LEMMA. $IA_2(\mathbf{A}_6) = \varnothing$.

Proof. Suppose $m \in IA_2(\mathbf{A}_6)$. We write $x \cdot y$ for $m(x, y)$. By idempotency we have $x \cdot x = x$ hence $x \sim\!\sim x \cdot (x+1)$ for x odd and $x -\!- x \cdot (x+1)$ for x even. Thus in $\mathbf{A}_6$ we have $x \cdot (x+1) \in \{x, x+1\}$.

Suppose that for some x_0 we have $x_0 \cdot (x_0+1) = x_0$. The case $x_0 \cdot (x_0+1) = x_0+1$ is handled symmetrically.

Then for x_0 odd:

$$x_0 = x_0 \cdot (x_0+1) -\!- x_0 \cdot (x_0+2) \sim\!\sim (x_0+1) \cdot (x_0+2) \in \{x_0+1, x_0+2\}$$

Since for x_0 odd there is no element $y \in A_6$ with $x_0 -\!- y \sim\!\sim x_0+2$, we must therefore have $(x_0+1) \cdot (x_0+2) = x_0+1$. Correspondingly, if x_0 is even, from

$$x_0 = x_0 \cdot (x_0+1) \sim\!\sim x_0 \cdot (x_0+2) -\!- (x_0+1) \cdot (x_0+2) \in \{x_0+1, x_0+2\}$$

we conclude as above that $(x_0+1) \cdot (x_0+2) = x_0+1$,
hence

$$\forall x \in A_6 \quad \underline{x \cdot (x+1) = x}$$

It follows:

$$\text{for } x \text{ odd:} \quad x = x \cdot (x+1) -\!- x \cdot (x+2) \sim\!\sim (x+1) \cdot (x+2) = x+1$$

$$x \quad \text{even:} \quad x = x \cdot (x+1) \sim\!\sim x \cdot (x+2) -\!- (x+1) \cdot (x+2) = x+1$$

so

$$\forall x \in A_6 \quad \underline{x \cdot (x+2) = x}$$

Then for x odd:

$$x = x \cdot x -\!- x \cdot (x-1) \sim\!\sim (x+1) \cdot (x-2) -\!- (x+1) \cdot (x-3) = (x+1) \cdot (x+3) = x+1$$

and for x even:

$$x = x \cdot x \sim\!\sim x \cdot (x-1) -\!- (x+1) \cdot (x-2) \sim\!\sim (x+1) \cdot (x-3) = (x+1) \cdot (x+3) = x+1$$

implies $\underline{x \cdot (x-1) = x}$ and $(x+1) \cdot (x-2) = x+1$ thus $\underline{x \cdot (x+3) = x}$
Finally from

$$x = x \cdot (x+3) — x \cdot (x+4) \rightsquigarrow (x+1) \cdot (x+4) = x+1 \quad \text{for } x \text{ odd, and}$$

$$x = x \cdot (x+3) \rightsquigarrow x \cdot (x+4) — (x+1) \cdot (x+4) = x+1 \quad \text{for } x \text{ even}$$

we infer $x \cdot (x+4) = x$.

Hence for all $x, y \in \mathbf{A}_6$ we have $x \cdot y = x$ which means that $m \notin IA_2(\mathbf{A}_6)$, a contradiction.

For the second step, to investigate $IA_n(\mathbf{A}_6)$ for $n > 2$ we prove a more general theorem about idempotent admissible functions:

2.2 THEOREM. *Let* $\mathbf{B}$ *be an arbitrary algebra such that* $IA_n(\mathbf{B}) \neq \emptyset$ *for some* $n \geq 2$. *Then either* $IA_2(\mathbf{B}) \neq \emptyset$ *or* $IA_3(\mathbf{B})$ *contains a Mal'cev-function, in particular* $\mathbf{B}$ *has permutable congruences.*

Proof. Suppose $IA_3(\mathbf{B})$ does not contain a Mal'cev-function. Let k be the smallest integer such that $IA_k(\mathbf{B}) \neq \emptyset$. We may suppose $k \geq 3$. Take an element $m \in IA_k(\mathbf{B})$ and define binary operations m_i for $i \leq k$ by

$$m_i(x, y) := m(x, \ldots, x, y, x, \ldots, x) \quad \text{where } y \text{ is at the i'th place.}$$

Claim. $\exists j \leq k \forall k \geq s \neq j \quad m_s(x, y) = x \quad (*)$

Obviously every m_i is compatible and idempotent. Therefore, and because $IA_2(\mathbf{B}) = \emptyset$ they have to be projections. Suppose $m_s(x, y) = y$ and $m_t(x, y) = y$ for $s \neq t$ then

$$p(x, y, z) := m(y, \ldots, y, x, y, \ldots, y, z, y, \ldots, y)$$

where x is at the s'th and z is at the t'th place, satisfies: $p(x, x, z) = m_t(x, z) = z$ and $p(x, y, y) = m_s(y, x) = x$ hence p is a Mal'cev-function. Thus the claim is proved and we can suppose w.l.o.g. that $j = 1$. Since $m \in IA_k(\mathbf{B})$, m is not a projection, in particular there exist elements $a_1, a_2, \ldots, a_k \in B$ with $m(a_1, a_2, \ldots, a_k) \neq a_1$.

For $b := a_k$ define a $(k-1)$-ary operation m_b by

$$m_b(x_1, x_2, \ldots, x_{k-1}) := m(x_1, x_2, \ldots, x_{k-1}, b)$$

Then $m_b(x, \ldots, x) = m(x, \ldots, x, b) = m_k(x, b) = x$ by $(*)$. Clearly m_b is admissible.

m_b cannot be a projection on the first argument because

$$m_b(a_1, a_2, \ldots, a_{k-1}) = m(a_1, a_2, \ldots, a_{k-1}, a_k) \neq a_1.$$

m_b cannot be an i'th projection for $1 < i \leq k-1$ because for any $a \neq b$ we have

$$m_b(b, \ldots, b, a, b, \ldots, b) = m(b, \ldots, b, a, b, \ldots, b, b) = m_i(b, a) = b \neq a.$$

Thus $m_b \in IA_{k-1}(\mathbf{B})$ contradicting the choice of m of minimal arity.

2.3 COROLLARY. *$IA_n(\mathbf{A}_6) = \varnothing$ for all $n \in \mathbf{N}$.*

Proof. This follows by lemma 2.1, theorem 2.2 and the fact that the congruences θ_1 and θ_2 of $\mathbf{A}_6$ do not permute.

Next we investigate which congruence equalities are satisfied in $\mathbf{A}_6$.

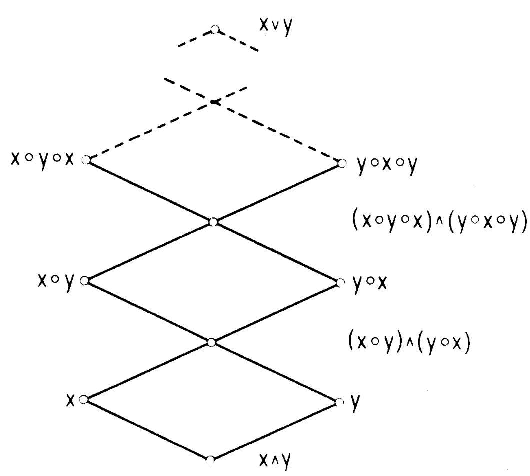
2.4 LEMMA. *$\mathfrak{C}(\mathbf{A}_6)$ satisfies every congruence equality which does not imply permutability. In particular every lattice equation holds in $\mathfrak{C}(\mathbf{A}_6)$.*

Proof. The second statement is obvious since $\mathfrak{C}(\mathbf{A}_6)$ is distributive. For the first statement suppose $e = g$ is a congruence equality not holding in $\mathfrak{C}(\mathbf{A}_6)$.

Since $\mathbf{A}_6$ has only two nontrivial congruences θ_1 and θ_2 we may w.l.o.g. assume that e and g contain only two variables, x and y.

Then for $S := \{x, y, x \wedge y, x \circ y, y \circ x, (x \circ y) \wedge (y \circ x), x \circ y \circ x, \ldots\}$ e and g are elements from S.

S carries a natural order:



If $e < g$ in S then the congruence equality $e = g$ implies the congruence equalities $e = f$ and $f = g$ for any $f \in S$ with $e \leq f \leq g$. Moreover since in (A_6) we have

$\theta_1 \circ \theta_2 \circ \theta_1 \wedge \theta_2 \circ \theta_1 \circ \theta_2 = \iota$ and since $e = g$ does not hold in (A_6) we may assume $e, g \leq (x \circ y \circ x) \wedge (y \circ x \circ y)$ in S.

Hence $e = g$ implies one of the following equalities: $(x \circ y \circ x) \wedge (y \circ x \circ y) = x \circ y$, $x \circ y = y \circ x$, $x \circ y = (x \circ y) \wedge (y \circ x)$, $(x \circ y) \wedge (y \circ x) = x$, $x = y$, $x = x \wedge y$.

All of those, however, trivially imply permutability.

Combining Corollary 2.3 and the preceding lemma we get

2.5 LEMMA. *Let $e = g$ be a congruence equality which is locally Mal'cev characterizable, then $e = g$ implies permutability.*

§3. The permutable case

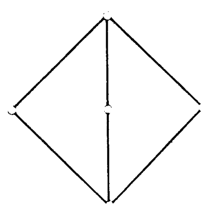
The congruence equalities which are left for consideration now have to imply permutability hence also modularity. In varieties congruence permutability is characterized by the existence of a ternary polynomial satisfying (α'). Thus if we can construct a finite algebra $\mathbf{A}$ such that

(i) every congruence equality which implies permutability but does not imply distributivity, holds in $\mathfrak{C}(\mathbf{A})$

(ii) there is no Mal'cev function on A which is compatible with all congruences of $\mathbf{A}$,

then, with the help of lemma 2.5 we will have shown that (β) is the only congruence equality which is locally Mal'cev-characterizable.

To get an algebra $\mathbf{A}$ as required above we first examine how $\mathfrak{C}(\mathbf{A})$ should look like. Since we are dealing with congruence equalities which imply permutability, hence modularity, $\mathfrak{C}(\mathbf{A})$ should be a modular lattice and all the elements of $\mathfrak{C}(\mathbf{A})$ should permute. Moreover $\mathfrak{C}(\mathbf{A})$ has to be nondistributive so it must contain the lattice $\mathcal{M}_3$ shown in the figure below as a sublattice.



3.1 LEMMA. *Let $\mathbf{A}$ be an algebra with permutable congruences. If $\mathfrak{C}(\mathbf{A}) \simeq \mathcal{M}_3$ then $\mathbf{A}$ satisfies condition* (i).

Proof. For a congruence equality $e = g$ define $\hat{e} = \hat{g}$ to be the lattice equation which arises by replacing every $\circ$ in e and in g by v. Then it is obvious that for any congruence equality $e=g$ which implies permutability we have that $e=g$ is equivalent to the conjunction of the congruence equality $x \circ y = y \circ x$ and the lattice equation $\hat{e}=\hat{g}$. Now let $e=g$ be any congruence equality holding in some nondistributive lattice of permuting equivalence relations, then $\hat{e}=\hat{g}$ holds in some modular nondistributive lattice. Therefore $\hat{e}=\hat{g}$ holds in $\mathcal{M}_3$, hence $e=g$ holds for $\mathfrak{C}(\mathbf{A})$.

Our aim is now, to construct a finite algebra $\mathbf{A}$ such that (i'): $\mathbf{A}$ has permutable congruences and $\mathfrak{C}(\mathbf{A}) \simeq \mathcal{M}_3$, and (ii) are satisfied.

We cite a lemma from Gumm [1]:

3.2 LEMMA. *Let θ_1, θ_2, θ_3 be permuting equivalence relations on a set S such that $\theta_i \circ \theta_j = \iota$ and $\theta_i \wedge \theta_j = \omega$ for all $i \neq j$. Then there exists a loop $\mathbf{Q}=(Q, \cdot, 1)$ and a bijection $g: Q \times Q \to S$ such that*

$$(x, y)\theta_1(x', y') \quad \text{iff} \quad x = x'$$

$$(x, y)\theta_2(x', y') \quad \text{iff} \quad y = y'$$

$$(x, y)\theta_3(x', y') \quad \text{iff} \quad x \cdot y = x' \cdot y',$$

if we are identifying S and $Q \times Q$ via the bijection g.

Moreover from theorem 3.8 of the cited paper we get:

3.3 LEMMA. *There exists a Mal'cev function compatible with θ_1, θ_2 and θ_3 if and only if $\mathbf{Q}$ as above is an abelian group.*

Hence an algebra satisfying (i') and (ii) can be constructed in only one way:

We may assume that all fundamental operations are chosen unary and that every map admitting θ_1, θ_2 and θ_3 is a fundamental operation. Define $\mathbf{A}_\mathbf{Q} := (Q \times Q, E)$ where $\mathbf{Q}$ is a loop which is not an abelian group and where F is the set of all maps from $Q \times Q$ into $Q \times Q$ which admit θ_1, θ_2, θ_3 from lemma 3.2.

f admitting θ_1 and θ_2 means:

$$f = (f_1, f_2) \quad \text{where} \quad f_1, f_2 \colon Q \to Q.$$

f admitting θ_3 means (compare [1], lemma 4.1):

$$(*): \qquad f_1(x) \cdot f_2(y) = f_1(x \cdot y) \cdot f_2(1) \qquad \text{for all } x, y \in Q.$$

Note that $(*)$ also implies:

$$f_1(1)\cdot f_2(x)=f_1(x)\cdot f_2(1).$$

$\mathbf{A_Q}$ is now uniquely determined by $\mathbf{Q}$, hence it remains to choose an appropriate loop (which is not an abelian group) such that $\mathbf{A_Q}$ has no other nontrivial congruences besides θ_1, θ_2 and θ_3.

If $\mathbf{Q}$ is a group $\mathbf{G}$, then the pairs of maps (f_1, f_2) satisfying $(*)$ can be easily determined:

Set $a:=f_1(1)$ and $b:=f_2(1)$, then $(*)$ implies:

$$a^{-1}f_1(x)a^{-1}f_1(y)=a^{-1}f_1(xy)$$

and

$$f_2(x)b^{-1}f_2(y)b^{-1}=f_2(xy)b^{-1}$$

Thus the fundamental operations of $\mathbf{A_G}$ are precisely the maps $f: G\times G\to G\times G$ defined by

$$f(x, y):=(a\Psi(x), \Psi(y)b) \quad \text{for fixed} \quad a, b\in G$$

and Ψ an endomorphism of $\mathbf{G}$.

Now we use a lemma which is due to B. Wolk:

3.4 LEMMA(Wolk). *Let* $\mathbf{G}$ *be a simple nonabelian group. Then* $\mathbf{A_G}$ *satisfies* (i').

Proof. We will show that for any nontrivial congruence Γ on $\mathbf{A_G}$ which is different from θ_1, θ_2 and θ_3 the definition $g(x):=y$ *iff* $(x, 1)\Gamma(1, y)$ yields a nontrivial automorphism g on $\mathbf{G}$ which is in the center of the automorphism group of $\mathbf{G}$. Since the automorphism groups of simple nonabelian groups have trivial center (see Zassenhaus [9]) such a Γ cannot exist.

So let Γ be a congruence on $\mathbf{A_G}$. By the definition of $\mathbf{A_G}$ we have:

$$(+): \qquad (x, y)\Gamma(u, v)\Rightarrow(a\psi(x), \psi(y)b)\Gamma(a\psi(u), \psi(v)b)$$

for all $x, y, u, v, a, b\in G$ and $\psi\in \mathrm{End}\ (\mathbf{G})$.

Define $(x, y)\in\hat{\Gamma}$ *iff* $(x, 1)\Gamma(1, y)$ then it is immediately clear from $(+)$ that $(x, y)\Gamma(u, v)$ iff $u^{-1}x\hat{\Gamma}vy^{-1}$, hence $\hat{\Gamma}$ uniquely determines Γ.

From $x\hat{\Gamma}y$ and $u\hat{\Gamma}v$ we can conclude $(x, y^{-1})\Gamma(1, 1)\Gamma(u, v^{-1})$ hence $u^{-1}x\hat{\Gamma}v^{-1}y$, which shows that $\hat{\Gamma}$ is a subgroup of $\mathbf{G}\times\mathbf{G}$ which is moreover fully invariant by (+).

Thus the projections $\hat{\Gamma}_1 := \{x \in G \mid \exists y, x\Gamma y\}$ and correspondingly $\hat{\Gamma}_2$ are fully invariant subgroups of $\mathbf{G}$. Since $\mathbf{G}$ is simple there are five cases to consider:

1. $\hat{\Gamma}_1 = \hat{\Gamma}_2 = \{1\}$; 2. $\hat{\Gamma}_1 = \{1\}, \hat{\Gamma}_2 = G$; 3. $\hat{\Gamma}_1 = G, \hat{\Gamma}_2 = \{1\}$;
4. $\hat{\Gamma}_1 = \hat{\Gamma}_2 = G$ and H_1 or $H_2 = G$ 5. $\hat{\Gamma}_1 = \hat{\Gamma}_2 = G$ and $H_1 = H_2 = \{1\}$,

where H_1 is the normal subgroup consisting of all elements y with $1\hat{\Gamma}y$. H_2 is defined symmetrically.

In the first four cases it is easily checked that Γ has to be ω, θ_1, θ_2, ι respectively. In the fifth case it follows that by $g(x) := y$ *iff* $x\hat{\Gamma}y$, a bijective map is defined which is an automorphism of $\mathbf{G}$ since $\hat{\Gamma}$ is a subgroup of $\mathbf{G}\times\mathbf{G}$. Moreover g is in the center of Aut ($\mathbf{G}$) since $\hat{\Gamma}$ is a fully invariant subgroup. Hence g has to be the identity mapping which is equivalent to saying $\Gamma = \theta_3$.

Starting with any finite simple nonabelian group $\mathbf{G}$ we can therefore construct the algebra $\mathbf{A_G}$ which by lemmas 3.1, 3.3 and 3.4 satisfy conditions (i) and (ii) at the beginning of the chapter. Together with lemma 2.5 therefore we get as result:

3.5 THEOREM. *The only nontrivial congruence equality which is locally Mal'cev characterizable is arithmeticity, i.e. permutability together with distributivity.*

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