MAL'CEV CONDITIONS IN SUMS OF VARIETIES AND A NEW MAL'CEV CONDITION

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1. Abstract

In [3] we announced the following result:

1.1. THEOREM. *Let E be a nontrivial equality in the terms $\circ$, $\wedge$, and $\vee$. There exist varieties K_1^E and K_2^E such that E is congruence-valid in K_1^E and in K_2^E but not in $K_1^E + K_2^E$.*

In this paper we are going to generalize this result to a greater class of Mal'cev conditions. Also we characterize the Mal'cev condition $C :\equiv \exists_p\ (p(x, y) = p(y, x))$ and show that C holds in $K_1 + K_2$ whenever C holds in K_1 and in K_2.

We use standard universal algebra terminology, see e.g. [1]. For varieties K_1 and K_2 of the same type Δ we define

$$K_1 + K_2 := \mathbf{HSP}(K_1 \cup K_2).$$

For a set Σ of equations, $Cn(\Sigma)$ denotes the set of all equations which follow from Σ, and $\mathrm{Mod}_\Delta(\Sigma)$ denotes the class of all algebras of type Δ in which all equations from Σ hold. If $K_1 = \mathrm{Mod}_\Delta(\Sigma_1)$ and $K_2 = \mathrm{Mod}_\Delta(\Sigma_2)$ we know by a classical result of Birkhoff that

$$K_1 + K_2 = \mathrm{Mod}_\Delta(Cn(\Sigma_1) \cap Cn(\Sigma_2)).$$

For the sequel we find it more natural to speak of 'properties of varieties' instead of classes of varieties.

Let S be a set, $a \in S$ and $\Delta := (n_i \mid i \in I)$ a type with the corresponding family of function symbols $F := (f_i \mid i \in I)$. Let S_Δ^a be the algebra of type Δ having S as underlying set where $f_i(s_1, \ldots, s_{n_i}) = a$ holds for each $i \in I$ and $s_1, \ldots, s_{n_i} \in S$. Clearly for a, $b \in S$ S_Δ^a is isomorphic to S_Δ^b, so we write S_Δ^* without specifying $* \in S$.

We list some obvious properties of S_Δ^*:

(i) Each equivalence relation on S_Δ^* is a congruence relation.

(ii) A map $\psi : S \to S$ is an endomorphism of S_Δ^* iff $\psi(*) = *$.

(iii) $U \subseteq S$ is a subalgebra of S_Δ^* iff $* \in U$. (If F does not contain any nullary function symbol the empty set will also be a subalgebra).

For a given type Δ let Ω_Δ be the class of all algebras S_Δ^* where S is a set. Obviously Ω_Δ is an equational class. For two polynomial symbols p and q we write $p \equiv q$ iff p and

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q are equal in the absolutely free algebra of the considered type Δ on countably many generators $x_1, \ldots, x_n, \ldots$.

The following lemma will be helpful:

1.2. LEMMA. *Let* $V := \mathrm{Mod}_\Delta(\Sigma)$ *be a variety of type* Δ. *Then the following are equivalent:*

(i) $2^*_\Delta \in V$ *(where* $2 := \{o, *\}$ *is a two-element set)*

(ii) $\Omega_\Delta \subseteq V$

(iii) *For all polynomial symbols p over V*

$$(p = x) \in \Sigma \text{ implies } p \equiv x.$$

Proof. (i)→(iii): Assume $2^*_\Delta \in V$ and p a polynomial symbol over V with $(p = x) \in \Sigma$ and $p \not\equiv x$. Then p cannot be a variable $y \neq x$, because V contains a two-element algebra. So p has the form $f_i(q_1, \ldots, q_{n_i})$ for a certain $i \in I$ and polynomials $q_1, \ldots, q_{n_i}$ over V. Thus

$$p(o, \ldots, o) = f_i(q_1, \ldots, q_{n_i})(o, \ldots, o) = * \neq o.$$

Therefore $p = x$ cannot hold in 2^*_Δ, so $(p = x) \notin \Sigma$.

(iii)→(ii): Assuming (iii) each equation in Σ is of the form $x = x$ or of the form $f_i(q_1, \ldots, q_{n_i}) = f_j(r_1, \ldots, r_{n_j})$. This implies that each algebra S^*_Δ satisfies all equations of Σ, thus $\Omega_\Delta \subseteq V$.

The following easy lemma should motivate why in the sequel we want to restrict our attention to properties which do not apply to Ω_Δ.

Let us first introduce the notion of congruence-equality:

DEFINITION. A *congruence-equality E* is an equation in variables $x_1, \ldots, x_n, \ldots$ and binary function symbols $\circ$, $\wedge$, and $\vee$. We say that E is *congruence-valid* in a variety V, or in short *E holds* in V, iff for every algebra $\mathscr{A} \in V$ the equality holds whenever the variables are interpreted as congruences on $\mathscr{A}$ and $\circ$, $\wedge$, and $\vee$, respectively, are interpreted as relational product, meet and join of congruences, resp. E is called nontrivial, iff E does not hold in every variety and there is a variety V, which is not the trivial variety containing only the one-element algebra, and E holds in V, see [7].

1.3. LEMMA. *Let* $V = \mathrm{Mod}_\Delta(\Sigma)$ *be a variety and* $\Omega_\Delta \subseteq V$. *Then:*

(i) *There is no nontrivial congruence-equality holding in V. If a lattice-equality holds in the congruence lattice of every algebra* $\mathscr{A} \in V$, *then it already holds in every lattice.*

(ii) *There is no nontrivial monoid-equality holding in the monoid of endomorphisms of every algebra* $\mathscr{A} \in V$.

Proof. (i): Whitman has shown in [6] that each lattice equality holding in every partition lattice already holds in every lattice, and we have already noticed that each partition on a set S is a congruence on S_Δ^*.

(ii): For each monoid M there is an embedding ψ into the endomorphism monoid of M_Δ^* where $M^* := M \cup \{*\}$ and $* \notin M$. For $a \in M$ define:

$$\psi(a): M_\Delta^* \to M_\Delta^* \quad \text{by} \quad \psi(a)(x) := \begin{cases} a \cdot x, & \text{if} \quad x \neq * \\ *, & \text{if} \quad x = * \end{cases}.$$

DEFINITION. A property α of varieties is called *essential*, iff there is a variety V for which α holds and if for all varieties V'

$$\text{if } \alpha \text{ holds in } V' \quad \text{then} \quad \Omega_\Delta \notin V'.$$

Now let $V = \mathrm{Mod}_\Delta(\Sigma)$ be an equational class of type Δ and let Δ' be a type containing Δ. The *extension* $\Delta'(V)$ of V to the type Δ' shall be the class of all algebras of type Δ' whose reducts to the type of Δ are from V. Obviously we have $\Delta'(V) = \mathrm{Mod}_{\Delta'}(\Sigma)$, so $\Delta'(V)$ is again an equational class. A property α of varieties is called *hereditary*, iff for each variety V of a type Δ and for each type $\Delta' \supseteq \Delta$

$$\text{if } \alpha \text{ holds in } V \quad \text{then} \quad \alpha \text{ holds in } \Delta'(V).$$

Now we are able to formulate the main theorem:

1.4. THEOREM. *Let α be an essential and hereditary property of varieties. There exist varieties V_1 and V_2 such that α holds in V_1 and in V_2 but not in $V_1 + V_2$.*

Proof. α is essential, so α holds in a nontrivial variety V. Let $\Delta = (m_i \mid i \in I)$ be the type of V and $H = (h_i \mid i \in I)$ be the corresponding family of function symbols and $V = \mathrm{Mod}_\Delta(\Sigma)$.

We duplicate the type of V by setting $J := 2 \times I$ and $\Delta' := (n_j \mid j \in J)$ with $n_{0,i} = n_{1,i} = m_i$ for all $i \in I$. Let f_i be the function symbol corresponding to $n_{0,i}$ and g_i the function symbol corresponding to $n_{1,i}$.

For $e \in \Sigma$ let e^f (resp. e^g) be the equation, obtained from e by replacing each h_i $(i \in I)$ by f_i (resp. g_i) and $\Sigma^f := \{e^f \mid e \in \Sigma\}$, $\Sigma^g := \{e^g \mid e \in \Sigma\}$.

Define $K^f := \mathrm{Mod}_{\Delta'}(\Sigma^f)$ and $K^g := \mathrm{Mod}_{\Delta'}(\Sigma^g)$.

As α is hereditary, α will hold in K^f and in K^g.

Assume now that α holds in $K^f + K^g$.

α is essential, so we may apply Lemma 1.2 to get a polynomial p over $K^f + K^g$ with $p \not\equiv x$ such that $p = x$ holds in $K^f + K^g$. As $K^f + K^g = \mathrm{Mod}_{\Delta'}(Cn(\Sigma^f) \cap Cn(\Sigma^g))$, the equation $p = x$ must be a consequence of both Σ^f and of Σ^g separately. We may assume that p is of the form $f_i(q_1, \ldots, q_{n_i})$ for a certain $i \in I$ and polynomials $q_1, \ldots, q_{n_i}$ over $K^f + K^g$. Take an algebra $\mathscr{A} \in K^g$ with $|\mathscr{A}| > 1$ and $* \in \mathscr{A}$.

Define a new algebra $\mathscr{A}'$ on the underlying set of $\mathscr{A}$ by:

(i) All operations g_i are defined as on $\mathscr{A}$.

(ii) All operations f_i map each tuple $(a_1, \ldots, a_{n_i}) \in \mathscr{A}^{ni}$ on the point $*$.

$\mathscr{A}'$ obviously belongs to K^g and therefore to $K^f + K^g$ but the equation $p = x$ cannot hold in $\mathscr{A}'$ because $\mathscr{A}'$ has more than one element.

2. Mal'cev conditions

For the following definitions see e.g. [4].

A *strong Mal'cev condition* is a formula of second order logic of the form

$$\exists_{p_0, \ldots, p_n} (\Sigma)$$

where Σ is a (universally quantified) set of equations in the polynomial symbols $p_0, \ldots, p_n$.

A *Mal'cev condition* is a countable disjunction $\mathsf{W}_{i \in \mathbf{N}} S_i$ of strong Mal'cev conditions S_i, $i \in \mathbf{N}$, such that for $i \leqslant j$ $S_i \Rightarrow S_j$.

A *weak Mal'cev condition* is a countable conjunction of Mal'cev conditions.

We will call a (weak) Mal'cev condition trivial, if it holds in every variety or in no nontrivial variety.

DEFINITION. Varieties V_1 and V_2 are called *equivalent*, if there exists an isomorphism $F: V_1 \to V_2$ of categories which commutes with the forgetful functors from V_1 (resp. V_2) into *Set*. That is, for any algebra $\mathscr{A} \in V_1$ and any homomorphism ψ between algebras from V_1 $F(\mathscr{A})$ and $\mathscr{A}$ and also $F(\psi)$ and ψ have the same underlying set.

We note that a weak Mal'cev condition which holds in a variety V must hold in each subvariety of V and in each variety W which is equivalent to V, see e.g. [4].

The *variety of pointed sets* is the (unique) variety of type (o), i.e. having one nullary operation.

Now for weak Mal'cev conditions we get a simpler version of theorem 1.4:

2.5 THEOREM. *If a nontrivial (weak) Mal'cev condition W does not hold in the variety of pointed sets, there exist varieties V_1 and V_2 such that W holds in V_1 and in V_2 but does not hold in $V_1 + V_2$.*

Proof. We only have to show that W is essential and hereditary. Clearly each weak Mal'cev condition is hereditary. Next note that the variety of pointed sets is just Ω_Δ with $\Delta = (\mathrm{o})$. Also if Δ and Δ' are nonempty types, Ω_Δ and $\Omega_{\Delta'}$ are equivalent varieties, so they are equivalent to the variety of pointed sets. Next assume that W is not essential Then there exists a variety V such that W holds in V and $V \supseteq \Omega_\Delta$ where Δ is the type of V. $\Delta \neq \phi$ as W is nontrivial. So W also holds in Ω_Δ and in $\Omega_{(0)}$, in the variety of pointed sets. This yields a contradiction.

For the proof of Theorem 1.1 we observe the following theorem of Wille [7]:

2.6. THEOREM (Wille). *Let E be a congruence equality in the terms* $\circ$, $\wedge$, *and* $\vee$. *There exists a weak Mal'cev condition W such that for any variety V*

E is congruence-valid in V iff W holds in V.

Now Theorem 2.5 and Lemma 1.3 immediately give us Theorem 1.1.

We remark that one also can show easily that a nontrivial congruence-equality is essential and hereditary, so one could also prove Theorem 1.1 without using Theorem 2.6.

Now we can easily list some properties of varieties for which Theorem 2.5, resp. Theorem 1.4 applies:

2.7 COROLLARY. *For the following properties $\alpha_1, \ldots, \alpha_7$ of varieties there exist varieties $V_{i,1}$ and $V_{i,2}$ such that α_i holds in $V_{i,1}$ and in $V_{i,2}$ but not in $V_{i,1}+V_{i,2}$; $1 \leqslant i \leqslant 7$.*

α_1: o-Regularity (*V is* o*-regular iff V has a nullary polynomial* o *and each congruence of each algebra $\mathscr{A} \in V$ is uniquely determined by its class containing* o.)

α_2: Lagrangian (*If $\mathscr{A} \leqslant \mathscr{B} \in V$ and $|\mathscr{B}|$ finite, then $|\mathscr{A}|$ is a divisor of $|\mathscr{B}|$*).

α_3: *Any homomorphic image of a finite $\mathscr{A} \in V$ has power dividing $|\mathscr{A}|$.*

α_4: *Any cosets of a congruence relation θ on a finite $\mathscr{A} \in V$ have the same power.*

α_5: *The number of fixedpoints of any endomorphisms of finite $\mathscr{A} \in V$ is a divisor of $|\mathscr{A}|$.*

α_6: Spec$(V) \subseteq J$ *for a fixed multiplicative closed set $J \subseteq \mathbf{N}$ with* $1 \in J \neq \mathbf{N}$. (Spec$(V) := \{n \in \mathbf{N} \mid \exists \mathscr{A} \in V (|\mathscr{A}| = n)\}$.

α_7: *V has no nontrivial finite algebras.*

Proof. Taylor has shown in [4] that each of the properties α_2–α_7 of varieties is equivalent to a weak Mal'cev condition, and it is obvious, that no one of these properties holds in the variety of pointed sets. For α_1 we may apply Theorem 4. Grätzer has shown in [2] that α_1 is a generalized Mal'cev condition.

3. Mal'cev conditions to which Theorem 2.5 does not apply

Theorem 2.5 and Corollary 2.7 show us that there are only very few Mal'cev conditions which hold in $V_1 + V_2$ whenever they hold in V_1 and in V_2. We will now give some such Mal'cev conditions, and then we are going to show that the converse of Theorem 2.5 is not true in general.

Consider the following strong Mal'cev conditions:

A: $\exists\alpha\ (\alpha(x)=\alpha(y))$

C_2: $\exists p\ (p(x, y)=p(y, x))$

C_n: $\exists p\ (p(x_1, x_2,\ldots, x_n)=p(x_2,\ldots, x_n, x_1))$; $\quad n\in\mathbf{N}$.

Condition A is due to Taylor. He has shown in [5] and in [4]:

3.8. THEOREM (Taylor). *For an equational class V the following are equivalent*:

(i) *A holds in V*

(ii) *Every endomorphism of any algebra in V has a fixed point*

(iii) *Every two subalgebras of any algebra in V have a point in common*

(iv) *If ϕ and ψ are endomorphisms of an algebra $\mathscr{A}\in V$ then there exists an $x\in\mathscr{A}$ such that $\phi(x)=\psi(x)$*

We are now going to prove a similar theorem for the condition C_2 (It may easily be generalized for C_n)

DEFINITION. An *involution* is an automorphism ψ with $\psi\circ\psi=\mathrm{id}$. (An *n-involution* is an automorphism ψ with $\psi^n=\mathrm{id}$).

3.9. THEOREM. *For an equational class V the following are equivalent:*

(i) *C_2 holds in V*

(ii) *If ψ is an endomorphism and $\psi\circ\psi$ has a fixed point then ψ has a fixed point*

(iii) *Each involution has a fixed point*

(iv) *If ψ and ϕ are involutions of $\mathscr{A}\in V$ and $\psi\circ\phi=\phi\circ\psi$ then there is an $x\in\mathscr{A}$ with $\psi(x)=\phi(x)$*

(v) *If for involutions ψ and ϕ of $\mathscr{A}\in V$ there is an $x\in\mathscr{A}$ with $\psi\circ\phi(x)=\phi\circ\psi(x)$ then there is a $y\in\mathscr{A}$ with $\psi(y)=\phi(y)$*

Proof. (i)→(ii): Assume $\psi\circ\psi(x)=x$, then $\psi(p(x, \psi(x))=p(\psi(x), \psi\circ\psi(x))= =p(\psi(x), x)=p(x, \psi(x))$, so $p(x, \psi(x))$ is a fixed point of ψ.

(ii)→(iii) is trivial.

(iii)→(i): Let $F_V(\{x, y\})$ be the free algebra in V on two generators x and y. Let ψ be the unique endomorphism of $F_V(\{x, y\})$ with $\psi(x)=y$ and $\psi(y)=x$. Then ψ is an involution and thus has a fixed point $a\in F_V(\{x, y\})$. So there exists a polynomial p over V such that $p(x, y)=a$. By the definition of $\psi:\psi(a)=\psi(p(x, y))= =p(\psi(x), \psi(y))=p(y, x)=a=p(x, y)$.

(v)→(iv) is again trivial.

(iv)→(iii): take $\psi=\mathrm{id}$.

(i)→(v): $\psi\circ\phi(x)=\phi\circ\psi(x)$ implies $\psi(p(\psi(x), \phi(x))=p(\psi^2(x), \psi\circ\phi(x))= =p(x, \psi\circ\phi(x))=p(x, \phi\circ\psi(x))=p(\phi\circ\psi(x), x)=\phi(p(\psi(x), \phi(x))$.

Now consider varieties V_1 and V_2 in which C_2 holds. So we get a polynomial f over V_1 and a polynomial g over V_2 such that

$$f(x, y)=f(y, x) \text{ holds in } V_1$$

and

$$g(x, y)=g(y, x) \text{ holds in } V_2.$$

But then $f(g(x, y), g(y, x))=f(g(y, x), g(x, y))$ holds in V_1+V_2, so $f(g, g(\pi_2^2, \pi_1^2))$ gives us a commutative polynomial over V_1+V_2, thus C_2 holds in V_1+V_2.

Similarly we can work with C_n and with A, so we get:

3.10. PROPOSITION. *If one of the conditions A, C_2 or C_n holds in varieties V_1 and V_2, it also holds in V_1+V_2.*

This suggests asking whether the converse of Theorem 2.5 is true. The answer is no.

To prove this we have to find a Mal'cev condition B which holds in the variety $\Omega_{(0)}$ and varieties K_1 and K_2 such that B holds in K_1 and K_2 but not in K_1+K_2.

Choose for B the following strong Mal'cev condition:

$$B: \qquad \exists p\ [p(x, p(y, z))=p(p(x, y), z) \wedge p(x, y)=p(y, x)].$$

3.11. PROPOSITION. *There exist varieties K_1 and K_2 such that B holds in K_1 and in K_2 but not in K_1+K_2.*

Proof. Take for a type $\Delta:=(2, 2)$ with corresponding function symbols f and g. Define:

$$K^f:=\text{Mod}_\Delta(\Sigma^f) \quad \text{with} \quad \Sigma^f:=\{f(x, f(y, z))=f(f(x, y), z), f(x, y)=f(y, x)\}.$$

and correspondingly K^g.

Obviously B holds in K^f and in K^g.

Assume B holds in K^f+K^g. Then there exists an associative commutative polynomial p over K^f+K^g.

First: p cannot be a projection (this is in fact the only point where we use the commutativity).

Second: p cannot be g, because g does not satisfy any nontrivial law in K^f. (Analogously p cannot be f.)

Therefore we may assume: p is of the form $f(q_1, q_2)$. The associative law for p is then:

$$\begin{aligned} &f(q_1(x, f(q_1(y, z), q_2(y, z))), q_2(x, f(q_1(y, z), q_2(y, z))))= \\ &\quad =f(q_1(f(q_1(x, y), q_2(x, y)), z), q_2(f(q_1(x, y), q_2(x, y)), z)). \qquad (*) \end{aligned}$$

Define an algebra $\mathscr{A}$ with the set $\mathbf{N}$ of natural numbers as underlying set by setting:

$$g(x, y) := \max\{x, y\} \quad \text{and} \quad f(x, y) := \max\{x, y\} + 1$$

g is associative and commutative, so $\mathscr{A} \in K^g$ and therefore $\mathscr{A} \in K^f + K^g$. For the rest of the proof we need:

3.12. LEMMA. *For every binary polynomial p over $K^f + K^g$ there is a pair (r, l) of natural numbers such that either*

(i) *for all $x, y \in \mathscr{A}$ with $y \geqslant x + r$: $p(x, y) = x + l$*

or

(ii) *for all $x, y \in \mathscr{A}$ with $y \geqslant x + r$: $p(x, y) = y + l$.*

Proof. We prove this by induction on the length of polynomials. For the projections and for f and for g the claim is clear. Assume: $p = f(p_1, p_2)$ and for p_1 and p_2 our claim is true. So there are pairs (r_1, l_1), resp. (r_2, l_2) associated to p_1, resp. p_2. There are four cases to consider:

Case 1. $p_1(x, y) = x + l_1$ for $x + r_1 \leqslant y$
$p_2(x, y) = x + l_2$ for $x + r_2 \leqslant y$.

Take $r = \max\{r_1, r_2\}$ and $l = \max\{l_1, l_2\}$, then

$$p(x, y) = f(p_1(x, y), p_2(x, y)) = x + l + 1 \quad \text{for} \quad x + r \leqslant y.$$

Case 2. $p_1(x, y) = x + l_1$ for $x + r_1 \leqslant y$
$p_2(x, y) = y + l_2$ for $x + r_2 \leqslant y$.

Take $r = \max\{r_1, r_2, l_1 - l_2\}$, then

$$f(p_1(x, y), p_2(x, y)) = f(x + l_1, y + l_2) = y + l_2 + 1 \quad \text{for} \quad x + r \leqslant y.$$

The remaining cases follow by symmetry.

Turning back to our assumption that there exists an associative and commutative polynomial p over $K^f + K^g$, we consider this polynomial in our algebra $\mathscr{A}$.

We have already remarked that p is of the form $f(q_1, q_2)$. By the above claim we find for p_1, resp. p_2 pairs (r_1, l_1), resp. (r_2, l_2) satisfying the conditions of Lemma 3.12.

Set $r := \max\{r_1, r_2\}$, $l := \max\{l_1, l_2\}$ and choose a, b and $c \in \mathbf{N}$ subject to the conditions

(i) $b \geqslant a + r$

(ii) $b \geqslant a + |l_1 - l_2|$

(iii) $c \geqslant b + l + r + 1$.

Then there are again four cases to consider as in the proof of the above lemma:

Now, however, we can actually compute both sides of the equation $(\underline{*})$, having substituted a, b and c for x, y and z, respectively:

	Left side of $(\underline{*})$	Right side of $(\underline{*})$
Case 1 yields	$a+l+1$	$a+2l+2$
Case 2 yields	$c+2l_2+2$	$c+l_2+1$
Case 3 yields	$c+2l+2$	$c+l+1$
Case 4 yields	$c+2l_1+2$	$c+l_1+1.$

In each of these cases the left hand side of $(\underline{*})$ and the right hand side of $(\underline{*})$ are different, so there cannot be an associative and commutative polynomial p over K^f+K^g.

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