Coalgebraic structure from weak limit preserving functors

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Abstract

Given an endofunctor F on the category of sets, we investigate how the structure theory of $\mathcal{S}et_F$, the category of F-coalgebras, depends on certain preservation properties of F. In particular, we consider preservation of various weak limits and obtain corresponding conditions on bisimulations and subcoalgebras. We give a characterization of monos in $\mathcal{S}et_F$ in terms of congruences and bisimulations, which explains, under which conditions monos must be injective maps.

Key words: Coalgebra, weak pullback, bisimulation.

1 Introduction

In recent years it has been discovered that a wide range of state based systems, amongst them various types of transition systems, automata, infinite and object oriented data structures, even topological spaces, can all be uniformly captured by the notion of *coalgebra*. It has turned out that the pertinent notions such as *bisimulation*, *finality*, *coinduction* and *cogeneration* can be defined uniformly for arbitrary coalgebras and their structure theory can be developed to quite some depth in the abstract setting of *coalgebras of type* F. Here F is an arbitrary endofunctor on the category of sets which serves as the *type functor* for the coalgebras under consideration. The above mentioned examples result from choosing for F the *powerset functor* $\mathcal{P}(-)$, the *power functor* $(-)^\Sigma$, the *product functor* $(-) \times \Sigma$, the *filter functor* $\mathcal{F}(-)$ (see [Gum98]) or various combinations thereof.

A major advantage of coalgebras is that the theory can naturally deal with nondeterminism and undefinedness, concepts which are hard, or even impossible, to treat algebraically.

The abstract theory of F-coalgebras has by now reached a certain kind of maturity, perhaps comparable to the state of affairs in universal algebra before the times of Mal'cev. The standard reference, which also includes many examples, has been by J. Rutten [Rut96]. Based on the important observation, that most relevant type functors share an extra property, namely they *preserve weak pullbacks*, much of the structure theory has been developed under this premise. Weak preservation of arbitrary pullbacks (that is limits of arbitrary sets of arrows with a common codomain) guarantees, amongst other things, that subcoalgebras of a coalgebra $\mathcal{A}$ are closed under intersection. This, in turn, plays an important structure theoretic role, since it allows one to work with one-generated subcoalgebras, that is subcoalgebras $\langle a \rangle$ of $\mathcal{A}$, generated by arbitrary elements $a \in \mathcal{A}$. In particular, every coalgebra is then a *conjunct sum* of *conjunctly irreducibles*, see [GS98], and the lattice of subcoalgebras is a sublattice of the powerset $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$.

However, amongst some researchers there has always been the uneasy feeling that assuming preservation of weak pullbacks trivializes interesting parts of the theory. For instance, it implies that both union and intersection of subcoalgebras are again subcoalgebras. In the dual field of *universal algebra*, one would rarely expect unions of subalgebras to be subalgebras. In fact such is the case essentially only with algebras whose nontrivial operations are unary. Moreover, there are viable examples of coalgebras (topological spaces, for instance) where one-generated subcoalgebras hardly ever exist.

For those reasons, in [Gum99] an attempt was made, to lay down an introduction to the general theory of coalgebras which abstains from any particular assumption on the type functor F. Surprisingly, the relevant structure theory could still be developed, including the isomorphism theorems and a coalgebraic version of Birkhoff's theorem. It turned out, that many proofs became considerably simpler, however further notions needed to enter the stage, such as e.g. *congruences* which previously had coincided with other notions, e.g. *bisimulation equivalence*.

Thus, not all structure theoretic results from [Rut96] are valid in the general case. For instance, monomorphisms need not be injective, the relational product $R \circ S$ of two bisimulations need not be a bisimulation and the preimage $\varphi^{-1}[\mathcal{U}]$ of a subcoalgebra $\mathcal{U} \leq \mathcal{B}$ under a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ need not be a subcoalgebra of $\mathcal{A}$. In this article, we shall identify the particular preservation properties of the functor F which are responsible for such structure theoretic properties.

In particular, we shall prove structure theoretic equivalents in the category $\mathcal{S}et_F$ of F-coalgebras to the following preservation properties of a type functor F:

- F weakly preserves pullbacks,
- F weakly preserves kernels,

- F weakly preserves pullbacks along injective maps,
- F weakly preserves pullbacks of injective maps.

When F does not preserve weak pullbacks, then monomorphisms need not be injective. The dual situation is well known in universal algebra, where epis are not necessarily surjective [1]. We give a simple structural criterion for the kernel $Ker(\varphi)$ of a homomomorphism φ that determines whether φ is a monomorphism or whether the monomorphism φ is injective.

2 Basic Notions

Let $F : \mathcal{S}et \to \mathcal{S}et$ be a functor on the category of sets. Such an F is usually called an *endofunctor*, but in our context we shall refer to it as a *type*.

An F*-coalgebra* (coalgebra of type F) is a pair $\mathcal{A} = (A, \alpha_A)$, consisting of a set A and a map $\alpha_A : A \to F(A)$. The set A is called the *carrier* or the *underlying set* and the map α_A is called the *co-operation* or the *structure map* of $\mathcal{A}$. In some contexts, it is suggestive to refer to A as the *state space* and to α_A as the *transition structure* of $\mathcal{A}$.

Given two F-coalgebras $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$, a map $\varphi : A \to B$ is called a *homomorphisms* if it preserves the structure, i.e. if it makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ F(A) \arrow[r, "F(\varphi)"] & F(B). \end{tikzcd}$$

This definition turns the class of all coalgebras of type F into a category, $\mathcal{S}et_F$. This category inherits all colimits from $\mathcal{S}et$, that is:

Theorem 2.1 ([Bar93]) *The forgetful functor from $\mathcal{S}et_F$ to $\mathcal{S}et$ creates colimits, in particular, in the category $\mathcal{S}et_F$ all coproducts and all coequalizers exist and are constructed as in $\mathcal{S}et$, the category of sets.*

The following theorem of Rutten can be used to prove that a map is a homomorphism:

Theorem 2.2 ([Rut96]) *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be coalgebras, $\varphi : \mathcal{A} \to \mathcal{C}$ a homomorphism and $f : A \to B$ and $g : B \to C$ maps with $g \circ f = \varphi$. Then*

(i) If f is a surjective homomorphism, then g is a homomorphism.

(ii) If g is an injective homomorphism, then f is a homomorphism.

[1] The natural embedding $\iota : \mathbb{Z} \to \mathbb{Q}$ is both mono and epi in the category of rings.

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

As a corollary, every bijective homomorphism φ is an isomorphism, that is, its inverse φ^{-1} is also a homomorphism.

2.1 Subcoalgebras

Subcoalgebras are defined in the usual way: If $\mathcal{A} = (A, \alpha)$ is a coalgebra, then a subset $U \subseteq A$ is called *closed*, if an F-coalgebra structure can be defined on U, so that the canonical inclusion map $\subseteq: U \to A$ becomes a homomorphism. U together with this structure map is called a subcoalgebra of $\mathcal{A}$ and we write $\mathcal{U} \leq \mathcal{A}$. It is easy to see that the mentioned coalgebra structure on U is unique, so that one often uses "closed set" and "subcoalgebra" synonymously.

The empty set is always closed, more generally, arbitrary unions of closed sets are closed ([Gum99]). Therefore, for any F-coalgebra $\mathcal{A} = (A, \alpha)$ and for any subset $S \subseteq A$, there is a largest subcoalgebra of $\mathcal{A}$ which is contained in S. We shall denote it by $[S]$ and call it the subcoalgebra *cogenerated by* S. We shall need:

Theorem 2.3 ([Gum99]) *Given a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, and given a subcoalgebra $\mathcal{U} \leq A$, then $\varphi[U] := \{\varphi(u) \mid u \in U\}$ is a closed subset of B, that is $\mathcal{U} \leq \mathcal{A}$ implies $\varphi[\mathcal{U}] \leq B$ and $\varphi : \mathcal{U} \to \varphi[\mathcal{U}]$ is a surjective homomorphism.*

If F preserves weak pullbacks, then preimages of subcoalgebras are subcoalgebras, too, that is: Given $\varphi : \mathcal{A} \to \mathcal{B}$ and $\mathcal{U} \leq \mathcal{B}$, then $\varphi^{-1}[U] := \{a \in A \mid \varphi(a) \in U\}$ is a subcoalgebra of $\mathcal{A}$ ([Rut96]). If F does not preserve weak pullbacks this is not true, as we shall see later.

2.2 Bisimulations

One of the most important notions of the theory of coalgebras is that of a *bisimulation*. Given coalgebras $\mathcal{A}$ and $\mathcal{B}$, a *bisimulation between $\mathcal{A}$ and $\mathcal{B}$* is a binary relation $R \subseteq A \times B$ on which a coalgebra structure $\delta : R \to F(R)$ can be defined, so that the canonical projections $\pi_1 : R \to \mathcal{A}$ and $\pi_2 : R \to \mathcal{B}$ become homomorphisms. If $\mathcal{A} = \mathcal{B}$, we speak of a bisimulation *on* $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[d, "\alpha_1"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\delta"] & B \arrow[d, "\alpha_2"] \\ F(A) & F(R) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(B) \end{tikzcd}$$

We shall need the following facts about bisimulations, which are proved in [Rut96]:

Theorem 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be coalgebras.*

(i) The union of bisimulations is a bisimulation, in particular, there is a largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$ between $\mathcal{A}$ and $\mathcal{B}$.

(ii) $\Delta_A := \{(a,a) \mid a \in A\}$ is always a bisimulation on $\mathcal{A}$.

(iii) If R is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, then

$$R^{-} := \{(y,x) \mid (x,y) \in R\}$$

is a bisimulation between $\mathcal{B}$ and $\mathcal{A}$.

(iv) Given a coalgebra $\mathcal{P}$ and two homomorphisms $\varphi_1 : \mathcal{P} \to \mathcal{A}$ and $\varphi_2 : \mathcal{P} \to \mathcal{B}$, then

$$(\varphi_1, \varphi_2)\mathcal{P} := \{(\varphi_1(p), \varphi_2(p)) \mid p \in \mathcal{P}\}$$

is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$. We call this the canonical bisimulation *arising from the 2-source $(\mathcal{P}, \varphi_1, \varphi_2)$.* [2]

As a consequence of this theorem, for every relation $R \subseteq A \times B$ there is always a largest bisimulation between $\mathcal{A}$ and $\mathcal{B}$ which is contained in R. We denote it by $[R]$. If R is reflexive or symmetric, then so is $[R]$.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $(id_{\mathcal{A}}, \varphi)A$ is a bisimulation according to the above theorem. This set is nothing but

$$G(\varphi) := \{(a, \varphi(a)) \mid a \in A\},$$

the *Graph of φ*.

2.3 Congruences

We define a *congruence relation* on a coalgebra $\mathcal{A}$ as the kernel of any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, more precisely:

Definition 2.5 *A binary relation θ on A is a congruence relation if there exists a coalgebra $\mathcal{B}$ and a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ so that*

$$\theta = Ker(\varphi) := \{(x,y) \in A \times A \mid \varphi(x) = \varphi(y)\}.$$

Let θ be an equivalence relation on a set A, then the map $\pi_\theta : A \to A/\theta$, mapping every element $a \in A$ to $[a]\theta$, its equivalence class modulo θ, is the coequalizer, in the category of sets, of the projection maps $\pi_i : \theta \to A$. Now assume that θ happens to be a bisimulation on the coalgebra $\mathcal{A}$, then the π_i are homomorphisms with respect to somepf appropriate coalgebra structure on θ. By theorem 2.1, π_θ is also the coequalizer in $\mathcal{S}et_F$ of π_1 and π_2, so there exists some coalgebra structure on A/θ for which $\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta$ is a

[2] A 2-source ist just a pair of morphisms with common domain.

homomorphism. We still have $\theta = Ker(\pi_\theta)$, so θ is a congruence, hence we get a result, due to Aczel and Mendler:

Theorem 2.6 ([AM89]) *A bisimulation R on $\mathcal{A}$ is a congruence relation, provided that R is reflexive, symmetric and transitive.*

Whenever the type functor preserves weak pullbacks, the converse of this theorem is also true ([Rut96]). In this case congruences are the same as bisimulation equivalences. Congruences have originally been introduced by Aczel and Mendler in 1989. As remarked earlier, most of the subsequent literature on coalgebras did assume preservation of weak pullbacks, so the notion of *congruence* has not received further attention.

2.4 A counterexample

The following example was also introduced in [AM89] to show that in general not every congruence needs to be a bisimulation. Since we shall reuse this example in the following sections, we present it here in detail:

Consider the functor $(-)^3_2 : \mathcal{S}et \to \mathcal{S}et$ which associates with every set A the set

$$A^3_2 := \{(x, y, z) \in A^3 \mid |\{x, y, z\}| \leq 2\}.$$

A map $f : A \to B$ is transformed into $(f)^3_2 : A^3_2 \to B^3_2$ by way of

$$(f)^3_2(x, y, z) := (f(x), f(y), f(z)).$$

Clearly, the 1-element set $\underline{1} = \{0\}$ can be equipped with a $(-)^3_2$-coalgebra structure. It is, in fact, terminal in the category of $(-)^3_2$-coalgebras.

Apart from this rather trivial example, it suffices to consider the two-element $(-)^3_2$-coalgebra $\mathcal{A} = (A, \alpha)$ with $A = \{a, b\}$, $\alpha(a) = (a, b, b)$ and $\alpha(b) = (b, b, a)$. There is a unique homomorphism $\varphi : \mathcal{A} \to \underline{1}$, and its kernel is $A \times A$. However, $A \times A$ is not a bisimulation, for its coalgebra structure δ would have to satisfy $\pi_1(\delta(a, b)) = \alpha(a) = (a, b, b)$ and $\pi_2(\delta(a, b)) = \alpha(b) = (b, b, a)$. This implies $\delta(a, b) = ((a, b), (b, b), (b, a)) \notin (A \times A)^3_2$. The argument shows that (a, b) (and similarly (b, a)) can not be contained in any bisimulation on $\mathcal{A}$. Therefore, the largest bisimulation on $\mathcal{A}$ is the diagonal Δ_A.

3 Epis and monos in $\mathcal{S}et_F$

A morphism which is epi (mono) in $\mathcal{S}et$ is trivially also epi (mono) in $\mathcal{S}et_F$. Epis, in any category, are just those morphisms for which the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & B \arrow[d, "id_B"] \\ B \arrow[r, "id_B"] & B \end{tikzcd}$$

is a pushout. Thus, the following result of Rutten is another consequence of theorem 2.1:

Theorem 3.1 ([Rut96]) *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. φ is epi in $\mathcal{S}et_F$ iff it is epi in $\mathcal{S}et$, i.e. surjective.*

The story is different for monos, and we shall see that monos in $\mathcal{S}et_F$ need not be injective. The following theorem, in fact, points out what is missing:

Theorem 3.2 *A homomorphism φ is mono iff $[Ker(\varphi)] = \Delta_{\mathcal{A}}$.*

Proof. Assume that $\varphi : \mathcal{A} \to \mathcal{B}$ is mono. Let $\pi_1, \pi_2 : Ker(\varphi) \to A$ be the canonical projection maps. Let $\tilde{\pi}_1, \tilde{\pi}_2 : [Ker(\varphi)] \to A$ be their restrictions to $[Ker(\varphi)]$. The latter set is a bisimulation on $\mathcal{A}$, so $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are coalgebra homomorphisms and $\varphi \circ \tilde{\pi}_1 = \varphi \circ \tilde{\pi}_2$. It follows $\tilde{\pi}_1 = \tilde{\pi}_2$, that is, $[Ker(\varphi)] = \Delta_A$.

Conversely, assume that $[Ker(\varphi)] = \Delta_A$ and assume that there are homomorphisms $\kappa_1, \kappa_2 : \mathcal{P} \to \mathcal{A}$ with $\varphi \circ \kappa_1 = \varphi \circ \kappa_2$. By theorem 2.4, $(\kappa_1, \kappa_2)\mathcal{P}$ is a bisimulation on $\mathcal{A}$, and it is clearly contained in $Ker(\varphi)$. By assumption then, $(\kappa_1, \kappa_2)\mathcal{P} \subseteq \Delta_A$ which implies that $\kappa_1 = \kappa_2$.

Corollary 3.3 *A monomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is injective iff $Ker(\varphi)$ is a bisimulation.*

3.1 An example

The coalgebra $\mathcal{A}$ on the base set $A = \{a, b\}$ from the previous section readily provides us with an example of a homomorphism that is both epi and mono but still not injective: The unique map $\varphi : \mathcal{A} \to \underline{1}$ is a surjective homomorphism, infact, $\underline{1}$ is final, but the kernel of φ is $A \times A$. The previous section shows that $\sim_{\mathcal{A}} = \Delta_A$, therefore $[Ker(\varphi)] = \Delta_A$.

A purely category theoretic characterization of injective homomorphisms can be given as follows:

Theorem 3.4 *A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is injective if and only if φ is an equalizer.*

Proof. The equalizer of two homomorphisms $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{B}$ is obtained from $\mathcal{E}_{\psi_1,\psi_2} = [\{a \in A \mid \psi_1(a) = \psi_2(a)\}]$, the largest subcoalgebra of $\mathcal{A}$ contained in the equalizer (in $\mathcal{S}et$) of ψ_1 and ψ_2. The equalizer of ψ_1 and ψ_2 is just the canonical embedding of the subcoalgebra $\mathcal{E}_{\psi_1,\psi_2}$ into $\mathcal{A}$. (For type functors F preserving weak pullbacks, this is shown in [Wor98], a proof for the general case can be found in [Gum99].)

Conversely, assume that $\varphi : \mathcal{A} \to \mathcal{B}$ is an injective homomorphism. In the category $\mathcal{S}et$, φ has a left inverse φ^- with $\varphi^- \circ \varphi = id_{\mathcal{B}}$. One checks that φ is the equalizer in $\mathcal{S}et$ of $f_1 := \varphi \circ \varphi^-$ and $f_2 = id_B$. Let $(\mathcal{P}, \pi_1, \pi_2)$ be the pushout of φ with itself. (B, f_1, f_2) is a competitor for this pushout, so there is a set map $h : P \to B$ with $h \circ \pi_i = f_i$ for $i = 1, 2$.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\kappa"] \arrow[drr, bend left, "\chi"] \arrow[ddr, bend right, "\chi"'] & & & \\
& A \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & B \arrow[d, "\pi_1"'] \arrow[ddr, bend left, "f_1 = \varphi \circ \varphi^-"] & \\
& B \arrow[r, "\pi_2"] \arrow[drr, bend right, "id_B"'] & P \arrow[dr, "h"] & \\
& & & B
\end{tikzcd}$$

We claim that φ is the equalizer in $\mathcal{S}et_F$ of π_1 and π_2. So let $\chi : \mathcal{Q} \to \mathcal{B}$ be a competitor of φ, that is χ is a homomorphism with

$$\pi_1 \circ \chi = \pi_2 \circ \chi.$$

It follows that

$$\begin{aligned} f_1 \circ \chi &= h \circ \pi_1 \circ \chi \\ &= h \circ \pi_2 \circ \chi \\ &= f_2 \circ \chi. \end{aligned}$$

Since φ is the equalizer, in $\mathcal{S}et$, of f_1 and f_2, there is a unique map $\kappa : Q \to A$ with $\varphi \circ \kappa = \chi$. Since χ and φ are homomorphisms, so is κ by theorem 2.2. Consequently, κ is the unique homomorphisms with $\varphi \circ \kappa = \chi$.

4 Special Functors

Most functors which we have mentioned in the introduction share an extra property which in the past was assumed in much of the structure theoretic investigations into coalgebras: They preserve *weak pullbacks*. We shall study this condition and a collection of somewhat weaker requirements on F, and we shall relate them to their structure theoretic consequences.

4.1 Split epis

Definition 4.1 *Amongst the objects of a category $\mathcal{C}$ we can define a relation $\preceq$ as follows:*
$A \preceq B :\Leftrightarrow$
there are morphisms $\tau : B \to A$ and $\delta : A \to B$ with $\tau \circ \delta = id_A$.

It follows that δ is mono and τ is epi. Such a τ is also called *split epi*. The relation $\preceq$ is a quasi ordering, i.e. $\preceq$ is

- reflexive:
  $\forall A \in \mathcal{C}.\ A \preceq A$,
- transitive:
  $\forall A, B, C \in \mathcal{C}.\ A \preceq B, B \preceq C \implies A \preceq C$.

$\preceq$ is preserved by arbitrary functors, that is, each functor F is

- monotone:
  $\forall A, B \in \mathcal{C}.\ A \preceq B \implies F(A) \preceq F(B)$.

In the category of sets we have $A \preceq B \iff |A| \leq |B|$, therefore, according to the Schröder-Bernstein theorem [3] $\preceq$ is

- anti-symmetric:
  $\forall A, B \in \mathcal{C}.\ A \preceq B, B \preceq A \implies A \cong B$.

Every split epi is an epi. In $\mathcal{S}et$, assuming the axiom of choice, we have the converse, i.e. every epi is a split epi.

4.2 Weak limit preservation

Given a diagram D in a category $\mathcal{C}$, a *weak limit of D* is given by a cone $(W, (w_i)_{i\in I})$ so that for every other cone $(Q, (q_i)_{i\in I})$ over D (that is for every competitor of $(W, (w_i)_{i\in I})$) there is at least one morphism $d : Q \to W$ with $w_i \circ d = q_i$ for all $i \in I$.

Definition 4.2 *Let $F : \mathcal{S}et \to \mathcal{S}et$ be a functor and D a diagram. We say that*

- F weakly preserves D-limits*, if F transforms every limit cone over D into a weak limit cone over $F(D)$, i.e. for every limit $(L, (\nu_i)_{i\in I})$ of the diagram D we get that $(F(L), (F(\nu_i))_{i\in I})$ is a weak limit of the transformed diagram $F(D)$.*
- F preserves weak D-limits *if it transforms every weak limit cone over D into a weak limit cone over $F(D)$.*

Fortunately, the fine linguistic difference between "F preserves weak limits" and "F weakly preserves limits" is easily seen to disappear in every category where all D-limits exist. This is an easy consequence of the following observation:

Lemma 4.3 *Let D be a diagram in an arbitrary category $\mathcal{C}$.*

(i) If $(W, (w_i)_{i\in I})$ is a weak limit of D and $W \preceq W'$ with split epi $\tau : W' \to W$, then $(W', (w_i \circ \tau)_{i\in I})$ is also a weak limit of D.

(ii) If the limit $(L, (p_i)_{i\in I})$ of D exists, then $(W, (w_i)_{i\in I})$ is a weak limit of D if and only if $w_i = p_i \circ \tau$ for some split epi $\tau : W \to L$, in particular, $L \preceq W$.

Thus, with respect to the order $\preceq$, introduced on the objects of a category, limits, if they exist, are just the infima of all weak limits.

[3] Given sets A and B and injective maps $f : A \to B$ and $g : B \to A$ then there is a bijection between A and B.

Corollary 4.4 *Let $\mathcal{C}$ be a category and D a diagram so that every D-limit exists in $\mathcal{C}$. Then F preserves weak D-limits if and only if F weakly preserves D-limits.*

Theorem 4.5 *Let C be a category in which every D-limit exists, then an endo-functor F preserves weak D-limits if and only if for every D-limit $(L, (p_i)_{i\in I})$ there exists a morphism δ from the limit $(Q, (q_i)_{i\in I})$ of $F(D)$ to $F(L)$, so that $F(p_i) \circ \delta = q_i$ for all i.*

Proof. Let $(L, (p_i)_{i\in I})$ be the limit of D and $(Q, (q_i)_{i\in I})$ the limit of $F(D)$. If $(F(L), (F(p_i))_{i\in I})$ is a weak limit of $F(D)$, then by definition there must be at least one morphism $\delta : Q \to F(L)$ with $F(p_i) \circ \delta = q_i$.

Conversely, if such a $\delta : Q \to F(L)$ exists, consider the unique morphism $\tau : F(L) \to Q$ with $F(p_i) = q_i \circ \tau$.

$$\begin{tikzcd} L \arrow[dr, "p_i"] & \\ & D_i \end{tikzcd} \qquad \begin{tikzcd} F(L) \arrow[d, bend left, "\tau"] \arrow[dr, "F(p_i)"] & \\ Q \arrow[u, dashed, bend left, "\delta"] \arrow[r, "q_i"'] & F(D) \end{tikzcd}$$

Then $q_i \circ \tau \circ \delta = q_i \circ id_Q$, so $\tau \circ \delta = id_Q$. Hence τ is split epi and $F(L)$ is a weak limit of $F(D)$.

4.3 Weak κ-pullbacks and their preservation

We illustrate the above situation in the case of a pullback (L, p_1, p_2) of two morphisms $f : A \to C$ and $g : B \to C$.

$$\begin{tikzcd} L \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Let Q the limit of the transformed diagram, then F weakly preserves pullbacks iff there is a map $\delta : Q \to F(L)$ with $F(p_i) \circ \delta = q_i$ for $i = 1, 2$.

$$\begin{tikzcd} Q \arrow[dr, shift left, "\delta"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\ & F(L) \arrow[ul, shift left, "\tau"] \arrow[r, "F(p_1)"'] \arrow[d, "F(p_2)"] & F(A) \arrow[d, "F(f)"] \\ & F(B) \arrow[r, "F(g)"'] & F(C) \end{tikzcd}$$

The pullback of two maps $f : A \to C$ and $g : B \to C$ in $\mathcal{S}et$ has as object the set $pb(f, g) = \{(a, b) \in A \times B \mid f(a) = g(b)\}$ and as morphisms the canonical projections π_1 and π_2. From this we get an easy criterion for a $\mathcal{S}et$-endofunctor to preserve weak pullbacks:

Proposition 4.6 ([Gum98]) *A functor $F : \mathcal{S}et \to \mathcal{S}et$ preserves weak pullbacks, iff for all maps $f : A \to C$ and $g : B \to C$ we have: Given $u \in F(A)$ and $v \in F(B)$ with $F(f)(u) = F(g)(v)$ then there exists $w \in F(\{(x, y) \mid f(x) = g(y)\})$ with $F(\pi_1)(w) = u$ and $F(\pi_2)(w) = v$.*

Corollary 4.7 *F weakly preserves the pullback of f and g iff the map $F(\pi_1) \times F(\pi_2) : F(pb(f, g)) \to pb(F(f), F(g))$ is onto.*

We will need to generalize the notion of pullback to consider pullbacks of a family of arrows with common codomain.

Definition 4.8 *Let κ be an ordinal. A κ-*sink *is a family $(f_i : A_i \to A)_{i\in\kappa}$ of morphisms with common codomain. A κ-pullback is the limit $(P, (\pi_i)_{i\in\kappa})$ of a κ-sink. A* weak κ-pullback *is a weak limit of a κ-sink.*

Thus, a 2-pullback is just an ordinary pullback. There has occasionally been slight confusion in the literature concerning the notion of "preservation of weak pullbacks". We know of functors preserving weak pullbacks, which do not preserve weak κ-pullbacks for $\kappa \geq \omega$. An example is the *filter functor*, see [Gum98].

The category of sets is *complete* and *cocomplete*, which is to say that all limits and all colimits exist. In particular, κ-pullbacks in $\mathcal{S}et$ exist and they are constructed similar as in the finite case:

Proposition 4.9 *Let $(f_k : A_k \to A)_{k\in\kappa}$ be a κ-sink. Then*

$$pb((f_k)_{k\in\kappa}) = \{(a_k)_{k\in\kappa} \mid \forall i, j \in \kappa. f_i(a_i) = f_j(a_j)\}$$

with canonical projections π_j defined as $\pi_j((a_k)_{k\in\kappa}) = a_j$ is the κ-pullback of the f_k in $\mathcal{S}et$. A functor F preserves weak κ-pullbacks iff for every family $(u_k)_{k\in\kappa} \in pb(F((f_k)_{k\in\kappa}))$ there exists an element $w \in F(pb((f_k)_{k\in\kappa}))$ with $F(\pi_k)(w) = u_k$ for all $k \in \kappa$.

Example 4.10 *Most functors considered so far preserve weak κ-pullbacks for arbitrary κ. Amongst those are:*

(i) *The constant functor $F(X) = A$ for a fixed set A.*

(ii) *The identity functor $\mathcal{I}d$.*

(iii) *$F(X) = A + X$ for a fixed set A.*

(iv) *$F(X) = X^\Sigma$ for a fixed set Σ.*

(v) *The power set functor $\mathcal{P}(-)$.*

The following lemma allows us to combine the above examples. Most of the practically relevant coalgebras have a type which arises in such a way.

Lemma 4.11 *If functors F and G preserve weak κ-pullbacks for some κ, then so do $F \circ G$, $F \times G$, and $F + G$.*

Example 4.12 *The functor $(-)^3_2$ does not preserve weak pullbacks. However, if at least one of $f : A \to C$, $g : B \to C$ is injective, then the functor weakly preserves the pullback of f and g. We say, that $(-)^3_2$ weakly preserves pullbacks* along injective maps.

Proof. Given $f : A \to C$, $g : B \to C$, $(u_1, u_2, u_3) \in A^3_2, and (v_1, v_2, v_3) \in B^3_2$ with $f(u_i) = g(v_i)$. We must find $w \in \{(x, y) \mid f(x) = g(y)\}^3_2$ so that $(\pi_1)^3_2(w) = u$ and $(\pi_2)^3_2(w) = v$. The only possibility is $w = ((u_1, v_1), (u_2, v_2), (u_3, v_3))$. It remains to show that $w \in (A \times B)^3_2$, i.e. that two of the components of w are equal. We may assume w.l.o.g. that $u_1 = u_2$. If $v_1 = v_2$ then we are done. The other case is, again w.l.o.g, that $v_2 = v_3$. If f is injective then $f(u_2) = g(v_2) = g(v_3) = f(u_3)$, so $(u_2, v_2) = (u_3, v_3)$, similarly, if g is injective then $(u_1, v_1) = (u_2, v_2)$.

To see that $(-)^3_2$ does not preserve weak pullbacks of arbitrary maps, choose $f = g$ the constant map $\{0, 1\} \to \{0\}$. For $u = (0, 0, 1)$ and $v = (1, 0, 0)$ it is impossible to find the required w with $(\pi_1)^3_2(w) = u$ and $(\pi_2)^3_2(w) = v$.

5 Preservation theorems

In the following we shall consider and characterize several particular properties which the functor F might have in regard to weak preservation of certain types of pullbacks.

Clearly, if F weakly preserves pullbacks, then

- F weakly preserves kernel pairs,
- F (weakly) preserves pullbacks along injective maps,
- F (weakly) preserves pullbacks of injective maps.

We shall study these conditions and characterize them by way of their structure theoretical consequences.

5.1 F weakly preserving pullbacks

There are many reasons why functors weakly preserving pullbacks entail nice structure theoretical properties. In purely category theoretical terms, such functors are characterized by the fact that they can be extended to functors on $\mathcal{R}el$, the category having as objects all sets and as morphisms all binary relations. This fact is due to Carboni, Kelly, and Wood [CKW90], for a further discussion see Rutten [Rut98].

The following theorem characterizes functors weakly preserving pullbacks in a coalgebraic context. The implications 1. → 2. → 3. are due to Rutten [Rut96].

Theorem 5.1 *For a functor F the following are equivalent:*

(i) *F weakly preserves nonempty pullbacks.*

(ii) *The pullback $pb(\varphi, \psi)$ of two F-homomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$.*

(iii) *In $\mathcal{S}et_F$, the relational product $R \circ S$ of two bisimulations R and S is a bisimulation.*

(iv) *If $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{D}$ are F-homomorphisms, and if R is a bisimulations between $\mathcal{C}$ and $\mathcal{D}$, then*

$$(\varphi, \psi)^{-}[R] := \{(a, b) \in A \times B \mid \varphi(a)\, R\, \psi(b)\}$$

is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$.

The key in proving the reverse direction of this and some later theorems is:

Lemma 5.2 *Let $F : \mathcal{S}et \to \mathcal{S}et$ be a functor and $f : A \to C$ and $g : B \to C$ be maps with $pb(f, g) \neq \emptyset$. Then the following are equivalent:*

(i) *F weakly preserves the pullback of f and g.*

(ii) *$\{(a, b) \mid f(a) = g(b)\}$ is a bisimulation for each coalgebra structure on A, B, and C for which f and g are homomorphisms .*

(iii) *For each pair $(a, b) \in pb(f, g)$ and for all coalgebra structures on A, B, and C making f and g homomorphisms there is a 2-source (Q, p_1, p_2) and an element $q \in Q$ with $f \circ p_1 = g \circ p_2$, $p_1(q) = a$ and $p_2(q) = b$.*

Proof. The equivalence of 2. and 3. is rather straightforward: If $pb(f, g)$ is a bisimulation, then it is an appropriate 2-source. Conversely, from a 2-source (Q, p_1, p_2) with $f \circ p_1 = g \circ p_2$ we get the canonical bisimulation $(p_1, p_2)Q$ as a subset of $pb(f, g)$. The extra condition guarantees that the sum of all such 2-sources has precisely $pb(f, g)$ as standard bisimulation.

The implication (1. $\to$ 2.) is due to Rutten([Rut96]): Let (P, π_1, π_2) be the pullback of the homomorphisms $f : A \to C$ and $g : B \to C$. Then $(F(P), F(\pi_1), F(\pi_2))$, by assumption, is a weak pullback of $F(f) : F(A) \to F(C)$ and $F(g) : F(B) \to F(C)$. Since f and g are homomorphisms, we calculate

$$\begin{aligned} F(f) \circ \alpha_A \circ \pi_1 &= \alpha_C \circ f \circ \pi_1 \\ &= \alpha_C \circ g \circ \pi_2 \\ &= F(g) \circ \alpha_B \circ \pi_2 \end{aligned}$$

$$\begin{tikzcd}
 & B \arrow[rr, "g"] \arrow[dd, "\alpha_B"] & & C \arrow[dd, "\alpha_C"] \\
P \arrow[ur, "\pi_2"] \arrow[rr, "\pi_1"] \arrow[dd, dashed, "\alpha_P"] & & A \arrow[ur, "f"'] \arrow[dd, "\alpha_A"] & \\
 & F(B) \arrow[rr, "F(g)"] & & F(C) \\
F(P) \arrow[ur, "F(\pi_2)"] \arrow[rr, "F(\pi_1)"'] & & F(A) \arrow[ur, "F(f)"'] &
\end{tikzcd}$$

This makes $(P, \alpha_A \circ \pi_1, \alpha_B \circ \pi_2)$ a competitor of the weak limit $(F(P), F(\pi_1), F(\pi_2))$ from which we get the desired mediating map α_P with $\alpha_A \circ \pi_1 = F(\pi_1) \circ \alpha_P$ and $\alpha_B \circ \pi_2 = F(\pi_2) \circ \alpha_P$.

2. $\rightarrow$ 1.: Let $f : A \rightarrow C$ and $g : B \rightarrow C$ be maps and (P, π_1, π_2) their pullback with $P \neq \emptyset$. Consider the pullback (Q, p_1, p_2) of $F(f)$ and $F(g)$. By theorem 4.5, we need to construct a map $\delta : Q \rightarrow F(P)$ with $F(\pi_i) \circ \delta = p_i$.

Take any $q \in Q$ and define structure maps α_A^q, α_B^q, and α_C^q on A, B, and C as the constant functions

$$\begin{aligned}
&\alpha_A^q = \lambda a.p_1(q), \\
&\alpha_B^q = \lambda b.p_2(q), \text{ and} \\
&\alpha_C^q = \lambda c.F(f)(p_1(q)) = \lambda c.F(g)(p_2(q)).
\end{aligned}$$

It is easy to see that f and g are homomorphisms with respect to these structures. We therefore find a structure map α_P^q on P turning P into a bisimulation. We finally set

$$\delta(q) = \alpha_P(p)$$

for an arbitrary $p \in P$. By construction,

$$\begin{aligned}
F(\pi_1)(\delta(q)) &= F(\pi_1)(\alpha_P^q(p)) \\
&= \alpha_A^q(\pi_1(p)) \\
&= p_1(q).
\end{aligned}$$

Similarly, $F(\pi_2) \circ \delta(q) = p_2(q)$, hence $p_i = F(\pi_i) \circ \delta$ as required.

$$\begin{tikzcd}
 & & B \arrow[rr, "g"] \arrow[dd] & & C \arrow[dd, "\alpha_C^q"] \\
 & P \arrow[ur, "\pi_2"] \arrow[rr, "\pi_1"] \arrow[dd, dashed, "\alpha_P^q"] & & A \arrow[ur, "f"'] \arrow[dd, "\alpha_A^q"] & \\
 & & F(B) \arrow[rr, "F(g)"] & & F(C) \\
Q \arrow[r, dotted, "\delta"] \arrow[urr, bend left, "p_2"] \arrow[rrr, bend right, "p_1"'] & F(P) \arrow[ur, "F(\pi_2)"] \arrow[rr, "F(\pi_1)"'] & & F(A) \arrow[ur, "F(f)"'] &
\end{tikzcd}$$

We now finish the proof of the theorem 5.1. The equivalence of 1. and 2. follows directly from the above lemma.

In order to prove (2.)$\rightarrow$(3), let R, resp. S, be bisimulations between $\mathcal{A}$ and $\mathcal{B}$, resp. between $\mathcal{B}$ and $\mathcal{C}$. The pullback of the projections $\pi_2^R : R \rightarrow B$

and $\pi_1^S : S \to B$ is

$$R \bowtie S := \{((a,b),(b,c)) \mid (a,b) \in R, (b,c) \in S\}.$$

By assumption, this can be equipped with a coalgebra structure, making $\pi_1^R \circ \pi_1$ and $\pi_2^S \circ \pi_2$ into homomorphisms. $R \circ S$ is nothing but $(\pi_1^R \circ \pi_1, \pi_2^S \circ \pi_2)(R \bowtie S)$, the canonical bisimulation for this 2-source.

$$\begin{tikzcd}
 & & R \bowtie S \arrow[dl, "\pi_1"'] \arrow[dr] \arrow[rr, dashed] & & R \circ S \arrow[dd] \arrow[ddllll, bend right] \\
 & R \arrow[dl, "\pi_1^R"'] \arrow[dr] & & S \arrow[dl] \arrow[dr, "\pi_2^S"] & \\
A & & B & & C
\end{tikzcd}$$

The implication (3.)$\to$(4.) follows from the observation that

$$(\varphi,\psi)^{-}[R] = (G(\varphi)) \circ R \circ (G(\psi))^{-},$$

whilst (4.)$\to$(1.) is due to the equality

$$pb(\varphi,\psi) = (\varphi,\psi)^{-}(\Delta_C).$$

5.2 F weakly preserving kernels

Recall that a *kernel* is the pullback of a morphism $f : A \to B$ with itself. We have seen above that the functor $(-)_2^3$ does not weakly preserve kernels.

Theorem 5.3 *F weakly preserves kernels of non-empty mappings if and only if every congruence relation is a bisimulation.*

Proof. Let R be a congruence on $\mathcal{A}$ with projection homomorphism $\pi_R : A \twoheadrightarrow A/R$. If F weakly preserves kernels, then $R = pb(\pi_R, \pi_R)$ is a bisimulation by lemma 5.2.

Conversely, let $f : A \to B$ be a non-empty mapping and $\pi_1, \pi_2 : Ker(f) \to A$ the canonical projection maps.

Given $u, v \in F(A)$ with $F(f)(u) = F(f)(v)$, we need to find some $w \in F(Ker(f))$ with $F(\pi_1)(w) = u$ and $F(\pi_2)(w) = v$.

If f is injective, then so is $F(f)$, yielding $u = v$. So, in any case, we can find a pair $(x,y) \in Ker(f)$ and a map $\alpha_A : A \to \{u,v\} \subseteq F(A)$ with $\alpha_A(x) = u$ and $\alpha_A(y) = v$. Obviously f is a homomorphism from (A, α_A) to (B, α_B), when α_B is the constant map with image $\{F(f)(u)\} \subseteq F(B)$. In particular, $Ker(f)$ is a congruence on A, and therefore, by our assumption, a bisimulation.

This yields a structure map $\delta : Ker(f) \to F(Ker(f))$ with

$$F(\pi_i) \circ \delta = \alpha_A \circ \pi_i.$$

Clearly, $w = \delta(x,y)$ is the required element.

Lemma 5.4 *If F weakly preserves kernels then the largest bisimulation $\sim_{\mathcal{A}}$ on a coalgebra $\mathcal{A}$ is transitive, in fact it is the largest congruence relation on $\mathcal{A}$.*

Proof. Theorem 5.3 implies that every congruence θ is contained in $\sim_{\mathcal{A}}$. On the other hand, consider the coequalizer ψ of the projection homomorphisms π_1 and π_2 :

$$\begin{tikzcd} \sim_{\mathcal{A}} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "\psi"] & C. \end{tikzcd}$$

Then $\sim_{\mathcal{A}} \subseteq Ker(\psi) \subseteq \sim_{\mathcal{A}}$, hence $\sim_{\mathcal{A}} = Ker(\psi)$ is a congruence relation.

Corollary 5.5 *If F weakly preserves kernels then every mono in $\mathcal{S}et_F$ is injective.*

Proof. If F weakly preserves kernels, then $Ker(\varphi) = [Ker(\varphi)]$, hence if $\varphi : \mathcal{A} \to \mathcal{C}$ is mono we have $Ker(\varphi) = [Ker(\varphi)] = \Delta_A$.

5.3 F weakly preserving pullbacks along injective maps

F is said to *weakly preserve pullbacks along injective maps* if F weakly preserves pullbacks $pb(f, g)$, whenever f or g is injective.

This condition on F is properly weaker than the condition of preserving arbitrary weak pullbacks, for we have seen that the functor $(-)^3_2$ has this property, yet it does not weakly preserve pullbacks of two arbitrary maps.

Lemma 5.6 *Assume that f is injective and that $pb(f, g) \neq \emptyset$. If F weakly preserves the pullback $pb(f, g)$, then F preserves the pullback of f and g.*

Proof. Let $f : A \to C$ and $g : B \to C$ be maps and $(pb(f, g), \pi_1, \pi_2)$ their pullback. In every category pullbacks along monos are mono, so if f is injective, then so is $\pi_2 : pb(f, g) \to B$. If this is not the empty map then, by the axiom of choice it is left invertible, and so is, consequently, $F(\pi_2)$. Now it is easy to check that a weak limit must in fact be a limit, once a single one of its projections is mono.

Theorem 5.7 *Let F be a $\mathcal{S}et$-endofunctor, then the following are equivalent:*

(i) F (weakly) preserves non-empty pullbacks along injective maps.

(ii) If $\mathcal{U} \leq \mathcal{B}$ and R is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, then

$$R^{-}[U] := \{a \in A \mid \exists u \in U.(a, u) \in R\}$$

is a subcoalgebra of $\mathcal{A}$.

(iii) If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{U} \leq \mathcal{B}$ is a subcoalgebra, then $\varphi^{-1}[U]$, the pre-image of $\mathcal{U}$ under φ, is a subcoalgebra of $\mathcal{A}$.

(iv) If $\mathcal{U} \leq \mathcal{A}$ and $\mathcal{V} \leq \mathcal{B}$, then the bisimulations between $\mathcal{U}$ and $\mathcal{V}$ are just the restrictions to $U \times V$ of bisimulations between $\mathcal{A}$ and $\mathcal{B}$.

Proof. 1. $\to$ 2.: Assume that F preserves non-empty pullbacks along injective maps and $R \subseteq A \times B$ is a bisimulation. The pullback in $\mathcal{S}et$ of $\leq: U \to B$ and $\pi_2^R : R \to B$ is $Q = \{(u,(a,u)) \mid u \in U, (a,u) \in R\}$. By assumption, Q must be a bisimulation, hence, there is a structure map on Q so that $\pi_2^Q : Q \to R$, and consequently, $\pi_1^R \circ \pi_2^Q : Q \to A$ a homomorphism. Its image, which is just $R^-[U]$, is a subcoalgebra of $\mathcal{A}$.

2. $\to$ 3.: This is a specialization with $R = G(\varphi)$.

3. $\to$ 1.: Let $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be homomorphisms with ψ injective. The epi-mono-factorization of ψ yields a subcoalgebra $\mathcal{B}'$ of $\mathcal{C}$ and an isomorphism $\tilde{\psi} : \mathcal{B} \to \mathcal{B}'$ so that $\psi = \leq_{B'} \circ \tilde{\psi}$. By assumption, $\mathcal{A}' := \varphi^{-1}[U]$ is a subcoalgebra of $\mathcal{A}$, and by theorem 2.2 the restriction φ' of φ to $\mathcal{A}'$ is a homomorphism. It follows that the canonical embedding $\leq: \mathcal{A}' \to \mathcal{A}$ together with the homomorphism $\tilde{\psi}^{-1} \circ \varphi' : \mathcal{A}' \to \mathcal{B}$ form a 2-source in $\mathcal{S}et_F$ whose canonical bisimulation is just the pullback in $\mathcal{S}et$ of φ and ψ.

1. $\to$ 4.: Every bisimulation between $\mathcal{U}$ and $\mathcal{V}$ is clearly a bisimulation between $\mathcal{A}$ and $\mathcal{B}$. Conversely, let R be a bisimulation between $\mathcal{A}$ and $\mathcal{B}$ and let $\leq_{\mathcal{U}}: \mathcal{U} \rightarrowtail \mathcal{A}$ and $\leq_{\mathcal{V}}: \mathcal{V} \rightarrowtail \mathcal{B}$ be the embeddings. Using the fact that pullbacks of injectives are injective, we are able to repeatedly take pullbacks along injective maps as shown in the following diagram where each square is a pullback:

$$\begin{tikzcd}
& A & & B & \\
U \arrow[ur, "\leq_{\mathcal{U}}"] & & R \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & & V \arrow[ul, "\leq_{\mathcal{V}}"'] \\
& P_1 \arrow[ul] \arrow[ur] & & P_2 \arrow[ul] \arrow[ur] & \\
& & P \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

With $P_1 = R \cap (U \times B)$ and $P_2 = R \cap (A \times V)$ we get that

$$P = P_1 \cap P_2 = R \cap (U \times V) = (\leq_{\mathcal{U}}, \leq_{\mathcal{V}})^-[R]$$

is a bisimulation, since all arrows are homomorphisms.

4. $\to$ 3.: Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an F-homomorphism and $\mathcal{U} \leq \mathcal{B}$. By assumption,

$$(G(\varphi)) \cap (A \times U) = \{(a, \varphi(a)) \mid \varphi(a) \in U\}$$

is a bisimulation, so

$$\pi_1 : (G(\varphi)) \cap (A \times U) \to A$$

is an F-homomorphismus whose image is nothing but $\varphi^{-1}[U]$. Consequently, this is a subcoalgebra of $\mathcal{A}$.

Recall that ∇_A is the largest congruence relation on a coalgebra A. We drop the index $_A$, if it is clear from the context.

Theorem 5.8 *If F preserves pullbacks along injectives, then the following are equivalent:*

(i) $\sim_{\mathcal{A}}$ *is transitive for all* $\mathcal{A} \in \mathcal{S}et_F$.

(ii) $\nabla_{\mathcal{A}} = \sim_A$ *for all* $\mathcal{A} \in \mathcal{S}et_F$.

Proof. $\nabla_{\mathcal{A}}$ is always transitive, so one direction is trivial. For the other direction, consider $a, a' \in A$ with $a \nabla a'$. We are going to show that $a \sim_{\mathcal{A}} a'$.

Let $\pi : \mathcal{A} \to \mathcal{A}/\nabla$ be the canonical projection and consider the sum $\mathcal{S} := \mathcal{A} + \mathcal{A}/\nabla + \mathcal{A}$ with its canonical embeddings ι_1, ι_2, and ι_3. Now π induces an endomorphism $\psi := [(\iota_2 \circ \pi), \iota_2, (\iota_2 \circ \pi)]$ on $\mathcal{S}$, satisfying

$$\psi \circ \iota_1 = \iota_2 \circ \pi = \psi \circ \iota_3.$$

Using the fact that the graph of ψ and its converse must be contained in $\sim_{\mathcal{S}}$, we obtain:

$$\iota_1(a) \sim_S \psi(\iota_1(a)) = \iota_2(\pi(a)) = \iota_2(\pi(a')) = \psi(\iota_3(a')) \sim_S \iota_3(a').$$

It is easy to see that $\iota_1(x) \sim_S \iota_3(x)$ for every $x \in A$, in particular, $\iota_3(a') \sim_{\mathcal{S}} \iota_1(a')$. By hypothesis, $\sim_{\mathcal{S}}$ is transitive, so with the above, we obtain $\iota_1(a) \sim_{\mathcal{S}} \iota_1(a')$. Theorem 5.7 allows us to conclude $a \sim_{\mathcal{A}} a'$.

We caution the reader that we do not claim that $\sim_A$ being transitive would imply $\sim_A = \nabla_A$. In fact, we have a counterexample to this stronger hypothesis:

Recall the earlier example of a $(-)^3_2$-coalgebra A with a homomorphism to a one-element coalgebra 1, where $\sim_A$ was the diagonal relation on A. In particular, $\sim_A$ is transitive, but $\nabla_A = A \times A \neq \sim_A$, even though the functor $(-)^3_2$ preserves pullbacks along injective maps.

Reusing this counterexample, we can construct a $(-)^3_2$ coalgebra whose largest bisimulation is not transitive: The proof of the theorem shows, that we can simply take $A + 1 + A$.

5.4 F (weakly) preserving κ-pullbacks of injective maps

The previous examples demonstrated that the functor $(-)^3_2$, which preserves pullbacks along injective maps, does not weakly preserve arbitrary pullbacks (not even kernels, i.e. pullbacks of identical maps). We start this section with an example of a functor $F : \mathcal{S}et \to \mathcal{S}et$ which fails to preserve weak pullbacks *along* injective maps, but will, nevertheless, preserve pullbacks *of* injective maps.

Example 5.9 *On a set A, define F identical to the powerset functor:*

$$F(A) := \mathcal{P}(A),$$

but on maps $f : A \to B$ define $F(f)$ on any $U \in \mathcal{P}(A)$ as

$$F(f)(U) := \begin{cases} f[U] & \text{if } f_{|U} \text{ is injective} \\ \emptyset & \text{otherwise.} \end{cases}$$

We leave it to the reader to check that F is a functor.

Obviously, $\mathcal{S}et_F$ and $\mathcal{S}et_{\mathcal{P}}$ agree on objects, that is, every $\mathcal{P}$-coalgebra is an F-coalgebra, and conversely. Also, the notion of *subcoalgebra* is the same in $\mathcal{S}et_F$ and in $\mathcal{S}et_{\mathcal{P}}$, since both categories have the same injective homomorphisms. In particular, F-subcoalgebras will be closed under intersection.

Consider now the following F-coalgebras:

$\mathcal{A} = (\{a_1, a_2, a_3\}, \alpha)$ with $\alpha(a_1) = \{a_2, a_3\}$ and $\alpha(a_2) = \alpha(a_3) = \emptyset$.
$\mathcal{B} = (\{b_1, b_2\}, \beta)$ with $\beta(b_1) = \beta(b_2) = \emptyset$.

It is easy to check that $\varphi(a_1) := b_1$ and $\varphi(a_2) := \varphi(a_3) := b_2$ defines a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. However, $\varphi^{-1}[\{b_1\}] = \{a_1\}$ is not a subcoalgebra of $\mathcal{A}$. By theorem 5.7, F does not preserve pullbacks along injective maps.

To see that F does preserve pullbacks *of* injective maps, in fact of an arbitrary collection $(f_k)_{k\in\kappa}$ of injective maps with a common codomain, one could directly check this, using the criterion from [Gum98], but it will also follow from the following characterization theorem. The only-if direction of this is again from [Rut96]:

Theorem 5.10 *F (weakly) preserves κ-pullbacks of injective maps if and only if the intersection of a κ-family of subcoalgebras is a subcoalgebra.*

Proof. The intersection of a family $(\mathcal{U}_k)_{k\in\kappa}$ of subcoalgebras of $\mathcal{A}$ is just the pullback of their embedding. If this intersection is empty, nothing is to prove. Otherwise, if F preserves pullbacks of these embeddings, the pullback is a bisimulation, so in this case, the intersection is a subcoalgebra.

Conversely, suppose that the intersection of a κ-family of subcoalgebras is a subcoalgebra. We shall present the proof for $\kappa = 2$, the general case is proven the same way. Let $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be injective coalgebra homomorphisms. We try to fulfill the condition of lemma 5.2 For any (a, b) with $\varphi(a) = \psi(b)$, we need to find a 2-source (Q, p_1, p_2) and a $q \in Q$ with $\varphi \circ p_1 = \psi \circ p_2$ and $p_1(q) = a$, $p_2(p) = b$.

We start with the epi-mono-factorizations $\leq \circ \tilde{\varphi} = \varphi$ and $\leq \circ \tilde{\psi} = \psi$. Then $\tilde{\varphi}$ and $\tilde{\psi}$ are isomorphisms with $\mathcal{A} \cong \mathcal{A}' = \tilde{\varphi}[A]$ and $\mathcal{B} \cong \mathcal{B}' = \tilde{\psi}[B]$.

The pullback, in $\mathcal{S}et$, of φ and ψ can be taken stepwise, as indicated in the following diagram. Since $\mathcal{A}' \cap \mathcal{B}'$ is a subcoalgebra, the innermost pullback is also a pullback in $\mathcal{S}et_F$. Continuing to the outside, we are taking pullbacks along isomorphisms, which always exist in $\mathcal{S}et_F$. Thus $(\mathcal{Q}, \pi_1, \pi_2)$ is a source as required, i.e. $\varphi \circ \pi_1 = \psi \circ \pi_2$, and $\pi_1(q) = x$, $\pi_2(p) = y$.

$$\begin{tikzcd}
A \arrow[r, "\tilde{\varphi}"] & A' \arrow[r, hook] & C \\
R \arrow[u] \arrow[r] & A' \cap B' \arrow[u, hook] \arrow[r, hook] & B' \arrow[u, hook] \\
Q \arrow[u] \arrow[r] & S \arrow[u] \arrow[r] & B \arrow[u, "\tilde{\psi}"']
\end{tikzcd}$$

We conclude this subsection with an example of a functor not preserving pullbacks of injective maps.

Example 5.11 *Define $F : \mathcal{S}et \to \mathcal{S}et$ on a set A as*

$$F(A) := \begin{cases} \mathcal{P}(A) \backslash \emptyset & |A| \leq 1 \\ \mathcal{P}(A) & \textit{otherwise,} \end{cases}$$

and on a map $f : A \to B$ as

$$F(f) := \begin{cases} \emptyset & A = \emptyset \\ (\mathcal{P}(f))_{|F(A)} & \textit{otherwise.} \end{cases}$$

One easily checks that F is a functor. However, the intersection of two subcoalgebras $\mathcal{A}$ and $\mathcal{B}$ of a given F-coalgebra $\mathcal{C}$ need not be a subcoalgebra of $\mathcal{C}$.

To see this, consider $\mathcal{C} = (\{a, b, c\}, \gamma)$ with $\gamma(a) = \{b\}$ and $\gamma(b) := \gamma(c) := \emptyset$. Then it is easy to see that $A := \{a, b\}$ and $B := \{b, c\}$ are subcoalgebras of $\mathcal{C}$, but $\{b\}$ is not.

5.5 Preservation of κ-pullbacks

If a functor F weakly preserves pullbacks of two maps, it need not weakly preserve pullbacks of infinitely many maps. In fact, the filter functor (see [Gum98]) preserves κ-pullbacks of injective maps if and only if $\kappa < \omega$.

Proposition 5.12 *If F weakly preserves 2-pullbacks then the subcoalgebras of an F-coalgebra form a topological space. Every homomorphism between F-coalgebras is continuous and open with respect to the corresponding topologies. Conversely, every topological space arises as the collection of subcoalgebras of type $\mathcal{F}$, where $\mathcal{F}$ is the filter functor.*

In a certain sense the converse is also true, see [Gum98]. On every topological space (X, τ) we can define a coalgebra $\mathcal{A}_\tau = (X, \alpha_X)$ so that the open sets of τ become exactly the subcoalgebras of $\mathcal{A}$ and the continuous open maps between (X, τ) and (Y, σ) are exactly the homomorphisms between $\mathcal{A} = (X, \alpha_X)$ and $\mathcal{B} = (Y, \alpha_Y)$. The type F is given by the "filter functor" which associates to any set X the set $\mathcal{F}(X)$ of all filters on X. For a given topological space (X, τ) we define the structure map α by

$$\alpha(x) := \mathcal{U}(x)$$

where $\mathcal{U}(x)$ is the neighborhood filter of the point x.

6 Conclusion

We have characterized variants of weak pullback preservation properties of type functors for coalgebras. In particular we have considered weak preservation of arbitrary pullbacks, of pullbacks *along* monos, and of pullbacks *of monos*. In each case we have isolated structure theoretic properties that are entailed by, or that in fact are equivalent to, such preservation assumptions. We have given examples that show these properties to be really different. Further, we have given a structure theoretic criterion for characterizing monomorphisms in $\mathcal{S}et_F$, and for establishing when a mono is injective.

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