Algebra Universalis, **19** (1984) 319–321 0002–5240/84/030319–03$01.50+0.20/0

Topological implications in n-permutable varieties

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It is well known that topological groups are Hausdorff whenver their topology is at least a T_0-space. We find universal algebraic reasons for this in the following two theorems:

THEOREM 1. *Let* **A** *be an algebra in an n-permutable variety. Any compatible T_0-topology on* **A** *is in fact T_1.*

THEOREM 2. *Let* **A** *be an algebra in a 3-permutable variety. Any compatible T_0-topology on* **A** *is in fact Hausdorff.*

The corresponding result for permutable varieties has been proven by W. Taylor [4], thus already capturing the group case. Further let us mention a related unpublished result by J. Hagemann:

THEOREM (Hagemann). *Let* **A** *be an algebra in a permutable variety. Every compatible preuniformity on* **A** *must already be a uniformity.*

The main purpose of this paper is to show how Nonstandard Analysis can be utilized to translate the above theorems into a purely algebraic context. In the end we have to deal with the familiar concepts of compatible relations on universal algebras. Notably, proving theorem 1 then amounts to showing that a compatible quasiorder on an algebra in an n-permutable variety must be trivial, a result which is due to S. Bulman-Fleming and W. Taylor [1]. For the concepts of Nonstandard analysis we refer the reader to the excellent introduction by M. Machover and J. Hirschfeld [3].

We write "$\sim$" for the relation of "closeness". Beware that this relation is neither symmetric nor transitive, however it is compatible with those algebraic terms which are continuous in the topology of **A**.

Presented by W. Taylor. Received March 14, 1983. Accepted for publication in final form July 28, 1983.

In this setting T_0 means: If a and b are standard elements of A with $a \sim b$ and $b \sim a$ then $a = b$.

T_1 means: If a and b are standard elements of A with $a \sim b$ then $a = b$.

Hausdorff means: If x is a nonstandard element of A and a and b are standard elements of A with $x \sim a$ and $x \sim b$ then $a = b$, compare [3].

An *n-permutable variety* possesses, according to Hagemann and Mitschke [2] ternary terms $p_0, p_1, \ldots, p_n$ satisfying the laws

$$x = p_0(x, y, z)$$
$$p_i(x, x, y) = p_{i+1}(x, y, y) \quad \text{for} \quad 0 \leq i < n,$$
$$p_n(x, y, z) = z$$

Proof of Theorem 1. Suppose $a \sim b$, we show that $p_i(b, a, a) = b$ for all $0 \leq i \leq n$, thus $a = b$ by the last equation. The case $i = 0$ is trivial, so by induction $p_i(b, a, a) = b$ whence $b = p_i(b, a, a) \sim p_i(b, b, a) = p_{i+1}(b, a, a)$ and clearly $p_{i+1}(b, a, a) \sim p_{i+1}(b, b, b) = b$. Thus, from the fact that **A**'s topology is T_0 we get $b = p_{i+1}(b, a, a)$.

Proof of Theorem 2. From the above we already know that A must be T_1. If $x \sim a$ and $x \sim b$ then, denoting the nonstandard extensions of the terms $p_0, \ldots, p_n$ by $p_0^*, \ldots, p_n^*$ we find:

$$a = p_1^*(a, x, x) \sim p_1^*(a, a, b) = p_2^*(a, b, b) = p_2(a, b, b).$$

From T_1 we get that $a = p_2(a, b, b)$, and similarly

$$b = p_2^*(x, x, b) \sim p_2^*(a, b, b) = p_2(a, b, b),$$

hence $b = p_2(a, b, b)$ by T_1 again, so $a = b$ as required.

The reader who may have translated the proofs into standard topological language will certainly agree that the nonstandard language is indeed more appropriate in this setting. With the nonstandard characterization of preuniform and uniform spaces in mind, Hagemann's theorem too becomes almost obvious:

A compatible preuniformity on **A** induces a reflexive, symmetric and compatible binary relation "$\approx$" on $\hat{A}$, the set of *-elements of **A**, see [3]. $\hat{\mathbf{A}}$ is always contained in the variety generated by **A**, so $\hat{\mathbf{A}}$ generates a permutable variety.

From well known facts about permutable varieties ([5]) the relation $\approx$ therefore must be transitive, which translates precisely back into the statement that the preuniformity is indeed a uniformity.

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