TYPES AND COALGEBRAIC STRUCTURE

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Abstract. We relate weak limit preservation properties of coalgebraic type functors F to structure theoretic properties of the class of all F-coalgebras.

1. Introduction

In his pioneering paper *Universal Coalgebra – a theory of state based systems* ([Rut00]), J. Rutten, has developed a theory, which is largely dual to Universal Algebra, when considered from a category theoretical standpoint. A *type* (or *signature*) is a functor $F : \mathcal{S}et \to \mathcal{S}et$ on the category of sets, and a *coalgebra of type* F is any pair $\mathcal{A} = (A, \alpha)$, with $\alpha : A \to F(A)$ being an arbitrary map, called the *structure map* of $\mathcal{A}$. With a natural notion of homomorphism, F-coalgebras form a category $\mathcal{S}et_F$. It turns out that coalgebras are ideally suited for describing important structures from computer science, such as Kripke structures, labeled transition systems, and various types of automata. For instance, to model nondeterministic automata with input alphabet Σ and a terminal set of states, one chooses $F(-) = \mathbb{P}(-)^\Sigma \times \underline{2}$. Here $\mathbb{P}(-)$ denotes the powerset functor, and $\underline{2}$ the constant functor with $\underline{2}(X) = 2 = \{0, 1\}$.

The success of the theory is not only that its development smoothly proceeds on such an abstract level, largely parallel to the general theory of universal algebra, but also that relevant notions and constructions from computer science, such as *bisimulation*, *coinduction*, *observational equivalence*, *minimization*, *co-recursive definitions*, to name just a few, have found universal coalgebraic explanations.

In order to carry this development through, Rutten, building on results and notions from Aczel and Mendler [AM89], Barr [Bar93, Bar94] and Lambek [Lam68], needed to assume two properties of the type functor F: that it should *preserve weak pullbacks*, and, somehow implicitly, that it should also preserve intersections. These assumptions, which we shall explain below, seem to be satisfied in all standard examples. Still, Rutten was careful to keep book, which of his proofs had actually made use of them.

From a mathematical standpoint, these assumptions appeared rather unmotivated, as already remarked in [Mos99]. Indeed, it turned out, that the essentials of the theory could be carried through for arbitrary type functors [Gum99]. Soon after, interesting applications were discovered, such as topological spaces, where the necessary type functor (the filter functor) does not preserve intersections [Gum01], and monoid labeled transition systems [GS], where the type functor does not preserve weak pullbacks, not even preimages, which are special cases of pullbacks.

These examples, indeed, lacked some desirable coalgebraic properties that seemed to require corresponding conditions on the type functor. For instance, in the general case, homomorphic preimages of subcoalgebras need not be subcoalgebras, congruences need not be bisimulations, and bisimulations need not be closed under relational composition. For bisimulations, which in many respects play the role of compatible relations in universal algebra, these shortcomings may actually be considered relevant. Even the largest bisimulation on a coalgebra need not be transitive in the general case, so as a consequence, "observational equivalence" occasionally falls short of being an equivalence relation.

In this paper we are wrapping up our investigation on the correspondence between preservation properties of the type functor with structural (co)algebraic, properties. Some of these results have been reported at conferences [GS00], some were first obtained in the second author's thesis [Sch01], others are for the first time presented here. Together they give a complete picture, confronting functorial preservation properties with equivalent (co)algebraic structural properties.

2. Categorical Notions

We need only basic category theoretic notions and facts, as found in the first few chapters of any textbook, such as e.g. [AHS90].

Recall that a mono $f : A \to B$ is called *regular mono*, if it is an equalizer, i.e. the limit of a parallel pair of arrows $g_1, g_2 : B \to C$. Similarly, a *regular epi* is a coequalizer. A morphism f is an isomorphism iff f is mono and regular epi iff it is epi and regular mono.

2.1. **Pullbacks, kernel pairs, preimages and intersections.** The *pullback* of two morphisms $f : A \to C$ and $g : B \to C$ is their limit, that is it consists of an object P together with morphisms $p_1 : P \to A$ and $p_2 : P \to B$ so that

(1) $f \circ p_1 = g \circ p_2$, and
(2) for every "competitor", that is, for every object Q with morphisms $q_1 : Q \to A$ and $q_2 : Q \to B$ satisfying $f \circ q_1 = g \circ q_2$, there is a unique morphism $d : Q \to P$ with $p_1 \circ d = q_1$ and $p_2 \circ d = q_2$.

If the uniqueness requirement for d is dropped, we call (P, p_1, p_2) a *weak pullback*.

$$\begin{tikzcd}
 & A \arrow[r, "f"] & C \\
 & P \arrow[u, "p_1"'] \arrow[r, "p_2"] & B \arrow[u, "g"'] \\
Q \arrow[uur, dashed, "q_1"] \arrow[ur, dotted, "d"] \arrow[urr, dashed, "q_2"'] & &
\end{tikzcd}$$

The pullback of f and g is called

kernel pair: if $A = B$ and $f = g$,
preimage: if g is a regular mono,
intersection: if both f and g are regular monos.

Observe that *weak* preimages are preimages. Similarly, weak intersections are intersections. One easily verifies:

Lemma 2.1. *If g, in a pullback diagram, is mono (right-invertible, regular mono), then so is the opposite arrow, p_1.*

(Weak) pullbacks can be pasted together, and pullbacks can be canceled on the right. The following lemma is readily verified([AHS90]):

Lemma 2.2. *Consider the following commutative diagram:*

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}$$

(1) *If both squares are (weak) pullbacks, then so is the outer rectangle.*
(2) *If the right square is a pullback, then the left square is a (weak) pullback iff the outer rectangle is a (weak) pullback.*

2.2. **The category $\mathcal{S}et$.** In the category $\mathcal{S}et$ of sets and mappings, monos are the injective maps. They are always regular and, if their domain is nonempty, they are left-invertible, too. Epis are the surjective maps. They are always regular epi, and, due to the axiom of choice, right-invertible, too.

Every set Q can be written as a sum of one-element sets: $Q \cong \sum_{i \in Q} 1$. As a consequence, condition (2) in the above definition of (weak) pullbacks only needs to be checked for $Q = 1$, that is we can replace it by

($\star$) for every $a \in A$ and $b \in B$ with $f(a) = g(b)$, there exists a (unique) $p \in P$ with $\pi_1(p) = a$ and $\pi_2(p) = b$.

The pullback of f with g always exists in $\mathcal{S}et$. It is given by

$$Pb(f, g) := \{(a, b) \in A \times B \mid fa = gb\}$$

together with the canonical projections π_1, and π_2.

Every map $f : A \to B$ can be factored as $f = \subseteq \circ f'$, where f' is surjective (i.e. right invertible) and $\subseteq$ injective, i.e. regular mono. Thus, by lemmas 2.1 and 2.2, each pullback can be built up, as in the following figure, from a pullback of surjective maps, two preimages along surjective maps, and an intersection. As a consequence we can restrict ourselves, considering only pullbacks of surjective maps. Moreover, w.l.o.g. we can assume that the injections are set inclusions.

$$\begin{tikzcd}
A \arrow[r, two heads, "f'"'] \arrow[rr, bend left, "f"] & \bullet \arrow[r, hook, "\subseteq"'] & C \\
\bullet \arrow[u, hook] \arrow[r, two heads] & \bullet \arrow[u, hook] \arrow[r, hook] & \bullet \arrow[u, hook, "\subseteq"] \\
P \arrow[u, two heads] \arrow[r, two heads] & \bullet \arrow[u, two heads] \arrow[r, hook] & B \arrow[u, two heads, "g'"] \arrow[uu, bend right, "g"']
\end{tikzcd}$$

2.3. **$\mathcal{S}et$-Functors.** In the following, let $F : \mathcal{S}et \to \mathcal{S}et$ be any functor. We begin with the simple observation that

$$X \neq \emptyset \implies F(X) \neq \emptyset,$$

unless F is the trivial functor with $F(Y) = \emptyset$ for every set Y. We shall disregard this trivial functor from our further considerations.

Next, we observe that F preserves monos whose domain is nonempty. This is because every mono $f : X \to Y$ with $X \neq \emptyset$ has a left-inverse g, hence by the properties of a functor

$$F(g) \circ F(f) = F(g \circ f) = F(id_X) = id_{F(X)},$$

so $F(g)$ is left inverse to $F(f)$.

Similarly, due to the axiom of choice, every surjective map is right invertible, so F also preserves epis.

Definition 2.3. F (weakly) preserves pullbacks*, if F transforms each pullback diagram into a (weak) pullback diagram. (Weak) preservation of kernel pairs, preimages and intersections are similarly defined.*

If F *weakly preserves pullbacks*, then it transforms every weak pullback diagram into a weak pullback diagram ([Rut00]). Thus, weak preservation of pullbacks is the same as *preservation of weak pullbacks.* Therefore, condition ($\star$) in section 2.2 translates into the following useful criterion:

Lemma 2.4. F preserves the weak pullback (P, p_1, p_2) *of* $f : A \to C$ *with* $g : B \to C$ *iff for all* $\tilde{a} \in FA$, $\tilde{b} \in FB$ *with* $(Ff)\tilde{a} = (Fg)\tilde{b}$ *there is some* $\tilde{p} \in F(P)$ *with* $(Fp_1)\tilde{p} = \tilde{a}$ *and* $(Fp_2)\tilde{p} = \tilde{b}$. F preserves the pullback *iff this* $\tilde{p}$ *is always unique.*

Remark 2.5. *Note that for preimages, i.e. when g is injective, we need only check the first equality,* $(Fp_1)\tilde{p} = \tilde{a}$*, since the second one follows from the fact that Fg is mono:*

$$(Fg)(Fp_2)\tilde{p} = (Ff)(Fp_1)\tilde{p} = (Ff)\tilde{a} = (Fg)\tilde{b}.$$

2.4. **Preservation of intersections.** We consider this special case first. Rather surprisingly, one gets preservation of finite intersections almost for free:

Proposition 2.6 (Trnková [Trn69])**.** *Every functor* $F : \mathcal{S}et \to \mathcal{S}et$ *preserves non-empty finite intersections. By redefining F on the empty set $\emptyset$ and on the empty maps* $\emptyset_X : \emptyset \to X$*, it can be made to preserve* all *(empty and non-empty) finite intersections.*

Elementary proofs for the fact that F preserves non-empty finite intersections can be found in [Man98] or [GS01a].

In order to redefine F on the empty set and on empty mappings, Trnková considers first the functor $\hat{1}$, which maps the empty set to itself and every nonempty set to the one-element set $1 = \{*\}$.

Let F' agree with F everywhere, except on the empty set and on the empty mappings. $F'(\emptyset)$ is defined to be the set of all natural transformations $\nu : \hat{1} \stackrel{\cdot}{\longrightarrow} F$. For each empty map $\emptyset_X : \emptyset \to X$, whenever $X \neq \emptyset$, define $F'\emptyset_X$ by $(F'\emptyset_X)(\nu) := \nu_A(*)$ for each ν.

Then F' preserves *all* finite intersections, and $F'\emptyset_X$ is injective for each set X, hence F' preserves *all* monos.

Since the above modification of F on the empty set and the empty mappings is not going to change the F-coalgebras, we will from now on assume that F preserves *all* finite intersections and all monos. A consequence of this "normalization" of F, is that

- we need not worry about empty pullbacks, and
- weak pullback preservation splits into two special cases,

as assured by the following proposition:

Theorem 2.7. *The following are equivalent:*

(1) F *weakly preserves pullbacks.*
(2) F *weakly preserves nonempty pullbacks*
(3) F *weakly preserves kernels and preimages.*

Proof. (1)⇒(2) and (1)⇒(3) are obvious. For (3)⇒(1), the trick is to factor the maps $f : A \to C$ and $g : B \to C$ through the sum $A + B$ as $f = [f, g] \circ e_1$ and $g = [f, g] \circ e_2$ where the e_i are the canonical embeddings into the sum and $[f, g] : A + B \to C$ is the sum morphism.

The pullback of f and g can be obtained by first taking the kernel pair of $[f, g]$, followed by two preimages and an intersection as is indicated in the following diagram. Lemma 2.2 guarantees that this process works.

$$\begin{tikzcd}
A \arrow[r, hook, "e_1"'] \arrow[rr, bend left, "f"] & A+B \arrow[r, "{[f,g]}"'] & C \\
\bullet \arrow[u] \arrow[r, hook] & \mathrm{Ker}[f,g] \arrow[u] \arrow[r] & A+B \arrow[u, "{[f,g]}"] \\
Pb(f,g) \arrow[u, hook] \arrow[r, hook] & \bullet \arrow[u, hook] \arrow[r] & B \arrow[u, hook, "e_2"] \arrow[uu, bend right, "g"']
\end{tikzcd}$$

Applying F, we obtain a diagram of the same shape. All weak limits of the constituent subsquares are weakly preserved by F, so applying lemma 2.2 again, we see that the outer square, i.e. the pullback of f with g is weakly preserved.

(2)⇒(1): Assume that $Pb(f, g) = \emptyset$. Then all four rectangles in the following diagram are either nonempty pullbacks or empty intersections:

$$\begin{tikzcd}
 & & & C \arrow[dl, hook] \\
A \arrow[r, hook, "e_1"] \arrow[urrr, bend left, "f"] & A+1 \arrow[r, "{[f,1]}"] & C+1 & \\
\emptyset \arrow[u, hook] \arrow[r, hook] & 1 \arrow[u, hook] \arrow[r, hook] & B+1 \arrow[u, "{[g,1]}"] & \\
\emptyset \arrow[u, hook] \arrow[r, hook] & \emptyset \arrow[u, hook] \arrow[r, hook] & B \arrow[u, hook] \arrow[uuur, bend right, "g"'] &
\end{tikzcd}$$

Applying F, which preserves *nonempty* pullbacks and *all* intersections, yields a similar such diagram where all image rectangles are again weak pullbacks. By lemma 2.2, $(F(\emptyset), F(\emptyset_A), F(\emptyset_B))$ is a weak pullback of $F([f, 1] \circ e_1)$ with $F([g, 1] \circ e_1)$. Since $Fe_1 : F(C) \to F(C + 1)$ is mono, $(F(\emptyset), F(\emptyset_A), F(\emptyset_B))$ is a pullback of Ff with Fg, as well. □

Obviously, the same proof works for kernels and preimages, too. Furthermore, the equivalence of (1) and (3) together with its proof remain true if all occurrences of "weakly" are deleted. For this case, Peter Freyd has given an alternative proof in the category mailing list ([ftp]).

The following examples show that the preservation of weak pullbacks, kernel pairs, preimages, and intersections by a functor F are indeed different properties:

Example 2.8. *The functor $(-)^3_2$, defined on a set X as $X3_2 := \{(x_1, x_2, x_3) \in X3 \mid x_1 = x_2,\ \text{or}\ x_1 = x_3,\ \text{or}\ x_2 = x_3\}$ and on maps componentwise, (see* [AM89]*), preserves preimages but does not weakly preserve kernel pairs.*

If $\mathcal{G}$ is a nontrivial abelian group, the functor $\mathcal{G}_\omega^{(-)}$ (see [GS]*) preserves weak kernel pairs but does not preserve preimages. The sum functor $(-)^3_2 + \mathcal{G}_\omega^{(-)}$ neither preserves preimages nor weak kernel pairs.*

3. Coalgebras

Any functor $F : \mathcal{S}et \to \mathcal{S}et$ is called a *type*. A *coalgebra of type* F is a pair $\mathcal{A} = (A, \alpha_A)$ consisting of a set A together with a map

$$\alpha_A : A \to F(A).$$

A is called the base set and α_A the structure map of $\mathcal{A}$.

If $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$ are coalgebras, then a *homomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a map $\varphi : A \to B$, making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ F(A) \arrow[r, "F(\varphi)"] & F(B) \end{tikzcd}$$

F-coalgebras with their homomorphisms form a category $\mathcal{S}et_F$. This category is cocomplete, and colimits are formed just as in $\mathcal{S}et$. In other words, the forgetful functor $U : \mathcal{S}et_F \to \mathcal{S}et$ which associates to each F-coalgebra $\mathcal{A}$ its underlying set A, *creates* colimits.

Isomorphisms in $\mathcal{S}et_F$ are the bijective homomorphisms, epis are the surjective homomorphisms. Monos, however, need not be injective. These and many of the following basic results about coalgebras can be found in [Rut00] or in [Gum99].

3.1. Subcoalgebras.

If $\mathcal{A} = (A, \alpha_A)$ is a coalgebra and U a subset of A, then there can be at most one structure map $\alpha_U : U \to F(U)$ turning the canonical embedding $\subseteq_U^A : U \to A$ into a homomorphism from $\mathcal{U} = (U, \alpha_U)$ to $\mathcal{A}$. In this case, we use the term *subcoalgebra* both for the subset U and for the coalgebra $\mathcal{U}$, and we write $\mathcal{U} \le \mathcal{A}$.

The set of all subcoalgebras of $\mathcal{A}$ is closed under arbitrary unions, so for any $X \subseteq A$ there is a largest subcoalgebra contained in X. It is called the subcoalgebra *cogenerated* by X and denoted $[X]$. Rather surprisingly, subcoalgebras are also closed under finite intersections, see [GS01a], hence they form a topology on A, where $[X]$ is the interior of X.

3.2. Homomorphic images, congruence relations.

If $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$ are coalgebras and $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{B}$ a surjective homomorphism, then $\mathcal{B}$ is called a *homomorphic image* of $\mathcal{A}$. Every homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ has a factorization $\varphi' : \mathcal{A} \twoheadrightarrow \varphi[U] \le \mathcal{B}$ as an epi followed by a subcoalgebra embedding, i.e. a mono which is injective. More general, if U is a subcoalgebra of $\mathcal{A}$, and $\varphi : \mathcal{A} \to \mathcal{B}$ a homomorphism, then $\varphi[U] := \{\varphi(u) \mid u \in U\}$ is a subcoalgebra of $\mathcal{B}$ and a homomorphic image of $\mathcal{U}$.

The following diagram lemma is useful in many situations:

Lemma 3.1 ([GS01c], First Diagram Lemma)**.** *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be F-coalgebras, $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{C}$ homomorphisms. If φ is surjective, then there is a (necessarily unique) homomorphism $\chi : \mathcal{B} \to \mathcal{C}$ with $\chi \circ \varphi = \psi$ iff $\ker(\varphi) \subseteq \ker(\psi)$.*

$$\begin{tikzcd} A \arrow[r, two heads, "\varphi"] \arrow[dr, "\psi"'] & B \arrow[d, dashed, "\chi"] \\ & C \end{tikzcd}$$

Congruence relations are defined as kernels of homomorphisms. Since colimits in $\mathcal{S}et_F$ are formed just as in $\mathcal{S}et$, the join of a family of congruence relations is the same as their join in the lattice of equivalence relations. In particular, there is always a largest congruence relation on any coalgebra $\mathcal{A}$, which we denote by $\nabla_{\mathcal{A}}$. In general, however, we have $\nabla_{\mathcal{A}}$ properly below $A \times A$.

3.3. **Bisimulations.** In the relevant computer science applications, bisimulations are the "indistinguishability relations" on states. Abstractly, a *bisimulation* between coalgebras $\mathcal{A}$ and $\mathcal{B}$ is a relation $R \subseteq A \times B$ which can be equipped with a coalgebra structure $\alpha_R : R \to F(R)$, so that the projections $\pi_A^R : R \to A$ and $\pi_B^R : R \to B$ are homomorphisms.

If R is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, its converse, $R^- := \{(b,a) \mid (a,b) \in R\}$ is a bisimulation between $\mathcal{B}$ and $\mathcal{A}$. If $\mathcal{A} = \mathcal{B}$, then R is called a bisimulation *on* $\mathcal{A}$. The diagonal $\Delta_A := \{(a,a) \mid a \in \mathcal{A}\}$ is always a bisimulation on $\mathcal{A}$.

The empty set $\emptyset \subseteq A \times B$ is always a bisimulation, and bisimulations are closed under arbitrary unions. Thus, there is always a largest bisimulation $\sim_{A,B}$ between given coalgebras $\mathcal{A}$ and $\mathcal{B}$. More generally, given any relation $G \subseteq A \times B$, then the *bisimulation cogenerated by* G is defined as the union of all bisimulations contained in G and denoted by $[G]_2$. If $G \subseteq A$ is reflexive and symmetric, then so is $[G]_2$.

Although bisimulations, in many respects, appear like 2-dimensional analogues to subcoalgebras, they are in general not closed under finite intersections.

The graph of a homomorphism is a bisimulation, in fact, a map $f : A \to B$ is a homomorphism if and only if its graph

$$G(f) := \{(a, f(a)) \mid a \in A\}$$

is a bisimulation. More generally:

Proposition 3.2 ([Rut00])**.** *If $\mathcal{Q}$ is any coalgebra and if $\varphi_1, \varphi_2 : \mathcal{Q} \to \mathcal{A}$ are homomorphisms, then $(\varphi_1, \varphi_2)[Q] := \{(\varphi_1(q), \varphi_2(q)) \mid q \in Q\}$ is a bisimulation.*

Even though this bisimulation can be obtained as the relational composition $G(\varphi)^- \circ G(\psi)$ of two bisimulations, we must caution the reader, that in general, bisimulations are not closed under composition.

3.4. **Regular congruences.** A bisimulation R which happens to be an equivalence relation, too, is a congruence relation. More generally:

Proposition 3.3 ([AM89])**.** *If R is a bisimulation on an F-coalgebra, then $\mathcal{E}q(R)$, the equivalence relation* generated *by R, is a congruence relation.*

For reasons that will become clear later, we call such congruences "regular", i.e. a congruence is regular, if it is generated, as an equivalence relation, by some bisimulation. In this case, θ is also generated by $[\theta]_2$, the largest bisimulation contained in θ. This is in fact a reflexive and symmetric relation, hence its transitive hull $[\theta]_2^*$ is a congruence relation below θ. So we have immediately:

Lemma 3.4. *A congruence θ is regular iff $\theta = [\theta]_2^*$.*

In the next example, we shall see that the largest congruence $\nabla_{\mathcal{A}}$ need not be regular. At the same time, we construct a homomorphism φ which is both epi and mono, but not an isomorphism.

Example 3.5. *Consider the functor $(-)^3_2$ from example 2.8 again, and the $(-)^3_2$-coalgebra $\mathcal{A} = (\{0,1\},\alpha)$ on the two-element set $\{0,1\}$, given by*

$$\alpha(x) = (0, x, 1).$$

Assume $(0,1) \in R$ for some bisimulation $R \subseteq A \times A$, then there must be a structure map $\rho : R \to (R)^3_2$ with π_1 and π_2 homomorphisms. With $\rho(0,1) = ((x_1,y_1),(x_2,y_2),(z_1,z_2))$ we obtain the conditions $(x_1,x_2,x_3) = ((\pi_1)^3_2 \circ \rho)(0,1) = (\alpha \circ \pi_1)(0,1) = (0,0,1)$, and similarly, $(y_1,y_2,y_3) = (0,1,1)$. But then $\rho(0,1) = ((0,0),(0,1),(1,1)) \notin R3_2$.

Hence $(0,1) \notin \sim_A$ and similarly $(1,0) \notin \sim_A$, so $\sim_A = \Delta_A$. With proposition 3.2, we conclude that for every coalgebra $\mathcal{Q}$ there is at most one homomorphism $\phi : \mathcal{Q} \to \mathcal{A}$. Consequently, each homomorphism φ with domain $\mathcal{A}$ is mono in $\mathcal{S}et_F$.

On the one-element set $\{\star\}$ there is a unique $(-)^3_2$-coalgebra structure, so the unique map $\varphi : A \to \{*\}$ is a surjective homomorphism with kernel $\nabla_{\mathcal{A}} := A \times A$.

By the above, φ is mono. Since φ is surjective, it is epi in $\mathcal{S}et_F$. Thus we have found a homomorphism, which is both mono and epi, but not an isomorphism.

4. Limits, and factorizations in $\mathcal{S}et_F$

The category $\mathcal{S}et_F$ is co-complete. In fact, all colimits are formed just like in the base category $\mathcal{S}et$. In categorical terms, *the forgetful functor creates colimits.* In particular, sums are defined canonically on the disjoint union of their base sets, and pushouts of two homomorphisms $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{C}$ are defined on the factor $(B + C)/\Theta$ where Θ is the equivalence generated by

$$(\varphi,\psi)[A] := \{(\varphi(a),\psi(a)) \mid a \in A\}.$$

Things are different for limits. Even though some limits exist, as we shall see, they are, in general, not created by the forgetful functor.

4.1. Equalizers.

The first type of limit that we consider is a equalizer:

Lemma 4.1 ([GS00]). *The equalizer of homomorphisms $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$ in $\mathcal{S}et_F$ is given by the largest subcoalgebra $[E]$ which is contained in their set-theoretical equalizer $E := \{a \in A \mid \varphi_1 a = \varphi_2 a\}$.*

Proof. E is the equalizer of the maps φ_1 and φ_2, so for the canonical embedding $\leq : [E] \to \mathcal{A}$ we clearly have $\varphi_1 \circ \leq = \varphi_2 \circ \leq$. Let $\psi : \mathcal{Q} \to \mathcal{A}$ be given with $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ then $\psi[Q]$ is a sub-coalgebra of $\mathcal{A}$, and it is contained in E. Consequently, $\psi[Q] \leq [E]$, hence ψ uniquely factors through $[E]$.

$$\begin{tikzcd}
{[E]} \arrow[r, "\leq"] & A \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & B \\
Q \arrow[u, dashed] \arrow[ur, "\psi"'] & &
\end{tikzcd}$$

□

4.2. **Monos and regular monos.** In this section we shall characterize monos, regular monos and regular epis in $\mathcal{S}et_F$.

In $\mathcal{S}et$, a map $f : X \to Y$ is mono, iff it is injective, which is to say: $\mathrm{Ker}\, f = \Delta_X$. Monos in $\mathcal{S}et_F$ need not be injective, as we saw in example 3.5. The following result from [GS00] shows how far away monos in $\mathcal{S}et_F$ may be from being injective:

Lemma 4.2 ([GS00])**.** *A homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *is mono iff* $[\mathrm{Ker}\,\varphi]_2 = \Delta_A$.

Proof. Assume that $\varphi : \mathcal{A} \to \mathcal{B}$ is mono. Let $\pi_1, \pi_2 : \mathrm{Ker}\,\varphi \to A$ be the canonical projection maps, and let $\tilde{\pi}_1, \tilde{\pi}_2 : [\mathrm{Ker}\,\varphi]_2 \to A$ be their restrictions to $[\mathrm{Ker}\,\varphi]_2$. The latter set is a bisimulation on $\mathcal{A}$, so $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are coalgebra homomorphisms and $\varphi \circ \tilde{\pi}_1 = \varphi \circ \tilde{\pi}_2$. It follows that $\tilde{\pi}_1 = \tilde{\pi}_2$, i.e. $[\mathrm{Ker}\,\varphi]_2 = \Delta_A$.

Conversely, assume that $[\mathrm{Ker}\,\varphi]_2 = \Delta_A$ and assume that there are homomorphisms $\kappa_1, \kappa_2 : \mathcal{P} \to \mathcal{A}$ with $\varphi \circ \kappa_1 = \varphi \circ \kappa_2$. Then $(\kappa_1, \kappa_2)[\mathcal{P}]$ is a bisimulation on $\mathcal{A}$, and it is clearly contained in $\mathrm{Ker}\,\varphi$. By assumption then, $(\kappa_1, \kappa_2)[\mathcal{P}] \subseteq \Delta_A$, which implies $\kappa_1 = \kappa_2$. $\square$

Injectivity gives us a useful stronger property:

Theorem 4.3. *A monomorphism* $\varphi : A \to \mathcal{B}$ *is regular mono, iff it is injective.*

Proof. As a consequence of lemma 4.1, regular monomorphisms must be injective.

Conversely, if $\varphi : \mathcal{A} \hookrightarrow \mathcal{B}$ is injective, it is regular mono in $\mathcal{S}et$. As such it is just the equalizer of its pushout (P, p_1, p_2) where $p_1 : B \to P$ and $p_2 : B \to P$ are maps. Since the forgetful functor creates colimits, there is a coalgebra structure on P, so that p_1 and p_2 are homomorphisms. It follows, that φ is the equalizer, in $\mathcal{S}et_F$, of these homomorphisms.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "p_1"] & P \\
\mathcal{A} \arrow[u, hook, "\varphi"] \arrow[r, hook, "\varphi"] & \mathcal{B} \arrow[u, "p_2"']
\end{tikzcd}$$

$\square$

An obvious corollary is:

Corollary 4.4. *Every morphism in* $\mathcal{S}et_F$ *has an epi-(regular mono) factorization.*

4.3. **Preimages.** We next show that preimages also exist in $\mathcal{S}et_F$.

Lemma 4.5. *The preimage of a regular mono* $\psi : \mathcal{V} \to B$ *along a morphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *exists in* $\mathcal{S}et_F$*. We may assume that* $\psi = \subseteq^B_V$*, then the preimage is given by* $[\varphi^-[V]]$*, the largest coalgebra contained in the inverse image* $\varphi^-[V]$ *of* V *under* φ*.*

Proof. We have seen that regular monos are injective. Hence ψ factors as $\psi = \subseteq^B_{\psi[V]} \circ \psi'$, with ψ' an isomorphism. Hence, from now on we assume $\psi = \subseteq^B_V$.

With $U := \varphi^-[V]$, we clearly obtain a commutative diagram in $\mathcal{S}et_F$:

$$\begin{tikzcd}
& \mathcal{A} \arrow[r, "\varphi"] & \mathcal{B} \\
& {[U]} \arrow[u, "\leq"'] \arrow[r] & \mathcal{V} \arrow[u, "\leq"'] \\
\mathcal{Q} \arrow[uur, dashed, "\varphi_1"] \arrow[ur, dotted] \arrow[urr, dashed, "\varphi_2"'] & &
\end{tikzcd}$$

Let $\mathcal{Q}$ with homomorphisms $\varphi_1 : \mathcal{Q} \to \mathcal{A}$ and $\varphi_2 : \mathcal{Q} \to \mathcal{V}$ be a competitor to $[U]$. Since U is the preimage of V in $\mathcal{S}et$, the map φ_1 must factor through the set U. Consequently, the image of the homomorphism φ_1, which is a subcoalgebra of $\mathcal{A}$, must be contained in U, hence in $[U]$. $\square$

It is tempting, to try extending this reasoning to the construction of arbitrary pullbacks. If $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are homomorphisms, one could consider $[P]_2$, the largest bisimulation contained in the set-theoretical pullback P. However, the coalgebra structure on $[P]_2$ is not uniquely determined, and given a competitor $(\mathcal{Q}, \psi_1, \psi_2)$ as above, the map $(\psi_1, \psi_2) : Q \to P$, even though it factors through $[P]_2$, need not be a homomorphism.

4.4. **Regular epis.** Epis in $\mathcal{S}et_F$ are exactly the surjective homomorphisms (see [Rut00]). But they need not be regular, as is witnessed once more by the homomorphism φ from example 3.5 which is both epi and mono, but not an isomorphism. So the question remains, what additional properties make an epi regular. The following theorem gives an answer and justifies a notion introduced earlier:

Theorem 4.6. *An epimorphism $\varphi : \mathcal{B} \twoheadrightarrow \mathcal{C}$ is a regular epi iff* $\operatorname{Ker}\varphi$ *is a regular congruence relation.*

Proof. Let φ be the coequalizer in $\mathcal{S}et_F$ of $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{B}$. Then the map φ is also the coequalizer in $\mathcal{S}et$ of the maps $\psi_1, \psi_2 : A \to B$, in particular,

$$\operatorname{Ker}\varphi = \mathcal{E}q((\psi_1, \psi_2)[A]).$$

By 3.2, $(\psi_1, \psi_2)[A]$ is a bisimulation, so $\operatorname{Ker}\varphi$ is a regular congruence.

Conversely, let $R \subseteq B \times B$ be a bisimulation on $\mathcal{B}$ with $\operatorname{Ker}\varphi = \mathcal{E}q(R)$. For the projection homomorphisms $\pi_1, \pi_2 : \mathcal{R} \to \mathcal{B}$ we have $\varphi \circ \pi_1 = \varphi \circ \pi_2$. If $\psi : \mathcal{B} \to \mathcal{D}$ is another homomorphism with $\psi \circ \pi_1 = \psi \circ \pi_2$, we must have $R \subseteq \operatorname{Ker}\psi$. Consequently, $\operatorname{Ker}\varphi = \mathcal{E}q(R) \subseteq \operatorname{Ker}\psi$, so ψ factors uniquely through φ by lemma 3.1.

$$\begin{tikzcd}
\mathcal{R} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathcal{B} \arrow[r, two heads, "\varphi"] \arrow[dr, "\psi"'] & \mathcal{C} \arrow[d, dashed] \\
 & & \mathcal{D}
\end{tikzcd}$$

$\square$

In contrast to corollary 4.4, not every homomorphism has a (regular epi)-mono factorization. This will follow from the following proposition:

Proposition 4.7. *A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ has a (regular epi)-mono factorization if and only if the canonical homomorphism $\varphi^\star : \mathcal{A}/[\operatorname{Ker}\varphi]_2^\star \to A/\operatorname{Ker}\varphi$ is mono.*

Proof. We can factor any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ as $\varphi = \leq \circ\, \varphi^\star \circ \varphi^r$ where $\varphi^r : \mathcal{A} \twoheadrightarrow A/[\operatorname{Ker}\varphi]_2^\star$ is the canonical homomorphism, whose kernel $[\operatorname{Ker}\varphi]_2^\star$ is the largest regular congruence relation contained in $\operatorname{Ker}\varphi$. In particular, φ^r is regular, so if $\varphi^\star$ is mono, then we have the desired factorization.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, two heads, "\varphi"] \arrow[dr, two heads, "\rho"'] \arrow[dd, "\varphi^r"'] & & \mathcal{B} \\
& \mathcal{A}/\operatorname{Ker}\rho \arrow[ur, tail, "\mu"'] \arrow[dl, dashed, leftrightarrow, "\iota"] & \\
\mathcal{A}/[\operatorname{Ker}\varphi]_2^\star \arrow[rr, two heads, "\varphi^\star"] & & A/\operatorname{Ker}\varphi \arrow[uu, hook, "\leq"']
\end{tikzcd}$$

Conversely, assume that $\varphi = \mu \circ \rho$ with ρ regular epi and μ mono. Obviously, $\operatorname{Ker}\rho \subseteq \operatorname{Ker}\varphi$, hence also $[\operatorname{Ker}\rho]_2 \subseteq [\operatorname{Ker}\varphi]_2$, and therefore

$$\operatorname{Ker}\rho = [\operatorname{Ker}\rho]_2^\star \subseteq [\operatorname{Ker}\varphi]_2^\star.$$

Suppose that $R \subseteq \operatorname{Ker}\varphi$ is a bisimulation, then R carries a coalgebra structure, so that the projections $\pi_1 : \mathcal{R} \to A$ and $\pi_2 : \mathcal{R} \to \mathcal{A}$ are homomorphisms with $\varphi \circ \pi_1 = \varphi \circ \pi_2$. Since $\varphi = \mu \circ \rho$ and μ is mono, it follows that $\rho \circ \pi_1 = \rho \circ \pi_2$, so $R \subseteq \operatorname{Ker}\rho$. This proves that $[\operatorname{Ker}\varphi]_2 \subseteq [\operatorname{Ker}\rho]_2$ whence

$$[\operatorname{Ker}\varphi]_2^\star \subseteq [\operatorname{Ker}\rho]_2^\star = \operatorname{Ker}\rho.$$

Consequently, we have an isomorphism $\iota : \mathcal{A}/Ker\rho \to \mathcal{A}/[\operatorname{Ker}\varphi]_2^\star$ with $\iota\circ\rho = \varphi^r$. Since ρ is epi, it follows that $\mu = \leq \circ\, \varphi^\star \circ \iota$. Hence we can suppress ι and assume that $\rho = \varphi^r$ and $\mu = \subseteq \circ\, \varphi^\star$. Since μ is mono, so is $\varphi^\star$. □

Example 4.8. *On the three-element set $\{a, b, c\}$ define the $(-)^3_2$-coalgebra structure $a \mapsto (a, b, b),\ b \mapsto (a, b, b),\ c \mapsto (c, c, b)$. Then $a \sim b$ but $a \not\sim c$ and $b \not\sim c$.*

Factoring by $\sim^\star = \sim$ we obtain a two-element coalgebra $\mathcal{B} = (\{\hat{b}, \hat{c}\}, \beta)$ where $\beta(\hat{b}) = (\hat{b}, \hat{b}, \hat{b})$ and $\beta(\hat{c}) = (\hat{c}, \hat{c}, \hat{b})$. Obviously, we have obtained "new" bisimilar elements $\hat{b}$ and $\hat{c}$.

Consider now the homomorphism $\varphi : \mathcal{A} \to \{\star\}$ to the one-element $(-)^3_2$-coalgebra. Then $\operatorname{Ker}\varphi = \nabla_{\mathcal{A}}$ and $[\operatorname{Ker}\varphi]_2^\star = \sim_A^\star$. But the unique homomorphism from $A/[\operatorname{Ker}\varphi]_2^\star \cong \mathcal{B}$ to $A/\operatorname{Ker}\varphi \cong \{\star\}$ is not mono, since its kernel contains a nontrivial bisimulation.

In contrast to corollary 4.4, we learn from this example:

Corollary 4.9. *Not every morphism in $\mathcal{S}et_{(-)_2 3}$ has a (regular epi)-mono factorization.*

5. Weak Preservation of Pullbacks

In this section we shall give a structure theoretical property which is equivalent to weak pullback preservation.

To begin with, consider a *source* in $\mathcal{S}et_F$, that is two coalgebra homomorphisms $\varphi_1 : \mathcal{A} \to \mathcal{B}_1$ and $\varphi_2 : \mathcal{A} \to \mathcal{B}_2$ with common domain. We know that their graphs and the converses of their graphs, in particular, $G\varphi_2$ and $(G\varphi_1)^-$, are bisimulations. The relational product of those is a bisimulation too, since

$$(G\varphi_1)^- \circ (G\varphi_2) = (\varphi_1, \varphi_2)[A].$$

Dually, consider a *sink* in $\mathcal{S}et_F$, i.e. a pair of homomorphisms with common codomain, $\psi_1 : \mathcal{A}_1 \to \mathcal{B}$ and $\psi_2 : \mathcal{A}_2 \to \mathcal{B}$. Then the relational product of the bisimulations $(G\psi_1)$ and $(G\psi_2)^-$ is

$$(G\psi_1) \circ (G\psi_2)^- = Pb(\psi_1, \psi_2),$$

which in general is *not* a bisimulation.

The following lemma contains the key observation:

Lemma 5.1. *Let $f : A \to C$ and $g : B \to C$ be maps with $Pb(f, g) \neq \emptyset$. Then the following are equivalent:*

(1) *F weakly preserves the pullback of f and g.*
(2) *$Pb(f, g)$ is a bisimulation for all F-coalgebra structures on A, B, and C, for which f and g are homomorphisms.*

Proof. (1) $\Rightarrow$ (2) is due to Rutten (c.f. [Rut00]): Assume that F weakly preserves the pullback (P, π_1, π_2) of homomorphisms $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$. Applying F to this pullback diagram we obtain a commutative square again, which forms the bottom of the cube in the following diagram.

$$\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dd, "\alpha_A"] & & C \arrow[dd, "\alpha_C"] \\
P \arrow[ur, "\pi_1"] \arrow[rr, "\pi_2"] \arrow[dd, dotted, "\alpha_P"] & & B \arrow[ur, "g"] \arrow[dd, "\alpha_B"] & \\
& F(A) \arrow[rr, "Ff"] & & F(C) \\
F(P) \arrow[ur, "F\pi_1"] \arrow[rr, "F\pi_2"] & & F(B) \arrow[ur, "Fg"] &
\end{tikzcd}$$

Since f and g are homomorphisms, we find

$$\begin{aligned}
Fg \circ \alpha_B \circ \pi_2 &= \alpha_C \circ g \circ \pi_2 \\
&= \alpha_C \circ f \circ \pi_1 \\
&= Ff \circ \alpha_A \circ \pi_1.
\end{aligned}$$

This means that P with maps $\alpha_B \circ \pi_1$ and $\alpha_A \circ \pi_2$ has become a competitor to the weak limit $F(P)$, so there is some map $\alpha_P : P \to F(P)$ with $F\pi_1 \circ \alpha_P = \alpha_A \circ \pi_1$ and $F\pi_2 \circ \alpha_P = \alpha_B \circ \pi_2$. Thus, α_P is a structure map on P with respect to which π_1 and π_2 are homomorphisms, i.e. P is a bisimulation.

(2)$\Rightarrow$(1): Given $f : A \to C$ and $g : B \to C$ with nonempty pullback (P, π_1, π_2), we check that $(F(P), F\pi_1, F\pi_2)$ is a weak pullback of Ff and Fg by verifying the conditions of lemma 2.4: Given $\tilde{a} \in F(A)$, $\tilde{b} \in F(B)$, and $\tilde{c} \in F(C)$ with

$$(Ff)(\tilde{a}) = \tilde{c} = (Fg)(\tilde{b}),$$

we need to find an element $\tilde{p} \in F(P)$ with $(F\pi_1)(\tilde{p}) = \tilde{a}$ and $(F\pi_2)(\tilde{p}) = \tilde{b}$.

On A define the constant coalgebra structure $\mathcal{A}^{\tilde{a}} = (A, \alpha_A)$, where $\alpha_A(x) := \tilde{a}$ for each $x \in A$. $\mathcal{B}^{\tilde{b}}$ and $\mathcal{C}^{\tilde{c}}$ are defined analogously. Then f and g are homomorphisms, so with our assumption, the pullback $P = (Gf) \circ (Gg)^{-}$ is a bisimulation. This means that there exists a coalgebra structure α_P on P so that $\pi_1 : \mathcal{P} \to \mathcal{A}^{\tilde{a}}$ and $\pi_2 : \mathcal{P} \to B^{\tilde{b}}$ are homomorphisms, i.e. for all $p \in P$ we have $(F\pi_1 \circ \alpha_P)(p) = (\alpha_A \circ \pi_1)(p) = \tilde{a}$ and $(F\pi_2 \circ \alpha_P)(p) = (\alpha_B \circ \pi_2)(p) = \tilde{b}$. Thus, for an arbitrarily chosen $p_0 \in P$ we set $\tilde{p} := \alpha_P(p_0)$ and verify $(F\pi_1)(\tilde{p}) = \tilde{a}$ and $(F\pi_2)(\tilde{p}) = \tilde{b}$. $\square$

If F weakly preserves pullbacks, it has been known (see [Rut00]) that the relational product of bisimulations is a bisimulation, and that this, in turn, implies that pullbacks of homomorphisms are bisimulations. With the help of lemma 5.1 and theorem 2.7, we obtain now the equivalence of these conditions:

Theorem 5.2. *For a functor $F : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

1. *F weakly preserves pullbacks,*
2. *For any two F-homomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ the pullback $Pb(\varphi, \psi)$ is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$.*
3. *The relational product $R \circ S$ of two bisimulations R and S is again a bisimulation.*

Proof. Lemma 5.1 gives us the equivalence of (1) and (2) for nonempty pullbacks. Theorem 2.7 allows us to drop "nonempty" in (1).

(2)⇒(3): Given coalgebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and bisimulations $R \subseteq A \times B$, and $S \subseteq B \times C$, then the projections π_A^R, π_B^R π_B^S, and π_C^S are homomorphisms. The pullback of π_B^R with π_B^S is $R \bowtie S := \{((a,b),(b,c)) \mid (a,b) \in R, (b,c) \in S\}$.

$$\begin{tikzcd}
 & & \mathcal{R} \bowtie \mathcal{S} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & \mathcal{R} \arrow[dl, "\pi_A^R"'] \arrow[dr, "\pi_B^R"] & & \mathcal{S} \arrow[dl, "\pi_B^S"'] \arrow[dr, "\pi_C^S"] & \\
\mathcal{A} & & \mathcal{B} & & \mathcal{C}
\end{tikzcd}$$

By assumption, this is a bisimulation, so there exists a coalgebra structure on $R \bowtie S$, turning the projections π_1 and π_2 into homomorphisms. Observe that

$$R \circ S = (\pi_A^R \circ \pi_1, \pi_C^S \circ \pi_2)[R \bowtie S],$$

which is a bisimulation by proposition 3.2.

(3)⇒(2): $Pb(f,g) = (Gf) \circ (Gg)^-$. □

Having obtained a coalgebraic characterization of weak pullback preservation, theorem 2.7 suggest to consider the cases separately, where F preserves preimages, resp. kernel pairs. Indeed, we have seen functors, preserving preimages, but not kernels, and functors preserving kernels, but not preimages. What are the corresponding structural properties of the F-coalgebras?

5.1. **Preservation of Preimages.**

Theorem 5.3 ([GS00]). *The following are equivalent:*

1. *F preserves preimages.*
2. *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{V} \leq \mathcal{B}$ a subcoalgebra, then $\varphi^{-1}[V]$ is a subcoalgebra of $\mathcal{A}$.*
3. *Given a bisimulation R between coalgebras $\mathcal{A}$ and $\mathcal{B}$ and subcoalgebras $\mathcal{U} \leq \mathcal{A}$ and $\mathcal{V} \leq \mathcal{B}$, then $R \cap (U \times V)$ is a bisimulation between $\mathcal{U}$ and $\mathcal{V}$.*
4. *Every homomorphism $\varphi : \mathcal{A} \to \mathcal{B} + \mathcal{C}$ splits its domain, i.e., $\varphi^-[B]$ and $\varphi^-[C]$ are subcoalgebras of $\mathcal{A}$ with $\mathcal{A} = \varphi^-[B] + \varphi^-[C]$.*

Proof. (1) ⇒ (2): Assume that F preserves preimages. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism and $\mathcal{V} \leq \mathcal{B}$ a subcoalgebra. By (i)⇒(ii) of theorem 5.2, the pullback

$$Pb(\varphi, \leq) = \{(a, \varphi(a)) \mid \varphi(a) \in V\}$$

is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, so its homomorphic image,

$$\pi_1[Pb(\varphi, \leq)] = \varphi^-[V]$$

is a subcoalgebra of $\mathcal{A}$.

(2) $\Rightarrow$ (3): Let $\mathcal{R}$ be the bisimulation $R \subseteq A \times B$, equipped with a coalgebra structure making $\pi_1 : \mathcal{R} \to \mathcal{A}$ and $\pi_2 : \mathcal{R} \to \mathcal{B}$ into homomorphisms. For a subcoalgebra $\mathcal{V} \leq \mathcal{B}$, we get from (2) that

$$R' := \pi_2^-[V] = R \cap (A \times V)$$

is a subcoalgebra of $\mathcal{R}$, in particular a bisimulation between $\mathcal{A}$ and $\mathcal{V}$. Continuing with R', we find by the same reasoning that $R' \cap (U \times V) = R \cap (U \times V)$ is a bisimulation between $\mathcal{U}$ and $\mathcal{V}$.

(3) $\Rightarrow$ (4): Given $\varphi : \mathcal{A} \to \mathcal{B} + \mathcal{C}$, the graph $(G\varphi)$ is a bisimulation, so

$$R_B := (G\varphi) \cap (A \times B), \text{ and } R_C = (G\varphi) \cap (A \times C)$$

are bisimulations. Hence $\varphi^-[B] = \pi_1[R_B]$ and $\varphi^-[C] = \pi_1[R_C]$ are disjoint subcoalgebras of $\mathcal{A}$, whose union is $\mathcal{A}$.

(4) $\Rightarrow$ (1): (This is the most complicated step): We may assume that $f : A \to B$ is surjective and $V \subseteq B$. Furthermore, we may assume that $V \neq \emptyset$. Put $U := f^-[V]$ and let f' be the restriction of f to U, i.e. $\subseteq_V^B \circ f' = f \circ \subseteq_U^A$. We need to check the condition of lemma 2.4:

Given $\tilde{a} \in F(A)$, $\tilde{b} \in F(B)$, and $\tilde{v} \in F(V)$ with $(Ff)(\tilde{a}) = \tilde{b} = (F \subseteq_V^B)(\tilde{v})$, we need to find an element $\tilde{u} \in F(U)$, so that $(F \subseteq_U^A)(\tilde{u}) = \tilde{a}$. The second equation, $(Ff')(\tilde{u}) = \tilde{v}$, is then automatically satisfied, by remark 2.5.

If $F(B - V) = \emptyset$ then $B - V = \emptyset$, so $U = A$ and $\tilde{u} = \tilde{a}$ will do. Otherwise, pick $\hat{b} \in F(B - V)$ and define a coalgebra structure α_B on B with

$$\alpha_B(x) := \textbf{ if } x \in V \textbf{ then } \tilde{b} \textbf{ else } \hat{b}.$$

Then V and $B - V$ are subcoalgebras of $\mathcal{B} = (B, \alpha_B)$.

We now pull the structure back to A. (Ff) is surjective, so pick some $\hat{a} \in F(A)$ with $(Ff)(\hat{a}) = \hat{b}$ and define a coalgebra $\mathcal{A} = (A, \alpha_A)$ by

$$\alpha_A(x) := \textbf{ if } x \in U \textbf{ then } \tilde{a} \textbf{ else } \hat{a}.$$

Obviously, f is a surjective homomorphism, so by (4), we get that U is a subcoalgebra of $\mathcal{A}$. Pick any $u_0 \in U$ and set $\tilde{u} := \alpha_U(u_0)$. Then

$$(F \subseteq_U^A)(\tilde{u}) = (F \subseteq_U^A \circ\, \alpha_U)(u_0) = \alpha_A(u_0) = \tilde{a},$$

as required. $\square$

5.2. **A class equation.** If $\mathcal{K}$ is a class of F-coalgebras, we denote by $\mathcal{H}(\mathcal{K})$ the class of all homomorphic images of coalgebras in $\mathcal{K}$, by $\mathcal{S}(\mathcal{K})$ the class of all subcoalgebras of coalgebras in $\mathcal{K}$. In general, for every class $\mathcal{K} \subseteq \mathcal{S}et_F$, we have $\mathcal{HS}(\mathcal{K}) \subseteq \mathcal{SH}(\mathcal{K})$.

If the type functor F preserves preimages, it is easy to check that the operators $\mathcal{H}$ and $\mathcal{S}$ commute, i.e. $\mathcal{HS}(\mathcal{K}) = \mathcal{SH}(\mathcal{K})$ for every class $\mathcal{K}$ of F-coalgebras (see [GS01b]). We do not know - but strongly conjecture - that the converse is also true, i.e. the commutation of $\mathcal{H}$ and $\mathcal{S}$ forces F to preserve preimages.

In this section we will prove this conjecture under the addditional assumption that $F(1)$ has more than one element. This result is from the second author's thesis and has been developed together with Alexander Schulz ([Sch]).

We first consider a special class of preimages:

Definition 5.4. *A* classifying preimage *is a preimage diagram of the form*

$$\begin{tikzcd} A \arrow[r, "\chi_U"] & 2 \\ U \arrow[u, hook, "\subseteq^A_U"] \arrow[r, "!_U"] & 1 \arrow[u, hook', "t"'] \end{tikzcd}$$

where $2 = \{t, f\}$*.* χ_U *is called the characteristic function of* U*.*

To any preimage diagram we can attach a classifying preimage, so that by lemma 2.2(1), the complete diagram becomes a classifying preimage, again.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\chi_V"] & 2 \\ U \arrow[u, hook, "\subseteq^A_U"] \arrow[r, "f'"] & V \arrow[u, hook, "\subseteq^A_V"'] \arrow[r, "!_V"] & 1 \arrow[u, hook', "t"'] \end{tikzcd}$$

Using (2) of the same lemma, we find:

Lemma 5.5. *If* F *preserves classifying preimages, then* F *preserves arbitrary preimages.*

With these preparations, we can state and prove the main result of this section:

Theorem 5.6 ([Sch01])**.** *If* $F(1) \ncong 1$ *then* F *preserves preimages if and only if* $\mathcal{HS}(\mathcal{K}) = \mathcal{SH}(\mathcal{K})$ *for each class* $\mathcal{K} \subseteq \mathcal{S}et_F$*.*

Proof. With notation as in definition 5.4, let a classifying preimage be given. Given elements $\tilde{a} \in F(A)$, $\tilde{c} \in F2$, and $\tilde{b} \in F1$ with $(F\chi_U)(\tilde{a}) = \tilde{c} = (Ft)(\tilde{b})$, it is enough (by lemma 2.4 and remark 2.5) to find an element $\tilde{u} \in FU$ with $(F \subseteq^A_U)\tilde{u} = \tilde{a}$.

Since $|F1| > 1$ and $F!_U$ is surjective, we find elements $\hat{b} \neq \tilde{b} \in F1$ and $\hat{u} \in F(U)$ with $(F!_U)(\hat{u}) = \hat{b}$. Put $\hat{a} := (F \subseteq^A_U)(\hat{u})$ and $\hat{c} := (F\chi_U)\hat{a} = (Ft)\hat{b}$.

$$\begin{tikzcd} FA \arrow[r, "F\chi_U"] & F2 \\ FU \arrow[u, "F\subseteq^A_U"] \arrow[r, "F!_U"] & F1 \arrow[u, "Ft"'] \end{tikzcd} \qquad \textit{with elements} \qquad \begin{tikzcd}[column sep=small] & \tilde{a} \arrow[rr, mapsto] & & \tilde{c} \\ \hat{a} \arrow[rr, mapsto] \arrow[ur, phantom, "\neq"] & & \hat{c} \arrow[ur, phantom, "\neq"] & \\ & \tilde{u} \arrow[uu, mapsto, dotted] \arrow[rr, mapsto, dotted] & & \tilde{b} \arrow[uu, mapsto] \\ \hat{u} \arrow[uu, mapsto] \arrow[rr, mapsto] & & \hat{b} \arrow[uu, mapsto] \arrow[ur, phantom, "\neq"] & \end{tikzcd}$$

We now define coalgebras $\mathcal{A} = (A, \alpha_A)$, $\mathbf{2} = (2, \alpha_2)$, and $\mathbf{1} = (1, \alpha_1)$ by

$$\alpha_A(x) := \begin{cases} \tilde{a} & \text{if } x \in U \\ \hat{a} & \text{else,} \end{cases} \qquad \alpha_2(x) := \begin{cases} \tilde{c} & \text{if } x = t \\ \hat{c} & \text{if } x = f, \end{cases} \quad \text{and} \qquad \alpha_1(t) := \tilde{b}.$$

Obviously, $\chi_U : \mathcal{A} \twoheadrightarrow \mathbf{2}$ is a surjective homomorphism and $\mathbf{1}$ is a subcoalgebra of $\mathbf{2}$. Thus $\mathbf{1} \in \mathcal{SH}(\mathcal{A})$.

By assumption, there must be a nonempty subcoalgebra $\mathcal{V}$ of $\mathcal{A}$ so that the unique map $!_V$ is a homomorphism from $\mathcal{V}$ to $\mathbf{1}$. V must be contained in $U \subseteq A$, for otherwise we would have a $v \in V - U$, leading to

$$\begin{aligned}
\tilde{b} &= (\alpha_1 \circ !_V)v \\
&= (F!_V \circ \alpha_V)v \\
&= (F!_A \circ F \subseteq_V^A \circ \alpha_V)v \\
&= (F!_A \circ \alpha_A)v \\
&= (F!_A)\hat{a} \\
&= (F!_A \circ F \subseteq_U^A)\hat{u} \\
&= (F!_U)\hat{u} \\
&= \hat{b}.
\end{aligned}$$

Thus we can choose any $u_0 \in V \subseteq U$ and obtain the desired element $\tilde{u} \in FU$ as $\tilde{u} := (F \subseteq_V^U)(\alpha_V(u_0))$, since

$$(F \subseteq_U^A)\tilde{u} = (F \subseteq_V^A)(\alpha_V(u_0)) = \alpha_A(u_0) = \tilde{a}.$$

□

5.3. **Preservation of Kernel Pairs.** The structure theoretical consequences that we obtain when F preserves kernel pairs are more important in coalgebraic theory than those that follow if F preserves preimages.

One of those consequences will be that the largest bisimulation $\sim_A$ is transitive, in fact it is the same as the largest congruence relation $\nabla_{\mathcal{A}}$. If, as often, bisimulation is interpreted as observational equivalence, we should expect these properties.

However, preservation of kernel pairs is a slightly stronger property, and we shall see in the next section, how to modify it to obtain an equivalential statement.

Theorem 5.7. *The following are equivalent.*

(1) *F weakly preserves kernel pairs.*
(2) *Every congruence is a bisimulation.*

Proof. (1)⇒(2): Let $\theta = \mathrm{Ker}\,\varphi$ for some surjective homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. By lemma 5.1 $\theta = Pb(\varphi, \varphi)$ is a bisimulation.

(2)⇒(1): Given a map $f : A \to C$, $\tilde{a}, \tilde{b} \in F(A)$ and $\tilde{c} \in C$ with $(Ff)\tilde{a} = \tilde{c} = (Ff)\tilde{b}$, lemma 2.4 requires us to find some $\tilde{p} \in F(\mathrm{Ker}\,f)$ with $(F\pi_1)\tilde{p} = \tilde{a}$ and $(F\pi_2)\tilde{p} = \tilde{b}$.

If f is injective, then so is Ff and $\tilde{a} = \tilde{b}$. In any case, we can find $x, y \in A$ with $fx = fy$, and a map $\alpha_A : A \to \{\tilde{a}, \tilde{b}\} \subseteq F(A)$ with $\alpha_A(x) = \tilde{a}$ and $\alpha_A(z) = \tilde{b}$ for all $z \neq x$.

$$\begin{tikzcd}
\theta \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[rr, "f"] & & C \arrow[ddd, "\alpha_C"] \\
(x,y) \arrow[d, dotted] & x \arrow[d] & y \arrow[d] & \\
\tilde{p} & \tilde{a} & \tilde{b} & \\
F\theta \arrow[r, shift left, "F\pi_1"] \arrow[r, shift right, "F\pi_2"'] & F(A) \arrow[rr, "Ff"] & & F(C)
\end{tikzcd}$$

Clearly, f becomes a homomorphism, if we define on C the constant coalgebra structure with $\alpha_C(z) = \tilde{c}$ for all $z \in C$. Now $\theta := \mathrm{Ker}\,f$ is a congruence relation, and a bisimulation by hypothesis (2). Hence we have a coalgebra structure ρ on $\mathrm{Ker}\, f$ with $F\pi_i \circ \rho = \alpha_A \circ \pi_i$ for $i = 1, 2$. We put $\tilde{p} := \rho(x, y)$ and check:

$$(F\pi_1)\tilde{p} = (F\pi_1 \circ \rho)(x, y) = (\alpha_A \circ \pi_1)(x, y) = \tilde{a},$$

and similarly, $(F\pi_2)\tilde{p} = \tilde{b}$. □

Corollary 5.8. *If F weakly preserves kernel pairs then*
(1) *every epi is regular, and*
(2) *every mono is regular.*

Proof. Every congruence is a bisimulation, hence it is a regular congruence. (2) follows from (1) in every category where each arrow has an epi-(regular mono) factorization (see e.g. [AHS90]). □

5.4. **Indistinguishability and Observational Equivalence.** Universal coalgebra can be considered as the theory of state based systems. In many practical applications of such systems, one is concerned whether two states can be distinguished by experiments or tests. If they cannot be told apart, they are called "bisimilar".

Aczel and Mendler have abstractly defined bisimulations as binary relations, compatible with the coalgebra structure. Every coalgebra $\mathcal{A}$ has a largest bisimulation $\sim_A$, and two elements a and b are called bisimilar, if $a \sim b$.

Bisimilarity has often been equated with *observational equivalence*, ([Rut00, Mos99]). Since most of the early papers on universal coalgebra assumed that the type functor F preserves weak pullbacks, this was justified, as we shall see. However, without such an assumption, it turns out that bisimilarity need not be transitive, hence not an equivalence relation.

But the notion of *observational equivalence* is very useful in many applications, so it should not be given up. For instance, one can identify observationally equivalent states and obtain an "equivalent" system with a minimal number of states.

A first attempt to define observational equivalence might be the transitive closure $\sim_A^\star$ of the largest bisimulation. By proposition 3.3, this is indeed a congruence. Unfortunately, however, the factor $\mathcal{A}/\sim_A^*$ could again have bisimilar states, as we have already seen in example 4.8.

Since homomorphisms of coalgebras are to preserve outcomes of experiments, the following definition seems to appropriate:

Definition 5.9. *Two states $a \in A$ and $b \in B$ are called* observationally equivalent*, if there is a coalgebra $\mathcal{C}$ and homomorphisms $\varphi : \mathcal{A} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{C}$ so that $\varphi(a) = \psi(b)$.*

Lemma 5.10. *Observational equivalence on a coalgebra $\mathcal{A}$ is given by the largest congruence relation $\nabla_{\mathcal{A}}$. Bisimilar states are observational equivalent.*

Proof. If $(x, y) \in \nabla_{\mathcal{A}}$ then x and y are clearly observationally equivalent. Conversely, given that $\varphi(x) = \psi(y)$, we take the pushout χ of φ and ψ and get $(x, y) \in \mathrm{Ker}(\varphi \circ \chi) \subseteq \nabla_{\mathcal{A}}$.

Due to proposition 3.3, we have $\sim_A \subseteq \sim_A^\star \subseteq \nabla_{\mathcal{A}}$. □

In this section, we shall be concerned with the question, which properties of the type functor F guarantee that bisimilarity is an equivalence relation or even agrees with $\nabla_{\mathcal{A}}$.

Theorem 5.11. *If F preserves preimages, then the following are equivalent:*

(1) $\sim_{\mathcal{A}}$ *is transitive for all* $\mathcal{A} \in \mathcal{S}et_F$.
(2) $\nabla_{\mathcal{A}} = \sim_A$ *for all* $\mathcal{A} \in \mathcal{S}et_F$.

Proof. $\nabla := \nabla_{\mathcal{A}}$ is always transitive, so one direction is trivial. For the other direction, consider $a, a' \in A$ with $a \nabla a'$. We are going to show that $a \sim_{\mathcal{A}} a'$.

Let $\pi : \mathcal{A} \to \mathcal{A}/\nabla$ be the canonical projection and consider the sum $\mathcal{S} := \mathcal{A} + \mathcal{A}/\nabla + \mathcal{A}$ with its canonical embeddings ι_1, ι_2, and ι_3. Now π induces an endomorphism $\psi := [(\iota_2 \circ \pi), \iota_2, (\iota_2 \circ \pi)]$ on $\mathcal{S}$, satisfying

$$\psi \circ \iota_1 = \iota_2 \circ \pi = \psi \circ \iota_3.$$

Using the fact that the graph of ψ and its converse must be contained in $\sim_{\mathcal{S}}$, we obtain:

$$\iota_1(a) \sim_S \psi(\iota_1(a)) = \iota_2(\pi(a)) = \iota_2(\pi(a')) = \psi(\iota_3(a')) \sim_S \iota_3(a').$$

It is easy to see that $\iota_1(x) \sim_S \iota_3(x)$ for every $x \in A$, in particular, $\iota_3(a') \sim_{\mathcal{S}} \iota_1(a')$. By hypothesis, $\sim_{\mathcal{S}}$ is transitive, so we obtain $\iota_1(a) \sim_{\mathcal{S}} \iota_1(a')$. Theorem 5.3 allows us to conclude $a \sim_{\mathcal{A}} a'$. □

From theorem 5.7 we obtain immediately:

Lemma 5.12 ([GS00])**.** *If F weakly preserves kernels, then bisimilarity is the same as observational equivalence.*

The converse of this lemma does not hold:

Example 5.13. *Consider the subfunctor $\mathbb{P}_4^+$ of the power set functor $\mathbb{P}$ given by*

$$\mathbb{P}_4^+(A) := \{U \subseteq A \mid 0 < |U| < 4\}.$$

To see that $\mathbb{P}_4^+$ does not weakly preserve kernel pairs, we consider the map

$$\text{even} : \{0, \dots, 5\} \to \{t, f\}.$$

For $U := \{0,1,2\}$ and $V := \{3,4,5\}$ we obviously have $(\mathbb{P}_4^+\,\text{even})(U) = \{t,f\} = (\mathbb{P}_4^+\,\text{even})(V)$, but there is no subset $W \subseteq \operatorname{Ker}$ even with less than 4 elements such that $(\mathbb{P}_4^+\pi_1)(W) = \pi_1[W] = \{0,1,2\}$ and $(\mathbb{P}_4^+\pi_1)(W) = \pi_1[W] = \{3,4,5\}$.

$\mathbb{P}_4^+$-coalgebras are just transition system where every element a has either 1, 2, or 3 successors. Given two such systems $\mathcal{A}$ and $\mathcal{B}$, elements $a \in A$ and $b \in B$, with successors $\alpha_A(a)$, resp. $\alpha_B(b)$, we can easily choose a set $W \subseteq \alpha_A(a) \times \alpha_B(b)$ of successor pairs, so that $\pi_1[W] = \alpha_A(a)$, $\pi_2[W] = \alpha_B(b)$, and $|W| < 4$. In this way we define a structure map on $A \times B$ so that the projections are homomorphisms - i.e. $A \times B$ is a bisimulation. In particular,

$$\sim_A = \sim_A^\star = \nabla_{\mathcal{A}} = A \times A$$

for every $\mathbb{P}_4^+$-coalgebra.

We will see now how the weak preservation of kernels can be expressed by greatest bisimulations. The key to this result is the following lemma.

Lemma 5.14. *Let $\mathcal{A} = (A, \alpha_A)$ be an F-Coalgebra, θ a congruence on $\mathcal{A}$ and $\pi_\theta : A \to A/\theta$ the canonical projection. Then $(A, (\pi_\theta, \alpha_A))$ is an $(A/\theta) \times F$-coalgebra on which θ is the largest congruence.*

From this lemma and theorem 5.7 we can conclude:

Proposition 5.15. *The following are equivalent:*

(1) F *weakly preserves kernel pairs.*
(2) *For any set* C *and any* $C \times F$*-coalgebra* $\mathcal{A}$ *we have* $\nabla_{\mathcal{A}} = \sim_{\mathcal{A}}$.

Proof. It is easy to see that F weakly preserves kernel pairs iff for every set C the functor $C \times F$ weakly preserves kernel pairs. Together with lemma 5.12 this gives (1)⇒(2).

To see (2)⇒(1), let θ by an F-congruence on the F-coalgebra $\mathcal{A} = (A, \alpha_A)$. By lemma 5.14, θ is the largest $(A/\theta)\times F$-congruence on $(A, (\pi_\theta, \alpha_A))$, so by assumption the largest $(A/\theta)\times F$-bisimulation. But then θ is also an F-bisimulation on $\mathcal{A}$ which suffices to prove (1) by theorem 5.7. □

6. Discussion

We have characterized preservation properties of set functors F by coalgebraic structure theorems for the category $\mathcal{S}et_F$ of F-coalgebras.

Preservation of intersections can be achieved by a standardization of the functor on the empty set and empty mappings, so we could always assume this. Weak preservation of pullbacks is then a combination of two easier preservation properties:

- (weak) preservation of preimages, and
- weak preservation of kernel pairs.

Both of these properties were characterized separately. For the class equation,

$$\mathcal{HS} = \mathcal{SH}$$

we proved that it is equivalent to F preserving preimages, provided $|F(1)| > 1$. For all practical purposes, this proviso captures the most important cases, since $|F(1)| = 1$ entails that the 1-element coalgebra is terminal. Nevertheless, must leave it as an open problem, whether $|F(1)| = 1$ together with $\mathcal{HS} = \mathcal{SH}$ entails that F preserves preimages.

The paper also characterized equalizers, preimages, monos, regular monos, and epis in the category $\mathcal{S}et_F$, and it describes under which condition arrows can be decomposed in a regular epi followed by a mono.

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