Some Characterizations of the Commutator

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Abstract

We start with a characterization of the modular commutator that was given by E. Kiss and the first author in [DK] and explore some of its consequences. Implicit in this characterization is that both the shifting lemma and the cube lemma from [Gu] yield descriptions of the commutator in terms of implications of identities. The shifting lemma translates into the well known term condition and the cube lemma yields a similar condition involving two terms. We give some applications, improving a result of [Gu] and propose to define the commutator in non-modular varieties using the construction from [DK]. Varieties satisfying $[\alpha, \beta] = 0$ and those satisfying $[\alpha, \alpha] = \alpha$ are then characterized. This work was done when the second author visited Lakehead University in February and March of 1986. E. Kiss has, through a different approach, independently obtained a result very similar to theorem 3.2 in a preprint of November 1986. The second author expresses his thanks to A. Day and the National Research Council of Canada for making his stay at Lakehead University both possible and pleasant.

1 Preliminaries

Given a homomorphism $\phi : A \to B$ and a congruence relation θ on A, we write $\vec{\phi}(\theta)$ or simply $\vec{\phi}\theta$ for the congruence relation on B generated by the

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direct images of pairs in θ :

$$\overrightarrow{\phi}(\theta) = con_B\left(\{(\phi x, \phi y) | (x, y) \in \theta\}\right)$$

This notation is justified by the observation that, given a homomorphism $\psi : B \to C$, then for all $\alpha \in Con(A)$,

$$\overrightarrow{(\psi \circ \phi)}\,(\alpha) = \overrightarrow{\psi}(\overrightarrow{\phi}(\alpha)).$$

If $\beta \ \in \ Con(B)$, the inverse image congruence on A is denoted by

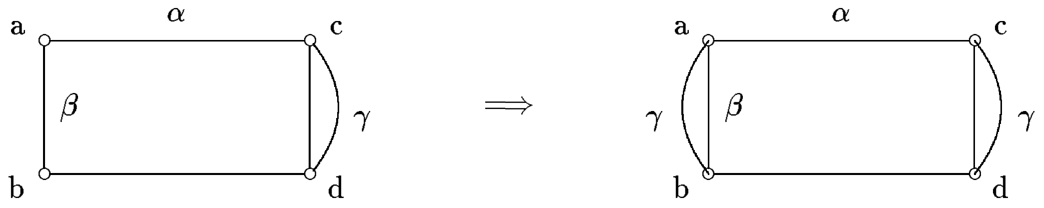
$$\overleftarrow{\phi}(\beta) := \left(\{(x, y) | (\phi(x), \phi(y)) \ \in \ \beta\}\right).$$

Thus the kernel of the homomorphism $\phi : A \to B$ is given by $Ker\phi = \overleftarrow{\phi}(0_B)$. Again we have the functorial property, for all $\gamma \ \in \ Con(C)$,

$$\overleftarrow{(\psi \circ \phi)}\,(\gamma) = \overleftarrow{\psi}(\overleftarrow{\phi}(\gamma)).$$

Except for chapter 4, we will assume that all algebras are contained in a congruence modular variety $\mathcal{V}$. Such varieties have been characterized by means of Mal'cev conditions in [D] and in [Gu]. A variety $\mathcal{V}$ is congruence modular iff in all algebras $A \in V$ the shifting lemma holds [Gu], that is:

For any three congruences α, β, γ with $\alpha \wedge \beta \leq \gamma$ we have



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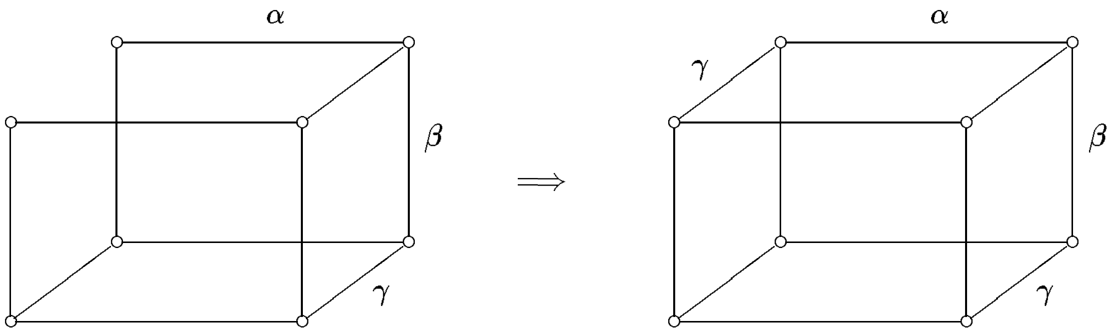
Here, nodes labeled x and y are connected by a line labeled with a congruence θ to express the fact that $(x, y) \in \theta$. Parallel lines are assumed to be labelled with the same congruence. The assumption that $\alpha \wedge \beta \leq \gamma$ may be dropped, but then $(a, b) \ \in \ \gamma \vee (\alpha \wedge \beta)$ must replace $(a, b) \ \in \ \gamma$ in the conclusion. If subalgebras of squares of A are also congruence modular then the cube lemma will hold. This is expressed similarly as:

If $\alpha \wedge \beta \leq \gamma$ then



For a variety of algebras, $\mathcal{V}$, both the shifting lemma and the cube lemma are individually equivalent to congruence modularity.

In a congruence modular variety, a commutator operation was defined by Hagemann and Herrmann [HH] on congruence lattices of algebras A in $\mathcal{V}$. The commutator of the congruences α and β is denoted $[\alpha, \beta]$, and can be described conveniently in the following way, as shown in [Gu]. Consider α as a subalgebra of $A \times A$ and let Δ_α^β be the congruence on α generated by the β-diagonals:

$$\Delta_\alpha^\beta := \vec{\nu}(\beta) = con_\alpha(\{\ ((x,x),(y,y)) \mid x\beta y\ \})$$

where $\nu : A \longrightarrow \alpha$ is the diagonal given by $\nu(a) = (a,a)$. Then

$$(a,b)\ \in\ [\alpha,\beta] \Leftrightarrow \exists w\ \in\ A : (a,w)\Delta_\alpha^\beta(b,w).$$

In our new notation, we have

$$[\alpha,\beta] = \vec{\pi}_1(Ker\pi_2 \cap \Delta_\alpha^\beta) = \vec{\pi}_1(Ker\pi_2 \cap \vec{\nu}(\beta)).$$

The most useful properties of the commutator are:

(i) $[\alpha,\beta] \leq \alpha \cap \beta$;
(ii) $[\alpha,\beta] = [\beta,\alpha]$;
(iii) $[\alpha, \bigvee \beta_i] = \bigvee [\alpha, \beta_i]$;
(iv) $\vec{\phi}[a,b] = [\vec{\phi}\alpha, \vec{\phi}\beta]$, if ϕ is an epimorphism.

In turn, the commutator $[\alpha,\beta]$ is the biggest multiplication on congruence lattices of a modular variety satisfying (i) through (iv). Note that (iv) may equivalently be formulated with inverse images of congruences:

(v) $\overleftarrow{\phi}[\alpha,\beta] = [\overleftarrow{\phi}\alpha, \overleftarrow{\phi}\beta] \vee Ker\phi$, if ϕ is an epimorphism.

We shall look at inverse images of commutators in free algebras $F(X+Y)$, where X and Y will be disjoint isomorphic sets. An element from $F(X+Y)$ will be a term $p(x_1,\ldots,x_k,y_1,\ldots,y_r)$ where the $x_i \in X$ and the $y_j \in Y$. We may sometimes list more variables than actually appear in p and therefore can always assume that $k = r$. We abbreviate $p(x_1,\ldots,x_k,y_1,\ldots,y_k)$ by $p(\mathbf{x};\mathbf{y})$.

2 Homomorphic sections

Let $\phi : A \twoheadrightarrow B$ be an epimorphism. A **section** of ϕ is a homomorphism $\mu : B \longrightarrow A$ with $\phi \circ \mu = id_B$. If μ is a section of ϕ then μ must be a monomorphism, so $\mu[B]$ is a subalgebra of A on which $\mu \circ \phi$ acts as the identity; in other words: $\mu \circ \phi$ is an idempotent endomorphism of A. Conversely, if $\psi : A \longrightarrow A$ is an endomorphism with $\psi \circ \psi = \psi$, then the inclusion of the subalgebra $B := \psi[A]$ into A is a section for ψ, considered as an epimorphism onto B. Given an epimorphism $\psi : A \twoheadrightarrow B$ with a section $\mu : B \longrightarrow A$, certain commutators can be easily described as in the following theorem of [DK] whose proof we include for completeness:

Theorem 2.1 *Let $\phi : A \twoheadrightarrow B$ be an epimorphism and $\mu : B \longrightarrow A$ a section of ϕ. Let β be a congruence relation on B. Then $[Ker\phi, \overrightarrow{\mu}(\beta)] = Ker\phi \cap \overrightarrow{\mu}(\beta)$.*

<u>Proof:</u> Let $\gamma = \overrightarrow{\mu}(\beta)$. The maps $\psi : A \longrightarrow Ker\phi$, given by $\psi(a) = (a, (\mu \circ \phi)(a))$, and the diagonal $\nu : A \longrightarrow Ker\phi$, given by $\nu(a) = (a,a)$ are homomorphisms with

$$\overrightarrow{\psi}(\gamma) = \overrightarrow{(\psi \circ \mu)}(\beta) = \overrightarrow{(\nu \circ \mu)}(\beta) = \overrightarrow{\nu}(\gamma) = \Delta^{\gamma}_{Ker\phi}$$

Therefore

$$\overrightarrow{\psi}(Ker\phi \cap \gamma) \subseteq Ker\pi_2 \cap \Delta^{\gamma}_{Ker\phi}$$

and

$$Ker\phi \cap \gamma = \overrightarrow{(\pi_1 \circ \psi)}(Ker\phi \cap \gamma) \subseteq \overrightarrow{\pi}_1(Ker\pi_2 \cap \Delta^{\gamma}_{Ker\phi}) = [Ker\phi, \gamma].$$

Thus $[Ker\phi, \vec{\mu}(\beta)] = Ker\phi \cap \vec{\mu}(\beta)$ as desired. $\square$

This theorem can in turn be used to give a definition of the modular commutator:

Theorem 2.2 *The commutator operation is characterized by the following two properties:*

(i) For every epimorphism $\phi : A \twoheadrightarrow B$ with a section $\mu : B \longrightarrow A$ and every $\beta \in Con(B)$, $[Ker\phi, \vec{\mu}(\beta)] = Ker\phi \cap \vec{\mu}(\beta)$.

(ii) For every epimorphism $\psi : A \twoheadrightarrow B$ and all congruences α, β on A , $\vec{\psi}[\alpha, \beta] = [\vec{\psi}(\alpha), \vec{\psi}(\beta)]$.

<u>Proof:</u> We need only prove the suffiency of the two properties. Given congruences $\alpha, \beta \in Con(A)$, consider the second projection $\pi_2 : \alpha \longrightarrow A$. Now π_2 has a section ν, given by $\nu(a) := (a, a)$, so on the algebra α we have

$$[Ker\pi_2, \vec{\nu}(\beta)] = Ker\pi_2 \cap \vec{\nu}(\beta),$$

and using property (ii) with $\phi = \pi_1$ we obtain

$$[\vec{\pi}_1(Ker\pi_2), \vec{\pi}_1(\vec{\nu}(\beta))] = \vec{\pi}_1(Ker\pi_2 \cap \vec{\nu}(\beta)).$$

Then

$$\begin{aligned} \vec{\pi}_1(Ker\pi_2) &= con_A(\{(\pi_1(a,b), \pi_1(c,b)) | a\alpha b, c\alpha b\}) \\ &= con_A(\{(a,c) | a\alpha c\}) \\ &= \alpha, \end{aligned}$$

and

$$\vec{\pi}_1(\vec{\nu}(\beta)) = \overrightarrow{(\pi_1 \circ \nu)}(\beta) = \beta.$$

$\square$

Therefore $[\alpha, \beta] = \pi_1(Ker\pi_2 \cap \vec{\nu}(\beta))$, and (i) and (ii) suffice to define the commutator in a congruence modular variety. Note that we did not have to assume congruence modularity to attain that definition. In order to compute the commutator of α and β in an algebra A then, a suitable preimage of A can be chosen where the commutator of (a preimage of) α and (a preimage of) β is calculated as their intersection and the resulting congruence is projected down into A. In the congruence modular case, such a suitable preimage algebra is the congruence α. In [DK], a much larger preimage algebra was considered, a free algebra with a generating set twice the cardinality of A.

We start with an algebra A and introduce three variables x_a, y_a, z_a for each $a \in A$. Define $X := \{x_a | a \in A\}$, Y and Z correspondingly. We consider $F(X+Y)$, $F(Z)$ and the canonical homomorphisms

$$F(X+Y) \xrightarrow{\nabla} F(Z) \xrightarrow{eval} A$$

given by extension from

$$\nabla(x_a) = \nabla(y_a) = z_a$$

and

$$eval(z_a) = a.$$

Given congruences α and β on A there are corresponding equivalence relations $\alpha_X = \{(x_a, x_b) | a\alpha b\}$ on X, and $\beta_X = \{(y_a, y_b) | a\beta b\}$ on Y , whose extensions to $F(X+Y)$ we denote by $\underline{\alpha}_X$ and $\underline{\beta}_X$. α_X and β_X are "cycle-independent" equivalence relations in the notation of [DK]. There it was shown that the commutator of congruences on free algebras, generated by cycle-independent equivalence relations on the generators, is equal to their intersection. For our purposes, it suffices to show directly, that $[\underline{\alpha}_X, \underline{\beta}_X] = \underline{\alpha}_X \cap \underline{\beta}_X$.

Lemma 2.3 *(i)* $[\underline{\alpha}_X, \underline{\beta}_Y] = \underline{\alpha}_X \cap \underline{\beta}_Y$;
(ii) $(eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X) = \alpha$;
(iii) $(eval \circ \nabla)^{\rightarrow}(\underline{\beta}_Y) = \beta$.

From this lemma, the characterization of [DK] follows immediately.

Theorem 2.4 $[\alpha, \beta] = (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \cap \underline{\beta}_Y)$.

Proof of lemma: For (i), consider the canonical $F(X+Y) \twoheadrightarrow F(X/\alpha_X + Y)$. This homomorphism has kernel $\underline{\alpha}_X$ and a section μ which is constructed as follows: For each element e from X/α_X pick a representative $\mu(e)$ from X. Extend μ by setting $\mu(y_a) = y_a$ to a map from $X/\alpha_X + Y$ to $X+Y$, then to a homomorphism $F(X/\alpha_X + Y) \rightarrow F(X+Y)$. Clearly $\vec{\mu}\,(\underline{\beta'}_Y) = \underline{\beta}_Y$, where β' is the congruence on $F(X/\alpha_X + Y)$ generated by β. Now the result follows from 2.1. Parts (ii) and (iii) are rather obvious. $\Box$

Given an element from $F(X+Y)$, i.e. a term $p(\mathbf{x};\mathbf{y})$ with $\mathbf{x}$ and $\mathbf{y}$ denoting sequences of variables from X, resp. Y, we denote by $p^A(\mathbf{x};\mathbf{y})$ its canonical value in A, i.e. $p^A(\mathbf{x};\mathbf{y}) := (eval \circ \nabla)p(\mathbf{x};\mathbf{y})$. Let $\varepsilon := Ker(eval \circ \nabla)$, then we get:

Lemma 2.5 $(a,b) \in [\alpha,\beta]$ *iff there exist terms,* $p(\mathbf{x};\mathbf{y})$ *and* $q(\mathbf{x};\mathbf{y}) \in F(X+Y)$ *such that*

(i) $p(\mathbf{x};\mathbf{y})\ \varepsilon \vee (\underline{\alpha}_X \cap \underline{\beta}_Y)\ q(\mathbf{x};\mathbf{y})$;

(ii) $p^A(\mathbf{x};\mathbf{y}) = a$ *and* $q^A(\mathbf{x};\mathbf{y}) = b$.

Proof: We have

$$\begin{aligned}
[\alpha,\beta] &= (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \cap \underline{\beta}_Y) \\
&= (eval \circ \nabla)^{\rightarrow}(eval \circ \nabla)^{\leftarrow}(eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \cap \underline{\beta}_Y) \\
&= (eval \circ \nabla)^{\rightarrow}(\varepsilon \vee (\underline{\alpha}_X \cap \underline{\beta}_Y)).
\end{aligned}$$

$\Box$

Let $p = p(\mathbf{x};\mathbf{y})$ and $q = q(\mathbf{x};\mathbf{y})$ be elements of $F(X+Y)$. For any $\theta \in Con(A)$, we can define a representative function $\sigma = \sigma_\theta : A \longrightarrow A$ satisfying $\sigma \circ \sigma = \sigma$, $\theta = Ker\sigma$, and $(a,\sigma(a)) \in \theta$ for all $a \in A$. For sequences of variables $\mathbf{x}$ in X and $\mathbf{y}$ in Y, we obtain new sequences $\mathbf{x}/\theta$ and $\mathbf{y}/\theta$ by replacing each x_a and y_a by $x_{\sigma(a)}$ and $y_{\sigma(a)}$ respectively. The relation $p\ \theta_X\ q$ simply says that $p(\mathbf{x}/\theta;\mathbf{y}) = q(\mathbf{x}/\theta;\mathbf{y})$ is an equation in the variety. Similarly, $p\ \theta_Y\ q$ says that $p(\mathbf{x};\mathbf{y}/\theta) = q(\mathbf{x};\mathbf{y}/\theta)$ is an equation in $\mathcal{V}$. Therefore, another way of saying $[\alpha,\beta] = 0$ is the following:

Lemma 2.6 $[\alpha,\beta] = 0$ *if and only if for any two terms* $p(\mathbf{x};\mathbf{y}), q(\mathbf{x};\mathbf{y})$ *with* $p(\mathbf{x}/\alpha;\mathbf{y}) = q(\mathbf{x}/\alpha;\mathbf{y})$ *and* $p(\mathbf{x};\mathbf{y}/\beta) = q(\mathbf{x};\mathbf{y}/\beta)$ *equations in* V*, we may deduce* $p^A = q^A$.

Proof: $[\alpha,\beta] = 0$ if and only if for all $(p,q) \in \underline{\alpha}_X \wedge \underline{\beta}_Y$, $p^A = q^A$. Now apply the above discussion. $\Box$

Corollary 2.7 *Suppose there are terms* p *and* q *such that the equations*

$$p(x,x,y,z) = q(x,x,y,z)$$

and

$$p(x, y, z, z) = q(x, y, z, z)$$

are satisfied, then for $a, b, c, d \in A$,

$$p(a, b, c, d)[con(a, b), con(c, d)]q(a, b, c, d).$$

Proof: Let $\alpha = con(a, b)$, $\beta = con(c, d)$, and consider the terms $p = p(x_a, x_b; y_c, y_d)$ and $q = q(x_a, x_b; y_c, y_d)$ in $F(X + Y)$. Clearly $p\ (\underline{\alpha}_X \wedge \underline{\beta}_Y)\ q$ so $p^A[\alpha, \beta]q^A$. $\Box$

Corollary 2.8 *An algebra* A *is abelian, if for any two terms* p, q *with* $p(x, x, z, u) = q(x, x, z, u)$ *and* $p(x, y, z, z) = q(x, y, z, z)$ *equations in* $V(A)$ *and any* $a, b, c, d \in A$ *we have* $p(a, b, c, d) = q(a, b, c, d)$.

As an example how to work with this characterization, consider the case of a group G with normal subgroups A and B . Let $a \in A$, $b \in B$ and let α be the congruence relation given by A, β the congruence relation given by B and define

$$p(\mathbf{x}; \mathbf{y}) := x_a x_1^{-1} y_1^{-1} y_b$$

and

$$q(\mathbf{x}; \mathbf{y}) := y_1^{-1} y_b x_a x_1^{-1}.$$

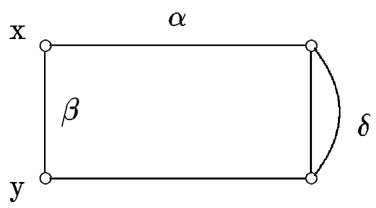
Then $p(\mathbf{x}/\alpha; \mathbf{y}) = y_1^{-1} y_b = q(\mathbf{x}/\alpha; \mathbf{y})$, and $p(\mathbf{x}; \mathbf{y}/\beta) = x_a x_1^{-1} = q(\mathbf{x}; \mathbf{y}/\beta)$; so

$$p^G = ab\ [\alpha, \beta]\ ba = q^G.$$

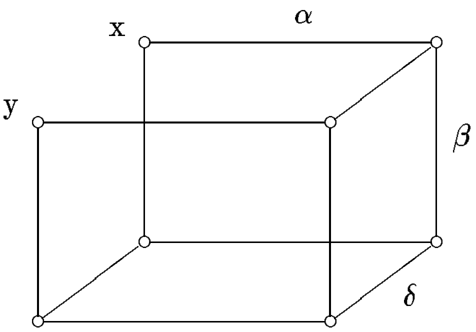
For rings use the terms $p = (x_a - x_0)(y_b - y_0)$ and $q = (y_b - y_0)(x_a - x_0)$.

3 Another characterization of the commutator

Both the shifting lemma and the cube lemma in the situations,



respectively



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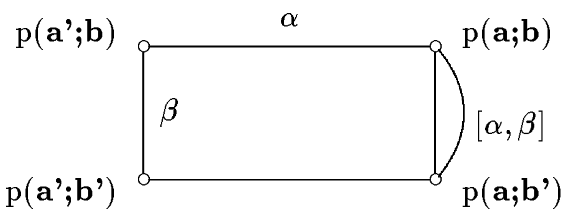
require that $\alpha \wedge \beta \leq \delta$ to throw (x, y) into δ or, equivalently establish that $(x, y) \in \delta \vee (\alpha \wedge \beta)$. In many situations one would rather need to establish the stronger result that $(x, y) \in \delta \vee [\alpha, \beta]$. This can indeed by done provided that the configurations have preimages in $F(X + Y)$, since there the required commutators are given by their meet, i.e. $[\underline{\alpha}_X, \underline{\beta}_Y] = \underline{\alpha}_X \wedge \underline{\beta}_Y$.

We shall look at such configurations and it will turn out that the shifting lemma leads us to the well known "term-condition" and, in the same way, the cube-lemma leads us to a second kind of term condition, involving two terms. Both these term conditions characterize the commutator. A condition very similar to the second one has also been found independently by E. Kiss [K]. He called it the "two-term-condition". His proof is based on a four-variable version of a difference term.

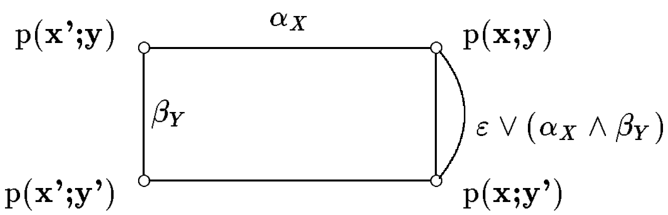
Finally, in this chapter we shall use the "two-term-condition" to strengthen somewhat a result connecting commutators with a homomorphism property of the original 3-variable version of the difference term of [Gu].

Theorem 3.1 (Term-condition (Gu, Fr McK)) *The commutator $[\alpha, \beta]$ is closed under the following condition: If $\mathbf{a}\alpha\mathbf{a}'$, $\mathbf{b}\beta\mathbf{b}'$ and $p(\mathbf{a}; \mathbf{b})[\alpha, \beta]p(\mathbf{a}; \mathbf{b}')$ then $p(\mathbf{a}'; \mathbf{b})[\alpha, \beta]p(\mathbf{a}'; \mathbf{b}')$.*

Proof: The stated relations give



Pulling this diagram back into $F(X+Y)$ we obtain



Thus the shifting lemma yields $p(\mathbf{x}';\mathbf{y}')\ \varepsilon\ \vee(\underline{\alpha}_X \wedge \underline{\beta}_Y) p(\mathbf{x}';\mathbf{y}')$ and, after projecting onto A, we get $p^A(\mathbf{a}';\mathbf{b})\ [\alpha,\beta]\ p^A(\mathbf{a}';\mathbf{b}')$. $\square$

In fact, it is known that the commutator is the *smallest* congruence closed under the above condition. This, however, does not seem to follow trivially from our definition of the commutator.

The cube-lemma may be used in an analogous way for the "two-term-condition". Here, our commutator definition gives us both directions with relative ease:

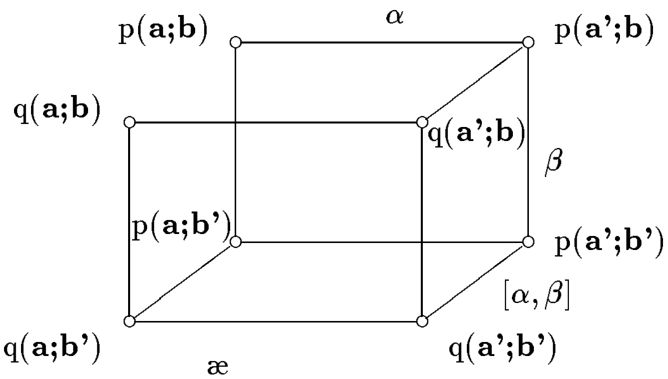
Theorem 3.2 *The commutator $[\alpha,\beta]$ is the smallest congruence relation closed under the following condition for $\mathbf{a}\alpha\mathbf{a}'$ and $\mathbf{b}\beta\mathbf{b}'$:*

If $p(\mathbf{a};\mathbf{b}')\ [\alpha,\beta]\ q(\mathbf{a};\mathbf{b}')$
$p(\mathbf{a}';\mathbf{b})\ [\alpha,\beta]\ q(\mathbf{a}';\mathbf{b})$
$p(\mathbf{a}';\mathbf{b}')\ [\alpha,\beta]\ q(\mathbf{a}';\mathbf{b}')$

then

$p(\mathbf{a};\mathbf{b})\ [\alpha,\beta]\ q(\mathbf{a};\mathbf{b})$.

<u>Proof:</u> To see that the commutator $[\alpha,\beta]$ is closed under the condition, consider the cube



By forming its preimage in $F(X+Y)$ as in the previous theorem, we obtain as before $p(\mathbf{x};\mathbf{y})\ \varepsilon \vee (\underline{\alpha}_X \wedge \underline{\beta}_Y)\ q(\mathbf{x};\mathbf{y})$, which, after projection onto A, yields the desired $p(\mathbf{a};\mathbf{b})\ [\alpha,\beta]\ q(\mathbf{a};\mathbf{b})$.

For the converse, assume $(u,v) \in [\alpha,\beta]$ and let ξ be the smallest congruence relation closed under the given condition. We have to show that $(u,v) \in \xi$.

Since $(u,v) \in [\alpha,\beta]$, by lemma 2.5 there exist $p,q \in F(X+Y)$ with $p^A = u$, $q^A = v$, and $p(\mathbf{x};\mathbf{y})\ \varepsilon \vee (\underline{\alpha}_X \wedge \underline{\beta}_Y) q(\mathbf{x};\mathbf{y})$. Thus there exists $r_0,\ldots,r_n$ with

$$
\begin{array}{ll}
r_0(\mathbf{x};\mathbf{y}) = p(\mathbf{x};\mathbf{y}), & \\
r_n(\mathbf{x};\mathbf{y}) = q(\mathbf{x};\mathbf{y}), & \\
r_i^A = r_{i+1}^A & \text{for } i \text{ even, and} \\
r_i(\mathbf{x};\mathbf{y})(\underline{\alpha}_X \wedge \underline{\beta}_Y) r_{i+1}(\mathbf{x};\mathbf{y}) & \text{for } i \text{ odd.}
\end{array}
$$

Thus, for i odd, the equations

$$r_i(\mathbf{x}/\alpha;\mathbf{y}) = r_{i+1}(\mathbf{x}/\alpha;\mathbf{y})$$
$$r_i(\mathbf{x};\mathbf{y}/\beta) = r_{i+1}(\mathbf{x};\mathbf{y}/\beta)$$

hold in the variety, and, since we only identify more variables,

$r_i(\mathbf{x}/\alpha;\mathbf{y}/\beta) = r_{i+1}(\mathbf{x}/\alpha;\mathbf{y}/\beta)$, as well.

Hence, passing back to A, the assumed condition on ξ implies that for i odd

$r_i^A \xi r_{i+1}^A$,

and thus $u = p^A \xi q^A = v$.

□

Finally we use the last theorem to improve a result of [Gu]. According to [Gu], Lemma 7.1, every congruence modular variety has a term t such that $t(x,y,y) = x$ is an equation in $\mathcal{V}$ and $t(a,a,b)\ [\alpha,\alpha]\ b$, if $a\alpha b$. Using the two-term-condition, we can show:

Theorem 3.3 *Let $x_i\ \alpha\ y_i\ \beta\ z_i$ for $i = 1,\dots,n$ and $[\alpha,\beta] \geq [\beta,\beta]$ then for every n-ary term f we have:*
$f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n))\ [\alpha,\beta]\ t(f(x_1,\dots,x_n),f(y_1,\dots,y_n),f(z_1,\dots,z_n))$

Proof: From the above mentioned properties of t we get the equalities
$f(t(x_1,z_1,z_1),\dots,t(x_n,z_n,z_n)) = t(f(x_1,\dots,x_n),f(z_1,\dots,z_n),f(z_1,\dots,z_n))$,
$f(t(y_1,z_1,z_1),\dots,t(y_n,z_n,z_n)) = t(f(y_1,\dots,y_n),f(z_1,\dots,z_n),f(z_1,\dots,z_n))$,
and
$f(t(y_1,y_1,z_1),\dots,t(y_n,y_n,z_n))\ [\beta,\beta]\ t(f(y_1,\dots,y_n),f(y_1,\dots,y_n),f(z_1,\dots,z_n))$.
Thus, since $[\beta,\beta] \leq [\alpha,\beta]$, theorem 3.2 yields:
$f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n))\ [\alpha,\beta]\ t(f(x_1,\dots,x_n),f(y_1,\dots,y_n),f(z_1,\dots,z_n))$.
$\square$

This improves a previous result [Gu], Theorem 9.1, where $\alpha \geq \beta$ had to be assumed instead of $[\alpha,\beta] \geq [\beta,\beta]$.

4 A commutator for arbitrary varieties.

We may try to use 2.4 to define the commutator in an arbitrary variety, i.e. define $[\alpha,\beta] := (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y)$. This approach has certain advantages, since it may be formulated category theoretically. This makes certain properties easily derived from "abstract nonsense":

1) $[\alpha,\beta] \leq \alpha \wedge \beta$, (sub-meet);

2) $\alpha \leq \alpha' \Rightarrow [\alpha,\beta] \leq [\alpha',\beta]$, (monotonicity);

3) $[\alpha,\beta] = [\beta,\alpha]$, (symmetry);

for any homomorphism $\phi : A \longrightarrow B, \alpha,\beta \in Con(A)$, and $\theta,\psi \in Con(B)$

4) $\overrightarrow{\phi}[\alpha,\beta] \leq [\overrightarrow{\phi}\alpha, \overrightarrow{\phi}\beta]$, (direct continuity);

which is equivalent under 1) through 3) to

5) $[\overleftarrow{\phi}\theta, \overleftarrow{\phi}\psi] \leq \overleftarrow{\phi}[\theta,\psi]$, (inverse continuity).

Join-distributivity and full compatibility with epimorphisms do not seem to hold in the general non-modular case, although we did not come up with any counterexamples. A weaker epimorphic condition does hold though, and it together with join-distributivity would supply full compatibility with epimorphisms.

Lemma 4.1 *For epimorphism $\phi : A \twoheadrightarrow B$, and congruences $\alpha, \beta \geq Ker\phi$, $\vec{\phi}[\alpha, \beta] = [\vec{\phi}\alpha, \vec{\phi}\beta]$.*

<u>Proof:</u> Let T, U, and V be the required variable sets isomorphic to B, and define $\theta := \vec{\phi}\,\alpha$, and $\psi := \vec{\phi}\,\beta$ in $Con(B).\phi$ induces natural epimorphisms,

$$m : F(Z) \twoheadrightarrow F(V) \;\text{ and }\; n : F(X+Y) \twoheadrightarrow F(T+U),$$

defined as extensions of $z_a \to v_{\phi a}$, $x_a \to t_{\phi a}$, and $y_a \to u_{\phi a}$. Since ϕ is a surjective set function, there exists a set-theoretical section $\mu : B \longrightarrow A$ with $\phi \circ \mu = id_B$. This set-theoretical section produces natural algebraic sections for these natural epimorphisms and the following commutative diagram.

$$\begin{tikzcd}
F(X+X) \arrow[r, shift left, "n"] \arrow[d, "\nabla_A"'] & F(T+U) \arrow[l, shift left, "v"] \arrow[d, "\nabla_B"] \\
F(Z) \arrow[r, shift left, "m"] \arrow[d, "eval_A"'] & F(V) \arrow[l, shift left, "u"] \arrow[d, "eval_B"] \\
A \arrow[r, "\phi"] & B
\end{tikzcd}$$

æ

Since $Ker\phi \leq \alpha$ and β, we obtain $\alpha = \overleftrightarrow{\phi\phi}\,\alpha = \overleftarrow{\phi}\,\theta$ and $\beta = \overleftarrow{\phi}\,\psi$. The naturality of our new homomorphisms then produces $\vec{n}\underline{\alpha}_X = \underline{\theta}_T$, $\vec{n}\underline{\beta}_Y = \underline{\psi}_U$, $\vec{v}\underline{\theta}_T \leq \underline{\alpha}_X$, and $\vec{v}\underline{\psi}_U \leq \underline{\beta}_Y$. Therefore,

$$\underline{\theta}_T \wedge \underline{\psi}_U \;=\; (n \circ v)(\underline{\theta}_T \wedge \underline{\psi}_U)$$

$$
\begin{aligned}
&\leq \vec{n}(\vec{v}(\underline{\theta}_T) \wedge \vec{v}(\underline{\psi}_U)) \\
&\leq \vec{n}(\underline{\alpha}_X \wedge \underline{\beta}_X) \\
&\leq \vec{n}(\underline{\alpha}_X) \wedge \vec{n}(\underline{\beta}_Y) \\
&= \underline{\theta}_T \wedge \underline{\psi}_U, \text{ and finally,}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\phi}[\alpha,\beta] &= (\phi \circ eval_A \circ \nabla_A)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y) \\
&= (eval_B \circ \nabla_B \circ n)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y) \\
&= (eval_B \circ \nabla_B)^{\rightarrow}(\underline{\theta}_T \wedge \underline{\psi}_U) \\
&= [\vec{\phi}\alpha, \vec{\phi}\beta].
\end{aligned}
$$

□

Corollary 4.2 *If the commutator also satisfies the join-distributivity condition*

$$[\alpha, \beta \vee \gamma] = [\alpha, \beta] \vee [\alpha, \gamma],$$

then for any epimorphism $\phi : A \twoheadrightarrow B$ *and* $\alpha, \beta \in Con(A)$

$$\vec{\phi}[\alpha,\beta] = [\vec{\phi}\alpha, \vec{\phi}\beta].$$

Proof: If $\theta = Ker\phi$, then by join-distributivity we get that

$$[\theta \vee \alpha, \theta \vee \beta] = [\theta, \theta \vee \beta] \vee [\alpha, \theta \vee \beta] = [\theta, \theta] \vee [\theta, \beta] \vee [\alpha, \theta] \vee [\alpha, \beta].$$

Joining with θ produces

$$\theta \vee [\theta \vee \alpha, \theta \vee \beta] = \theta \vee [\alpha, \beta],$$

and applying $\vec{\phi}$ gives us

$$\vec{\phi}[\theta \vee \alpha, \theta \vee \beta] = \vec{\phi}(\theta \vee [\theta \vee \alpha, \theta \vee \beta]) = \vec{\phi}(\theta \vee [\alpha, \beta]) = \vec{\phi}[\alpha, \beta].$$

4.1 then produces the desired result. □

Theorem 4.3 *Let* X *and* Y *be disjoint sets,* $\alpha \in Eq(X)$ *and* $\beta \in Eq(Y)$. *Then in* $F(X+Y)$ *we have :*

$$[\underline{\alpha}_X, \underline{\beta}_Y] = \underline{\alpha}_X \cap \underline{\beta}_Y.$$

Proof: Suppose $(p,q) \in \underline{\alpha}_X \cap \underline{\beta}_Y$, then $p(\mathbf{x}/\alpha;\mathbf{y}) = q(\mathbf{x}/\alpha;\mathbf{y})$ and $p(\mathbf{x};\mathbf{y}/\beta) = q(\mathbf{x};\mathbf{y}/\beta)$ are equations in V. Let $A = F(X^1 + Y^1)$ and $B = F(X^2+Y^2)$ be two disjoint copies of $F(X+Y)$. Let $\alpha^1, \alpha^2, \beta^1, \beta^2$ be the equivalence relations on X^1, X^2, Y^1, Y^2 corresponding to α and β, and let similarly $(\underline{\alpha}_X)^1$ and $(\underline{\beta}_Y)^2$ be the equivalences on A and B corresponding to $\underline{\alpha}_X$ and to $\underline{\beta}_Y$. In $F(A+B)$ we must now consider the congruence relations $\underline{\alpha}_A$ and $\underline{\beta}_B$ generated by $(\underline{\alpha}_X)^1$ and $(\underline{\beta}_Y)^2$. Next we modify $p(\mathbf{x},\mathbf{y})$ and $q(\mathbf{x},\mathbf{y})$ by replacing the $\mathbf{x}$'s and the $\mathbf{y}$'s by the corresponding variables from X^1 and Y^2, yielding terms $\hat{p} = p(\mathbf{x}^1;\mathbf{y}^2)$ and $\hat{q} = q(\mathbf{x}^1;\mathbf{y}^2)$. We now have that $p(\mathbf{x}^1/\alpha^1;\mathbf{y}^2) = q(\mathbf{x}^1/\alpha^1;\mathbf{y}^2)$ and $p(\mathbf{x}^1;\mathbf{y}^2/\beta^2) = q(\mathbf{x}^1;\mathbf{y}^2/\beta^2)$ hence $\hat{p}\ \underline{\alpha}_A \cap \underline{\beta}_B\ \hat{q}$. Finally, since $(eval \circ \nabla)\hat{p} = p$ and $(eval \circ \nabla)\hat{q} = q$, it follows that $(p,q) \in (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_A \cap \underline{\beta}_B) = [\underline{\alpha}_X, \underline{\beta}_Y]$. □

Corollary 4.4 *For any set U and congruence relations, α and β, on $F(U)$, which are generated by equivalence relations on U, we have*

$$[\alpha,\beta] = \Delta^{\rightarrow}(\vec{\mu_1}\alpha \wedge \vec{\mu_2}\beta)$$

where

$$\Delta : F(U) + F(U) \longrightarrow F(U)$$

is the codiagonal and

$$\mu_i : F(U) \longrightarrow F(U) + F(U),\ \mathrm{i}\ =\ 1,2$$

are the canonical injections.

Proof: We now have $A := F(U)$, X, Y, and Z set-isomorphic to A, and the following commutative diagram,

$$\begin{tikzcd} F(X+Y) \arrow[r, "eval + eval"] \arrow[d, "\nabla"'] & F(U)+F(U) \arrow[d, "\Delta"] \\ F(Z) \arrow[r, "eval"] & F(U) \end{tikzcd}$$

where $(eval + eval)$ takes X and Y back to $F(U)$.

Now

$$
\begin{aligned}
[\alpha,\beta] &= (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y) \\
&= (\Delta \circ (eval + eval))^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y) \\
&\leq \Delta^{\rightarrow}((eval + eval)^{\rightarrow}(\underline{\alpha}_X) \wedge (eval + eval)^{\rightarrow}(\underline{\beta}_Y)) \\
&= \Delta^{\rightarrow}(\vec{\mu_1}\,\alpha \wedge \vec{\mu_2}\beta) \\
&= \Delta^{\rightarrow}[\vec{\mu_1}\,\alpha, \vec{\mu_2}\beta], \text{ by the theorem} \\
&\leq [\Delta^{\rightarrow}\,\vec{\mu_1}\,\alpha, \Delta^{\rightarrow}\,\vec{\mu_2}\beta], \text{ by continuity} \\
&= [\alpha,\beta].
\end{aligned}
$$

□

Corollary 4.5 *Join-distributivity is equivalent to the following condition on congruences on $F(X+Y)$:*
For all equivalence relations α on X, and β,γ on Y,

$$\underline{\alpha}_X \wedge (\underline{\beta}_Y \vee \underline{\gamma}_Y) \leq (\underline{\alpha}_X \wedge \underline{\beta}_Y) \vee (\underline{\alpha}_X \wedge \underline{\gamma}_Y).$$

<u>Proof:</u> By the theorem, $\underline{\alpha}_X \wedge (\underline{\beta}_Y \vee \underline{\gamma}_Y)$, $(\underline{\alpha}_X \wedge \underline{\beta}_Y)$, and $(\underline{\alpha}_X \wedge \underline{\gamma}_Y)$ are the commutators of their respective congruences in $F(X+Y)$. Thus the condition is necessary. Conversely, if it holds, we have for any A in the variety and any $\alpha,\beta,\gamma \in Con(A)$,

$$
\begin{aligned}
[\alpha,\beta \vee \gamma] &= (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \wedge (\underline{\beta}_Y \vee \underline{\gamma}_Y)) \\
&\leq (eval \circ \nabla)^{\rightarrow}((\underline{\alpha}_X \wedge \underline{\beta}_Y) \vee (\underline{\alpha}_X \wedge \underline{\gamma}_Y)) \\
&= (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\beta}_Y) \vee (eval \circ \nabla)^{\rightarrow}(\underline{\alpha}_X \wedge \underline{\gamma}_Y) \\
&= [\alpha,\beta] \vee [\alpha,\gamma].
\end{aligned}
$$

□

This join-distributivity condition can be expressed as a Mal'cev type implication, thus we think that it is false in general.

Corollary 4.6 *If the variety is congruence modular, then the commutator is join-distributive.*

Proof: In a modular lattice, any distributive equality between three variables implies that these variables generate a distributive sublattice. It is an easy calculation to see that for any disjoint sets, X and Y, and any equivalence relations, $\alpha \in Eq(X)$ and $\beta, \gamma \in Eq(Y)$, in $Con(F(X+Y))$:

$$\underline{\beta}_Y \wedge (\underline{\alpha}_X \vee \underline{\gamma}_Y) = \underline{\beta}_Y \wedge (\underline{\alpha}_X \vee (\underline{\beta}_Y \wedge \underline{\gamma}_Y)).$$

□

The extreme cases: $[\alpha, \beta] = 0$ and $[\alpha, \beta] = \alpha \wedge \beta$ can be characterized though.

Theorem 4.7 *The following are equivalent for a variety, $\mathcal{V}$:*

(i) *$\mathcal{V}$ satisfies $[\alpha, \beta] = \alpha \cap \beta$;*
(ii) *$\mathcal{V}$ satisfies $[\alpha, \alpha] = \alpha$;*
(iii) *There are quaternary terms $p_0, \ldots, p_n$ such that $\mathcal{V}$ satisfies:*

$$\begin{array}{ll} x = p_0(x,y,z,u), \text{ and } p_n(x,y,z,u) = y; & \\ p_i(x,y,x,y) = p_{i+1}(x,y,x,y), & \text{for } i \text{ odd;} \\ p_i(x,x,y,z) = p_{i+1}(x,x,y,z) & \text{and} \\ p_i(x,y,z,z) = p_{i+1}(x,y,z,z), & \text{for } i \text{ even.} \end{array}$$

Proof: (i) $\Leftrightarrow$ (ii) is obvious, since $[\alpha \wedge \beta, \alpha \wedge \beta] \leq [\alpha, \beta] \leq \alpha \wedge \beta$.

Assume (ii). Then in the two-generated free algebra, $A := F(a, b)$ we have by 4.4:

$$(a,b) \in [con_A(a,b), con_A(a,b)] = \nabla^{\rightarrow}(con_B(x,y) \wedge con_B(z,u))$$

where $B := F(x,y,z,u)$, and ∇ is given by the extension of $\nabla x = \nabla z = a$ and $\nabla y = \nabla u = b$. If $\varepsilon := Ker\nabla = con_B(x,z) \vee con_B(y,u)$, then we have as in 2.5:

$$(a,b) \in [con_A(a,b), con_A(a,b)] \text{ iff } (x,y) \in (\varepsilon \vee (con_B(x,y) \wedge con_B(z,u))).$$

From this, (iii) easily follows.

Conversely, assume there are quaternary terms as in (iii). Then for any $A \in V$, any $\theta \in Con(A)$, and any $(a,b) \in \theta$, we have in $F(X+Y)$, the terms $p_i(x_a, x_b, y_a, y_b)$ for $i = 1, \ldots, n$. The assumed properties of these terms give us

$$p_i \varepsilon p_{i+1} \qquad \text{for i even}$$
$$p_i(\theta_X \wedge \theta_Y) p_{i+1} \qquad \text{for i odd.}$$

Therefore $(a, b) \in [\theta, \theta]$ by 2.5. $\square$

Corollary 4.8 *A congruence semidistributive variety satisfies $[\theta, \theta] = \theta$.*

<u>Proof:</u> Note that the quaternary terms in 4.3 express the fact that, in $F(x, y, z, u)$, the congruences $\alpha = con(x, z) \vee con(y, u)$, $\beta = con(x, y)$, $\gamma = con(z, u)$ satisfy

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \leq \alpha \vee (\beta \wedge \gamma),$$

which is also implied by semidistributivity, since $\alpha \vee \beta = \alpha \vee \gamma$. $\square$

Corollary 4.9 *If $[\theta, \theta] = \theta$ then there exist ternary terms $t_0, \ldots, t_n$ with*

$$\begin{aligned}
&x = t_0(x, y, z) \text{ and } t_n(x, y, z) = y; \\
&t_i(x, y, x) = t_{i+1}(x, y, x), \text{ for } i \text{ odd}; \\
&t_i(x, x, y) = t_{i+1}(x, x, y), \text{ and} \\
&t_i(x, y, y) = t_{i+1}(x, y, y) \quad \text{for } i \text{ even.}
\end{aligned}$$

<u>Proof:</u> Just identify the second and fourth variables. $\square$

Note that this Mal'cev condition is also used in [HM] to characterize congruence-semidistributivity in locally finite varieties.

Theorem 4.10 *A variety satisfies $[\mu, \nu] = 0$ if and only if for any two disjoint sets, X and Y, any equivalence relations, α on X and β on Y, and any two terms $p(\mathbf{x}; \mathbf{y})$ and $q(\mathbf{x}; \mathbf{y})$ in $F(X + Y)$:*

if

$$p(\mathbf{x}/\alpha; \mathbf{y}) = q(\mathbf{x}/\alpha; \mathbf{y}) \text{ and } p(\mathbf{x}; \mathbf{y}/\beta) = q(\mathbf{x}; \mathbf{y}/\beta)$$

then

$$p(\mathbf{x}; \mathbf{y}) = q(\mathbf{x}; \mathbf{y}).$$

Corollary 4.11 *The following varieties satisfy $[\mu, \nu] = 0$:*

0) Sets;

1) Semigroups;

2) Rectangular bands [i.e. (xy)z = x(yz) = xz];

3) Semilattices;

4) any absolutely free variety.

One should note that this "commutator" is not invariant with respect to subvarieties: $A \in W \leq V$, and $\alpha, \beta \in Con(A)$ imply $[\alpha,\beta]_W \geq [\alpha,\beta]_V$. Indeed, the largest binary operation defined on $Con(A)$ for each A in $\mathcal{V}$ that is submeet and satisfies the full epimorphism condition is clearly varietal dependent. In the same vein as 4.5 one can show that invariance with respect to subvarieties is equivalent to the following condition on congruences on $F(X+Y)$ with $X = Y$:

For all equivalence relations $\alpha, \beta \in Eq(X)$ and fully-invariant congruences, $\Sigma \in Con(F(X+X))$,

$$(\Sigma \vee \underline{\alpha}_X) \wedge (\Sigma \vee \underline{\beta}_Y) = \Sigma \vee (\underline{\alpha}_X \wedge \underline{\beta}_Y).$$

It is easy to see that we always have $\underline{\alpha}_X \vee (\Sigma \wedge \underline{\beta}_Y) \geq \Sigma \wedge (\underline{\alpha}_X \vee \underline{\beta}_Y)$, thus, a similar argument as in 4.6 shows that the commutator in a congruence modular variety is invariant with respect to subvarieties.

Epilog: The "real" commutator for an arbitrary variety of algebras still lacks an algebraic description. The full epimorphism property implies that it is smaller than ours; how much smaller is an open problem.

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