Toeplitz C*-algebras over non-convex cones and pseudo-symmetric spaces

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0 Introduction

Toeplitz operators and Toeplitz C*-algebras for multi-variable domains of holomorphy have been studied extensively in the literature, notably for tube domains [MR], symmetric domains [U1] and Reinhardt domains [SSU], [CM]. In [U2], a systematic framework for studying Toeplitz C*-algebras for non-smooth domains has beed developed based on the notion of *polar decomposition* with respect to a group action. The above mentioned cases arise for euclidean vector groups (tube domains), compact tori (Reinhardt domains) and non-commutative compact Lie groups (symmetric domains). However, due to advances in representation theory (discrete series of semi-simple Lie groups, non-Riemannian symmetric spaces, Gelfand-Gindikin program [GG]) it is important to allow more general groups (neither compact nor commutative) in the polar decomposition and also to treat non-pseudoconvex domains for which the relevant "Hardy spaces" consist of $\bar{\partial}$-closed cohomology classes [G1].

The purpose of this paper is to provide a general framework for studying Toeplitz C*-algebras over "S-bicircular domains" Ω where S is a Lie group and Ω may be pseudoconcave. The basic idea is to relate the structure theory for the associated Toeplitz C*-algebra $\mathcal{T}(S)$ to the fundamental results of non-commutative C*-algebra duality (coactions and cocrossed products [LPRS]). As an application, we then determine the ideal structure of $\mathcal{T}(S)$ explicitly for an important example of a non-convex tube domain, namely the symmetric 2×2-matrices of negative determinant (this is the exterior of the double light cone in 3 dimensions). In a subsequent paper this analysis will be generalized to cover arbitrary pseudoconcave domains defined for semi-simple Jordan algebras, and also for non-commutative groups such as $SL(2, \mathbf{R})$. In view of this we will formulate many Propositions for a general Lie group S, although in this paper only the case of euclidean vector groups is considered in depth.

1 Tube domains and quasi-symmetric embedding

Let $\underline{S}$ be the Lie algebra and exp the exponential map of a Lie group S. For $s \in S$ let $Int(s)$ be the mapping $x \longmapsto sxs^{-1}$ and put $Ad(s) := T_e\,(Int\,(s))$. A

[1] Supported by NSF

cone $V \subset i\underline{S}$ is called invariant if for all $s \in S$

$$Ad(s)(V) = V.$$

By general Lie theory it follows that

$$Int(s)(\exp(V)) = \exp(V) \tag{1.1}$$

for all $s \in S$. The *tube domain* $S \odot V$ corresponding to S and V is defined by

$$S \odot V := S \cdot \exp(V). \tag{1.2}$$

This is a subset of the complexification of S. In the commutative *vector group* case the Lie group is denoted by

$$S = \underline{S} = iX, \tag{1.3}$$

where X is a euclidean space $\mathbf{R}^n$. Then the invariance condition is fulfilled for every cone $V \subset i\underline{S} = X$, and we have for such cones

$$S \odot V = iX \oplus V. \tag{1.4}$$

Example. For the Jordan algebra

$$X := Sym(n, \mathbf{R}) := \{A \in M(n, \mathbf{R}) \mid A^T = A\}$$

of symmetric matrices there exist for $k = 0, ..., n$ the open cones

$$V_k := \{A \in Sym(n, \mathbf{R}) \ invertible \mid A \text{ has exactly k positive eigenvalues}\}.$$

These cones satisfy

$$\bigcup_{k=0,\dots,n}^{\cdot} V_k \underset{dense}{\subset} Sym(n, \mathbf{R}). \tag{1.5}$$

One sees at once that only V_0 and V_n are convex cones. By [AG] and [K1] we have the following:

Lemma 1.1 *The cones in $Sym(n, \mathbf{R})$ are (right) homogeneous spaces*

$$\begin{aligned} V_0 &= SO(0,n) \setminus Gl^+(n, \mathbf{R}), \\ V_k &= SO(k, n-k) \setminus Gl^+(n, \mathbf{R}), \ k = 1, ..., n-1, \\ V_n &= SO(n,0) \setminus Gl^+(n, \mathbf{R}) \end{aligned}$$

and for the corresponding tube domains we have

$$\begin{aligned} iX \oplus V_0 &= U(0,n) \setminus Sp(2n, \mathbf{R}), \\ iX \oplus V_k &\underset{dense}{\subset} U(k, n-k) \setminus Sp(2n, \mathbf{R}), \ k = 1, ..., n-1, \\ iX \oplus V_n &= U(n,0) \setminus Sp(2n, \mathbf{R}). \end{aligned}$$

Via suitable involutions the spaces $U(k,n-k)\backslash Sp(2n,\mathbf{R})$ are quasi-symmetric and even symmetric in case $k=0,n$. Thus we obtain the *quasi-symmetric embedding* of $iX \oplus V_k$ for $k \neq 0,n$ [AG].

Later we will treat in detail the cone given by the exterior of the *double light cone,* which arises when $n=2$ and $k=1$. So we summarize the results for that special case:

Example 1.2 For $n=2$ we have:

$$
\begin{array}{rclrcl}
i\mathbf{R}^3 \oplus V_{--} & = & U(0,2)\backslash Sp(4,\mathbf{R}); & V_{--} & = & SO(0,2)\backslash Gl^+(2,\mathbf{R}), \\
i\mathbf{R}^3 \oplus V_{+-} & \underset{dense}{\subset} & U(1,1)\backslash Sp(4,\mathbf{R}); & V_{+-} & = & SO(1,1)\backslash Gl^+(2,\mathbf{R}), \\
i\mathbf{R}^3 \oplus V_{++} & = & U(2,0)\backslash Sp(4,\mathbf{R}); & V_{++} & = & SO(2,0)\backslash Gl^+(2,\mathbf{R}).
\end{array}
$$

2 Hardy spaces and Gindikin cohomology

Usually one defines for a locally compact unimodular group S with a Haar measure λ and a convex invariant cone V the *Hardy space* of the tube domain $S \odot V$ as

$$H^2(S\odot V) := \left\{ h : S \odot V \to \mathbf{C} \ holomorphic \ \mid \ \|h\|^2 := \sup_{z \in S\odot V} \int\limits_S |h(zs)|^2 ds < \infty \right\}.$$

For a nonconvex cone in $\mathbf{R}^n$, Gindikin modified this concept in [G1] to obtain the cohomology space $H_2^{(d)}(S \odot V)$, where d is a concavity index for V. To obtain the Fourier transform of this space, define the *unitary dual* $\hat{S}$ of S as the set of all irreducible continuous representations of S. In the vector group case $S = iX$ we have $\widehat{(iX)} = X^{\#}$ (linear dual) and the dual cone is do define in the sense of duality introduced by [G1]. By Fourier theory [SW] we get an isometrical isomorphism

$$\mathcal{F} : \quad L^2(iX) \xrightarrow{\approx} L^2(X^{\#}), \tag{2.1}$$

given by

$$(\mathcal{F}h)(\xi) = \hat{h}(\xi) := c \int_X e^{i\xi x} h(ix) dx \tag{2.2}$$

for all $h \in L^2(iX)$ and $\xi \in X^{\#}$, where c is a constant. For the closed cone

$$\Lambda := \hat{V}$$

the multiplication operator χ_Λ on $L^2(X^\#)$ maps onto the closed subspace $L^2(\Lambda)$. Since $\|\chi_\Lambda(\phi)\| \leq \|\phi\|$, $(\chi_\Lambda)^* = \chi_\Lambda$ and $\chi_\Lambda\chi_\Lambda = \chi_\Lambda$, χ_Λ is a projection of $L^2(X^\#)$ onto $L^2(\Lambda)$. Because of the isometrical property of $\mathcal{F}$ the space

$$H^2_\Lambda(iX) := \mathcal{F}^{-1}\left(L^2(\Lambda)\right) = \left\{h \in L^2(iX) \ \mid \hat{h}(\xi) = 0 \text{ for almost all } \xi \in X^\#\backslash\Lambda\right\}$$

is a closed subspace of $L^2(iX)$. Putting $E = \widehat{\chi_\Lambda}$ the following diagram commutes

$$\begin{array}{cccccc} & L^2(iX) & \xrightarrow{\mathcal{F}} & L^2(X^\#) & & \\ l_E & \downarrow & & \downarrow & \chi_\Lambda & \\ & H^2_\Lambda(iX) & \xrightarrow{\mathcal{F}} & L^2(\Lambda) & , & \end{array} \tag{2.3}$$

where l_E is left convolution. For a convex cone it is well known, that taking the limit function h defined by

$$h(s) := \lim_{S\odot V\ni z\to e} h(s\cdot z), \tag{2.4}$$

gives an isomorphism of $H^2(S\odot\Lambda)$ to $H^2_\Lambda(S)$ [O1]. For nonconvex cones $V \subset X = \mathbf{R}^n$ we have by [G1, Theorem 1] the following description of the Hardy space

$$H_2^{(d)}(iX \oplus V) \cong H^2_\Lambda(iX) \tag{2.5}$$

using the Fourier transform (2.1) and a suitable boundary evaluation.

3 C^*-algebras, coactions and cocrossed products

We will now apply the methods of "non-commutative C^*-duality" [LPRS] to the construction of representations of Toeplitz algebras. We will use the symbol f not only for a function in $L^\infty(S)$ but also for the multiplication operator in $\mathcal{L}\left(L^2(S)\right)$ (bounded operators) defined by f.

Definition 3.1 The (spatial) group C^*-algebra of S is defined as

$$C^*(S) := C^*\left(l_f \mid \ f \in L^1(S)\right) \subset \mathcal{L}\left(L^2(S)\right),$$

where l_f is the left convolution operator defined by $f \in L^1(S)$. For an element $u \in M^1(S)$, the algebra of bounded complex measures on S (the linear dual of $\mathcal{C}_0(S)$ [D2, 1.38]), we define the left convolution operator on $L^2(S)$ by

$$l_u(h)(s) := \int_S u(dx)h(x^{-1}s), \ s \in S.$$

Then the group W^*-algebra of S is

$$W^*(S) := W^*\left(l_u \mid \ u \in M^1(S)\right) \subset \mathcal{L}\left(L^2(S)\right).$$

As a bounded left convolution operator, l_E commutes with all right translations, denoted by γ_s, so [S1, p. 261] yields

$$l_E \in W^*(S). \tag{3.1}$$

For any Banach algebra A there is the following contractive action of A on its linear dual space $A^{\#}$:

$$A^{\#} \otimes A \longrightarrow A^{\#}, \ (\alpha, a) \longmapsto \alpha \cdot a \tag{3.2}$$

defined for all $b \in A$ by

$$\langle \alpha \cdot a, b\rangle := \langle \alpha, ab\rangle.$$

An easy consequence is the continuous action of the *Fourier algebra* $A(S)$ on $W^*(S)$ in the above sense, since by [E1] $A(S)$ is the predual of $W^*(S)$, i.e., $A(S)^{\#} = W^*(S)$, via the pairing

$$\langle l_u, f\rangle := \int_S u(ds) f(s). \tag{3.3}$$

In that case we have for $f \in A(S)$ and $u \in M^1(S)$

$$l_u \cdot f = l_{uf} \tag{3.4}$$

and obtain by contractivity

$$||l_u \cdot f||_{W^*(S)} = ||l_{uf}||_{W^*(S)} \leq ||f||_{A(S)} \cdot ||l_u||.$$

Define the C^*-algebra

$$\begin{aligned} C^*_\Lambda(S) &:= C^*\left(l_E \cdot f \mid f \in A(S)\right) \\ &= C^*\left(l_{fE} \mid f \in A(S)\right) \subset W^*(S). \end{aligned} \tag{3.5}$$

In case S is not compact, the constant function $\mathbf{1}$ is not contained in $A(S) \subset \mathcal{C}_0(S)$ and $l_E \notin C^*_\Lambda(S)$. Specializing to the commutative case $S = iX$, we have [E1, p. 12]

$$A(iX) = \left\{\hat{\phi} \mid \phi \in L^1(X^{\#})\right\}, \tag{3.6}$$

and

$$\|\hat{\phi}\|_{A(iX)} = \|\phi\|_{L^1(X^\#)}.$$

Via the calculation

$$\begin{aligned} Ad(\mathcal{F})(C^*_\Lambda(iX)) &= Ad(\mathcal{F})C^*\left(l_{fE} \mid f \in A(iX)\right) \\ &= C^*\left(l_{\hat{f}}(\hat{E}) \mid f \in A(iX)\right) \overset{3.6}{=} C^*\left(l_{\phi}(\hat{E}) \mid \phi \in L^1(X^\#)\right) \\ &= C^*\left(\phi \star \chi_\Lambda \mid \phi \in L^1(X^\#)\right) =: \mathcal{C}_0^\Lambda(X^\#) \end{aligned}$$

we can pass to the dual picture, and obtain

Lemma 3.2

$$\mathcal{C}_0^\Lambda(X^\#) \subset \mathcal{U}_b(X^\#) := \left\{\phi : X^\# \to \mathbf{C} \mid \phi \textit{ uniformly continuous and bounded}\right\}.$$

Proof. Let $\psi = \phi \star \chi_\Lambda \in \mathcal{C}_0^\Lambda(X^\#)$ and let (ϕ_n) be a sequence in $\mathcal{C}_c(X^\#)$ with $\phi_n \to \phi$ in the L^1-norm. Then Young's inequality gives

$$\|\phi_n \star \chi_\Lambda - \phi \star \chi_\Lambda\|_\infty \leq \|\phi_n - \phi\|_1 \cdot \|\chi_\Lambda\|_\infty = \|\phi_n - \phi\|_1 \to 0, \tag{3.7}$$

which proves that

$$\phi_n \star \chi_\Lambda =: \psi_n \to \psi \text{ in the } L^\infty - \text{norm.} \tag{3.8}$$

Now we calculate for $\xi, \eta \in X^\#$:

$$(\phi_n \star \chi_\Lambda)(\xi) = \int\limits_{X^\#} \phi_n(\xi - \tau)\chi_\Lambda(\tau)d\tau = \int\limits_{\Lambda} \phi_n(\xi - \tau)d\tau = \int\limits_{\xi - \Lambda} \phi_n(\tau)d\tau, \tag{3.9}$$

hence

$$\begin{aligned} |\psi_n(\xi) - \psi_n(\eta)| &= \Big|\int\limits_{\xi-\Lambda} \phi_n(\tau)d\tau - \int\limits_{\eta-\Lambda} \phi_n(\tau)d\tau\Big| = \Big|\int\limits_{(\xi-\Lambda)\triangle(\eta-\Lambda)} \phi(\tau)d\tau\Big| \\ &\leq \|\phi_n\|_\infty \cdot |K_n \cap ((\xi - \Lambda) \triangle (\eta - \Lambda))|, \end{aligned}$$

where K_n is the compact support of ϕ_n, $\triangle$ denotes the symmetric difference and $|\cdot|$ denotes the measure. Since

$$K_n \cap ((\xi - \Lambda) \triangle (\eta - \Lambda)) \to \emptyset \text{ for } \|\xi - \eta\| \to 0$$

the ψ_n are uniformly continuous. The boundedness is clear by Young's inequality. Since $\mathcal{U}_b(X^\#)$ is closed under uniform convergence we have proved the assertion.

As a consequence there is a continuous action of $X^\#$ on $\mathcal{C}_0^\Lambda(X^\#)$ by right translations given by

$$X^\# \times \mathcal{C}_0^\Lambda(X^\#) \longrightarrow \mathcal{C}_0^\Lambda(X^\#),\ \xi \propto \psi(\eta) := \psi(\eta + \xi). \tag{3.10}$$

This action can be described by the embedding

$$\rho : \mathcal{C}_0^\Lambda(X^\#) \longrightarrow \tilde{M}(\mathcal{C}_0^\Lambda(X^\#) \otimes \mathcal{C}_0(X^\#)) = \mathcal{C}_b\left(X^\#, \mathcal{C}_0^\Lambda(X^\#)\right) \tag{3.11}$$

where

$$\rho(\psi)(\xi) = \xi \propto \psi.$$

Here $\otimes$ is the spatial C^*-tensor product and $\tilde{M}$ denotes the (restricted) multiplier algebra [LPRS]. Because

$$Ad(\mathcal{F}^*) : \mathcal{C}_0^\Lambda(X^\#) \cong C_\Lambda^*(iX)$$

there is also an action

$$\hat{\rho} : C_\Lambda^*(iX) \longrightarrow \tilde{M}(C_\Lambda^*(iX) \otimes \mathcal{C}_0(X^\#)).$$

By [LPRS, 2.2.(4)] this is equivalent to the existence of a *coaction* of iX on $C_\Lambda^*(iX)$, given as the mapping

$$\delta : C_\Lambda^*(iX) \longrightarrow \tilde{M}(C_\Lambda^*(iX) \otimes C^*(iX))$$

defined by

$$\delta(a) = (1 \otimes \mathcal{F}^*)\hat{\rho}(a)(1 \otimes \mathcal{F}),\ a \in C_\Lambda^*(iX). \tag{3.12}$$

This coaction can also be defined in the non-commutative case as the restriction of the standard W^*-coaction $\delta(l_s) = l_s \otimes l_s$ $(s \in S)$, where l_s denotes left translation. Thus we can define the corresponding *cocrossed product*

$$A \otimes_\delta \mathcal{C}_0(S) := C^*\left(\delta(a)(1 \otimes f) \mid a \in A,\ f \in \mathcal{C}_0(S)\right), \tag{3.13}$$

where $A := C_\Lambda^*(S)$.

4 Representations of $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$

First recall the following fundamental result [NT, Theorem A.1(b)]:

Theorem 4.1 *There is a bijection between the nondegenerate representations μ of the Fourier algebra $A(S)$ on a Hilbert space H and the unitaries $W \in \mathcal{B}(H) \bar{\otimes} W^*(S)$, fulfilling the condition*

$$(W \otimes 1)(W \otimes 1)^\sigma = (1 \otimes W_S)(W \otimes 1)(1 \otimes W_S^*) \tag{4.1}$$

on $H \otimes L^2(S \times S)$. It is given by

$$\mu(f) = S_f(W) \text{ for } f \in A(S). \tag{4.2}$$

Here, for an operator W on $L^2(S \times S) = L^2(S) \otimes L^2(S)$ we put $W^\sigma := \Sigma W \Sigma^*$, with $(\Sigma\Phi)(s,t) := \Phi(t,s)$, $\Phi \in L^2(S \times S)$, W_S is the unitary operator on $L^2(S \times S)$, given by

$$(W_S \Phi)(s,t) := \Phi(s, s^{-1}t) \tag{4.3}$$

and the *slice map* S_f is the mapping

$$\left(\sum_{i=1}^n a_i \otimes b_i \right) \longmapsto \sum_{i=1}^n a_i f(b_i), \tag{4.4}$$

extended to the multiplier algebra $M(A \otimes B)$ for an arbitrary $f \in B^*$ [LPRS, 1.5]. Note that for $f \in A(S)$ the multiplication operator $M_f = S_f(W_S)$.

Definition 4.2 Let δ be a coaction of a Lie group S on a C^*-algebra A. A pair (π, W) consisting of a nondegenerate representation π of A on a Hilbert space H and a unitary $W \in \mathcal{B}(H) \bar{\otimes} W^*(S)$, corresponding by theorem 4.1 to a nondegenerate representation μ of $A(S)$ on H, is called a *covariant pair* for the system (A, S, δ), if for all $a \in A$ the condition

$$\overline{(\pi \otimes i)}\,(\delta(a)) = W\,(\pi(a) \otimes 1)\,W^* \tag{4.5}$$

holds. Here the bar denotes strict extension. In this case one obtains a nondegenerate representation $\Pi(\pi, \mu)$ of the cocrossed product $A \otimes_\delta \mathcal{C}_0(S)$ satisfying

$$\delta(a)(1 \otimes f) \longmapsto \pi(a) \cdot \mu(f), \tag{4.6}$$

for all $a \in A,\ f \in \mathcal{C}_0(S)$. The mapping

$$(\pi, \mu) \longmapsto \Pi(\pi, \mu) \tag{4.7}$$

is a bijection between covariant pairs of representations of (A, S, δ) and nondegenerate representations of $A \otimes_\delta \mathcal{C}_0(S)$ [LPRS, Theorem 3.7].

Our aim is to define for every face of $\Lambda = \hat{V}$ a covariant pair of representations of $(C^*_\Lambda(S), S, \delta)$ in the vector group case $S = iX$. In general we will deal with cones $\Lambda = T \cdot \Lambda_0$, where Λ_0 is a *polyhedral* cone (in a lower dimensional space $X_0 \subset X$) and T is a compact group acting by linear transformations on X and hence on $X^\#$.

Lemma 4.3 *There is a strongly continuous right action of T on $\mathcal{C}_0^\Lambda(X^\#)$ defined by*

$$(\psi \cdot t)(\xi) := \psi(t\xi) \tag{4.8}$$

for all $t \in T$ and $\xi \in X^\#$.

Proof. It suffices to show continuity for generators $\psi = \phi \star \chi_\Lambda$, $\phi \in L^1(X^\#)$ at $t = e$. Since $\mathcal{C}_c^\infty(X^\#)$ is a dense subset of $L^1(X^\#)$ we may assume $\phi \in \mathcal{C}_c^\infty(X^\#)$ and $K = B_R(0) \subset X^\#$ contains the support of ϕ. Now

$$|\psi(t\xi) - \psi(\xi)| \leq ||(\phi \star \chi_\Lambda) \cdot t - \phi \star \chi_\Lambda||_\infty \leq ||\phi \cdot t - \phi||_1 ||\chi_\Lambda||_\infty. \tag{4.9}$$

The mean value theorem implies $\phi \cdot t \to \phi$ uniformly as $t \to e$. Therefore $||\phi \cdot t - \phi||_1 \to 0$ and the assertion follows.

Define for the faces F of Λ_0 the "annihilator"

$$F^\perp := \{x \in iX_0 \ ; \ \langle x, F\rangle = 0\}, \tag{4.10}$$

let $T_F \subset T$ be the stabilizer group of F in T and

$$S_F := T_F \cdot F^\perp. \tag{4.11}$$

The faces of Λ are all faces F of Λ_0 rotated by the action of T/T_F and have the form tF, for some $t \in T/T_F$.

In our case where $\Lambda = \widehat{V_{+-}}$, a 3-dimensional cone of convex rank 2 (c.f. Example 1.2), we have $T = \mathbf{T}$ (1-torus) and $\Lambda_0 = (\mathbf{R}^+)^2$, with faces $\Lambda'_+ = \mathbf{R}^+ \times 0$ (1-dimensional face of rank 1), $\Lambda'_- = 0 \times \mathbf{R}^+$ (perpendicular face of rank 1) and $\Lambda'' = (0)$ is the vertex (rank 0) (c.f. figure 4.1). Note that the space of all 1-dimensional faces of Λ is parametrized by $\mathbf{T} \,\dot{\cup}\, \mathbf{T}$ (disjoint union), in particular this space is not connected. Such a phenomenon does not occur for convex cones.

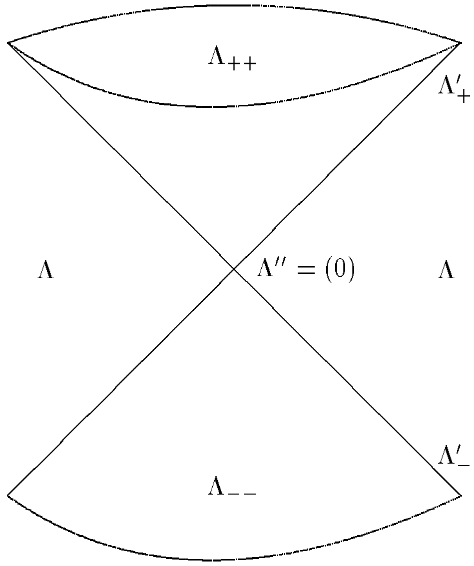

figure 4.1

We define a nondegenerate representation μ_{tF} of $A(iX)$ on $L^2(tS_F)$ corresponding to a face tF of Λ by the restriction

$$\mu_{tF} : f \longmapsto f \mid_{tS_F} =: \bar{f}.$$

It is easy to show that μ_{tF} is nondegenerate, because $A(iX)$ is a dense subset of $\mathcal{C}_0(iX)$ by [E1, Prop 3.7].

The construction of the associated representation π_{tF} (4.6) requires more work, and in fact constitutes the main technical part of this paper, carried out in the vector group setting. Because $Ad(\mathcal{F}) : C^*_\Lambda(iX) \to \mathcal{C}_0^\Lambda(X^\#)$ and $\mathcal{F} \mid_{L^2(tS_F)}: L^2(tS_F) \to L^2((tS_F)^\#)$ are isomorphisms we need to find a nondegenerate representation $\hat{\pi}_{tF}$ of $\mathcal{C}_0^\Lambda(X^\#)$ on $L^2((tS_F)^\#)$.

Theorem 4.4 *Let $(\xi_n) \subset tX_0^\#$ be a sequence with*

1. *$\xi_n \mid_{tS_F} \to \bar{\xi} \in (tS_F)^\#$,*
2. *$pr_{\langle tF \rangle}(\xi_n) \to +\infty$ relative to the order determined by tF. Here $\langle tF \rangle$ is the linear span of tF.*

For short we write $\xi_n \stackrel{tF}{\to} \bar{\xi} \in (tS_F)^{\#}$. *Then the limit*

$$\bar{\psi}(\bar{\xi}) = \lim_{\xi_n \stackrel{tF}{\to} \bar{\xi}} \psi(\xi_n) \tag{4.12}$$

exists for all $\psi \in \mathcal{C}_0^{\Lambda}(X^{\#})$.

In case $F = \Lambda'' = (0)$, we have $S_F = S$ and π_F is the natural representation. Now consider the 1-dimensional case $F = \Lambda'_+ = (\mathbf{R}^+ \times 0)$ with perpendicular face $\Lambda'_- = (0 \times \mathbf{R}^+) =: (\Lambda'_+)_\perp$ (similarly for Λ'_-). Then $(S_{\Lambda'})^{\#} = \langle \Lambda'_\perp \rangle$.

Proposition 4.5 *Let* (ξ_n) *be a sequence in* Λ *with* $\xi_n \stackrel{t\Lambda'}{\to} \xi'$. *Suppose* $\xi' \in \widehat{V_{++}}$. *Then* $\chi_{\xi_n+\Lambda}$ *converges almost everywhere to the characteristic function of*

$$\xi' + (X^{\#} \backslash (\langle t\Lambda' \rangle + \widehat{V_{++}})) \cup C \tag{4.13}$$

where C *is a subset of the tangential plane on* Λ *through* $t\Lambda'$. *A similar conclusion holds for* $\widehat{V_{--}}$.

Proof. Since C has measure zero it suffices to show

$$\lim_{n\to\infty} \chi_{\xi_n+\Lambda}(\eta) = \chi_{H(\xi')}(\eta)$$

for all η not in the hyperplane through ξ' and orthogonal to $t\Lambda'_\perp$. Here

$$H(\xi') := \xi' + (X^{\#} \backslash (\langle t\Lambda' \rangle + \widehat{V_{++}})) \tag{4.14}$$

is the closed halfspace determined by the conditions $\xi' \in \partial H(\xi')$, $\partial H(\xi')$ is orthogonal to $t\Lambda'_\perp$ and $\xi' + \widehat{V_{++}} \not\subset H(\xi')$. Since $\xi_n \stackrel{t\Lambda'}{\to} \xi'$ we introduce cartesian coordinates according to $t\Lambda'$ and $t\Lambda'_\perp$ and write $\xi_n = (\xi_n^1, \xi_n^2, \xi_n^3)$, with $t\Lambda'_\perp \ni \xi_n^1 \to \xi'$, $t\Lambda' \ni \xi_n^2 \to \infty$ and $\xi_n^3 = 0$. For $\eta \in X^{\#}$, $\xi_n - \eta = (\xi_n^1 - \eta^1, \xi_n^2 - \eta^2, -\eta^3)$ converges to $(\xi' - \eta^1, \infty, -\eta^3)$. Thus we get for sufficiently large n

$$\begin{array}{lll} \xi_n - \eta \in \Lambda & \text{if} & \eta^1 < \xi' \\ \xi_n - \eta \notin \Lambda & \text{if} & \eta^1 > \xi' \end{array}$$

implying

$$\chi_{\xi_n - \Lambda}(\eta) = \begin{cases} 1 & \text{if} \quad \eta^1 < \xi' \\ 0 & \text{if} \quad \eta^1 > \xi' \end{cases} \quad \text{for all } n \text{ large enough.}$$

Theorem 4.6 *For each 1-dimensional face $tF = t\Lambda'$, there is a nondegenerate representation $\hat{\pi}_{tF}$ of $\mathcal{C}_0^\Lambda(X^\#)$ on $L^2((tS_F)^\#)$ determined by*

$$\hat{\pi}_{tF} : \psi = \phi \star \chi_\Lambda \longmapsto \psi', \ \psi'(\xi') := \int\limits_{H(\xi')} \phi(\xi) d\xi$$

Proof. Consider the set

$$\mathcal{A} := \{\psi \in \mathcal{U}_b(X^\#) \mid \lim_{\xi_n \overset{t\Lambda'}{\to} \xi'} \psi(\xi_n) =: \psi'(\xi') \text{ exists for all } \xi' \in \langle t\Lambda'_\perp \rangle\}.$$

Clearly $\mathcal{A}$ is a *-algebra and $\psi \mapsto \psi'$ is a contractive *-homomorphism. To show that $\mathcal{A}$ is a closed algebra, consider a sequence ψ_m in $\mathcal{A}$ converging uniformly to a function ψ. Since ψ_m is a uniform Cauchy sequence it is bounded by c and we get $|\psi(\xi)| = \lim_{n\to\infty} |\psi_n(\xi)| \le c$ for all $\xi \in X^\#$. Therefore a subsequence $(\psi(\xi_{m_k}))$ converges to a M in $\mathbf{C}$. For arbitrary $\epsilon > 0$, choose m_0 so that $\|\psi - \psi_{m_0}\| < \frac{\epsilon}{6}$, an l so that $|M - \psi(\xi_{n_l})| < \frac{\epsilon}{6}$ and $|\psi'_{m_0}(\xi') - \psi_{m_0}(\xi_{n_l})| < \frac{\epsilon}{6}$, and n_0 so that $|\psi'_{m_0}(\xi') - \psi_{m_0}(\xi_n)| < \frac{\epsilon}{6}$ for all $n > n_0$. Then for $n > n_0$

$$\begin{aligned} &|M - \psi(\xi_n)| = |M - \psi(\xi_{n_l})| + |\psi(\xi_{n_l}) - \psi_{m_0}(\xi_{n_l})| \\ &\quad + \quad |\psi_{m_0}(\xi_{n_l})| - \psi'_{m_0}(\xi')| + |\psi'_{m_0}(\xi') - \psi_{m_0}(\xi_n)| + |\psi_{m_0}(\xi_n) - \psi(\xi_n)| < \epsilon. \end{aligned}$$

Therefore $M = \lim_{\xi_n \overset{t\Lambda'}{\to} \xi'} \psi(\xi_n)$ exists. By Proposition 4.5 and dominated convergence we get for all $\psi = \phi \star \chi_\Lambda, \ \phi \in L^1(X^\#)$:

$$\psi'(\xi') := \lim_{\xi_n \overset{t\Lambda'}{\to} \xi'} \psi(\xi_n) = \int\limits_{H(\xi')} \phi(t) dt. \tag{4.15}$$

Hence the generators of $\mathcal{C}_0^\Lambda(X^\#)$ are contained in $\mathcal{A}$. Since $\mathcal{A}$ is closed, $\mathcal{C}_0^\Lambda(X^\#) \subset \mathcal{A}$ and, by contractivity, the limit function $\psi' \in \mathcal{U}_b(\langle t\Lambda'_\perp \rangle)$ for all $\psi \in \mathcal{C}_0^\Lambda(X^\#)$. To see that the representation on $L^2(\langle t\Lambda'_\perp \rangle) = L^2((t\Lambda'^\perp)^\#)$ is nondegenerate, consider a function $\phi \in L^1(X^\#)$ with the properties $\int_{H(0)} \phi(\xi) d\xi = 1$ and $\phi(\xi) = 0$ for all ξ in the exterior of the half-space $H(0)$. Then $\hat{\pi}(\phi \star \chi_\Lambda)(\vartheta) = 0$ implies $\vartheta = 0 \in L^2(\langle t\Lambda'_\perp \rangle)$.

The corresponding representation π_{tF} of $C^*_\Lambda(iX)$ is given by applying $Ad(\mathcal{F})$ and $Ad(\mathcal{F} \mid_{\langle tS_F \rangle})$ and has the explicit form:

Corollary 4.7 *For a 1-dimensional face $tF = t\Lambda'$ there exists a representation π_{tF} of $C^*_\Lambda(iX)$ on $L^2(tS_F)$ determined by*

$$\pi_{tF}(l_{fE}) = l_{\bar{f}E_{tF}}, \tag{4.16}$$

where $\bar{f} := f \mid_{tS_F}$ and $E_{tF} := (\widehat{\chi_{\Lambda \cap tF_\perp}})$. Here $tF_\perp$ is the perpendicular face.

Proof. Since $\mathcal{C}_c(X^\#)$ is dense in $L^1(X^\#)$, we may assume $\phi \in \mathcal{C}_c(X^\#) \subset L^2(X^\#)$. Since $L^2(iX) = L^2(\mathbf{R}^2) \otimes L^2(tS_F)$, we may assume $\hat{\phi} = g \otimes h,\ g \in \mathcal{C}_c(\mathbf{R}^2),\ h \in \mathcal{C}_c(tS_F)$. Since $|e^{ix_0\beta}| \leq 1$ for all $x_0 \in \mathbf{R}^2, \beta \in X^\#$, the function $g_\beta :\ x_0 \longmapsto e^{ix_0\beta} g(x_0)$ is an element of $L^2(\mathbf{R}^2)$ and

$$g_\beta(x) = \int\limits_{\mathbf{R}^2} e^{-i\xi_0 x} \hat{g}_\beta(\xi_0) d\xi_0 = \int\limits_{\mathbf{R}^2} e^{-i\xi_0 x} \int\limits_{\mathbf{R}^2} e^{i\xi_0 x_0} g_\beta(x_0) dx_0 d\xi_0, \tag{4.17}$$

hence

$$g(0) = e^{i0\beta} g(0) \overset{(Def)}{=} g_\beta(0) = \int\limits_{\mathbf{R}^2} 1 \int\limits_{\mathbf{R}^2} e^{i\xi_0 x_0} g_\beta(x_0) dx_0 d\xi_0 \tag{4.18}$$

and we get, using (4.15)

$$\begin{aligned}
\psi'(\xi') &= g(0) \int_{\xi_1 \leq \xi', \xi_1 \in \langle tF_\perp \rangle} \hat{h}(\xi_1) d\xi_1 \\
&= g(0) \int\limits_{\xi' - tF_\perp} \hat{h}(\xi_1) d\xi_1 = g(0) \int\limits_{\langle tF_\perp \rangle} \chi_{tF_\perp}(\xi' - \xi_1) h(\xi_1) d\xi_1 \\
&= g(0) \cdot (\chi_{tF_\perp} \star \hat{h})(\xi'),
\end{aligned}$$

with the convolution relative $\langle tF_\perp \rangle$. Applying $Ad(\mathcal{F})$ we get

$$Ad(\mathcal{F})(\psi') = g(0) \cdot l_{h\hat{\chi}_{tF_\perp}} = l_{g(0)h\hat{\chi}_{tF_\perp}} \tag{4.19}$$

and the assertion follows.

Finally, let $F = \Lambda_0 = (\mathbf{R}^+)^2$.

Proposition 4.8 *Let (ξ_n) be a sequence in Λ with $\xi_n \overset{\Lambda_0}{\to} 0$. Then*

$$\phi \star \chi_\Lambda(\xi_n) \to \hat{\phi}(0)$$

for all $\phi \in L^1(X^\#)$.

Proof. Let $\eta \in X^\#$. Then $\xi_n - \eta = (\xi_n^1 - \eta_1, \xi_n^2 - \eta_2, -\eta_3)$ converges to infinity in the first two components. Therefore $\chi_{\xi_n - \Lambda}$ converges pointwise to the constant function 1 on $X^\#$. Dominated convergence implies

$$\phi \star \chi_\Lambda(\xi_n) = \int\limits_{X^\#} \phi(\tau) \chi_{\xi_n - \Lambda}(\tau) d\tau \to \int\limits_{X^\#} \phi(\tau) d\tau = \hat{\phi}(0),\ \phi \in L^1(X^\#). \tag{4.20}$$

Remark 4.9 By the symmetric properties of Λ one sees, that $\chi_{\xi_n - \Lambda}$ converges to 1 for *every* sequence $\xi_n \overset{t\Lambda_0}{\to} 0$ with $t \in T$.

Theorem 4.10 *There exists a character, denoted by π_Λ $(=\pi_{\Lambda_0})$, of $C^*_\Lambda(iX)$ on $L^2(0)=\mathbf{C}$, determined by*

$$\pi_\Lambda : l_{fE} \longmapsto f(0). \tag{4.21}$$

Proof. The same techniques as in the proof of Theorem 4.6 together with (4.20) show that

$$\hat{\pi}_\Lambda : \psi = \phi \star \chi_\Lambda \longmapsto \hat{\phi}(0)$$

is a representation of $\mathcal{C}_0^\Lambda(X^\#)$ on $\mathbf{C}$. Applying $Ad(\mathcal{F})$ to the equation $\widehat{fE} = \phi \star \chi_\Lambda$ yields

$$l_{fE} \mapsto f(0) \in \mathbf{C}.$$

Since $f(0) \neq 0$ for some $f \in A(S)$, the character (4.21) is non-zero.

Let $\bar{S}$ be the space tS_F and put $l_{\bar{f}\bar{E}} = \pi_{tF}(l_{fE})$. We will now show the covariance condition of Definition 4.2 for the representation π_{tF} associated with any face tF of Λ.

Theorem 4.11 *For all g in $A(S)$ we have*

$$(\pi_{tF} \otimes i)^-(\delta(l_{gE})) = W_{\bar{S}}(\pi_{tF}(l_{gE}) \otimes 1)W^*_{\bar{S}},$$

where

$$W_{\bar{S}}\Phi(s,r) = \Phi(s, r-s) \ \textit{(additive notation)}$$

for all $s \in \bar{S}, r \in S$.

Proof. Because $(\pi_{tF} \otimes i)^-(\delta(l_{gE}))$ and $W_{\bar{S}}(\pi_{tF}(l_{gE}) \otimes 1)W^*_{\bar{S}}$ commute with right translations $\gamma_s \otimes \gamma_r$, $(s,r) \in \bar{S} \times S$ both are elements of $W^*(\bar{S}) \bar{\otimes} W^*(S) = W^*(\bar{S} \times S)$. Therefore it's enough to show, that the Fourier transform of both define the same multiplication operator on $L^2(\bar{S} \times S)$. First we compute

$$\begin{aligned}
&W_{\bar{S}}(l_{\bar{g}\bar{E}} \otimes 1)W^*_{\bar{S}}(\phi \otimes \psi)(s,r) = (l_{\bar{g}\bar{E}} \otimes 1)W^*_{\bar{S}}(\phi \otimes \psi)(s, r-s) \\
&= \int_{\bar{S}} \bar{g}\bar{E}(x)\, W^*_{\bar{S}}(\phi \otimes \psi)(s-x, r-s)\, dx = \int_{\bar{S}} \bar{g}\bar{E}(x)\, \phi(s-x)\psi(r-x)\, dx.
\end{aligned}$$

Hence, putting $s - x =: y$, $r - x =: z$ and $\bar{\eta} := \eta\,|_{\bar{S}}$,

$$\begin{aligned}
&\mathcal{F}(W_{\bar{S}}(l_{\bar{g}\bar{E}} \otimes 1)W^*_{\bar{S}}(\phi \otimes \psi))(\bar{\xi}, \eta) \\
&= \int_{\bar{S}}\int_S\int_{\bar{S}} \bar{g}\bar{E}(x)\phi(s-x)\psi(t-x)e^{-i\bar{\xi}(s)}e^{-i\eta(t)}dx\, ds\, dt \\
&= \int_{\bar{S}}\int_S \phi(y)\psi(z)e^{-i\bar{\xi}(y)}e^{-i\eta(z)} \int_{\bar{S}} \bar{g}\bar{E}(X)e^{-i\bar{\xi}(x)}e^{-i\bar{\eta}(X)}dx, \\
&= \mathcal{F}(\phi \otimes \psi)(\bar{\xi}, \eta) \cdot \mathcal{F}(\bar{g}\bar{E})(\bar{\xi} + \bar{\eta}).
\end{aligned} \tag{4.22}$$

On the other hand we get $\delta(l_{gE}) = \lim \sum\limits_i l_{u_i} \otimes l_{v_i} \in (C^*_\Lambda(S) \otimes C^*(S))^-$ (strict closure), hence $(\pi_{tF} \otimes i)^-(\delta(l_{gE})) = \lim \sum\limits_i \pi_{tF}(l_{u_i}) \otimes l_{v_i}$ and

$$Ad(\mathcal{F})((\pi_{tF} \otimes i)(\delta(l_{gE})))(\bar{\xi}, \eta) = \lim \sum_i Ad(\mathcal{F})(\pi_{tF}(l_{u_i}))(\bar{\xi}) \otimes \mathcal{F}(v_i)(\eta)$$

$$\begin{aligned} &= \lim_{\xi \xrightarrow{tF} \bar{\xi}} \lim \sum_i \mathcal{F}(u_i)(\xi) \otimes \mathcal{F}(v_i)(\eta) = \lim_{\xi \xrightarrow{tF} \bar{\xi}} Ad(\mathcal{F})(\delta(l_{gE}))(\xi, \eta) \\ &= \lim_{\xi \xrightarrow{tF} \bar{\xi}} \mathcal{F}(gE)(\xi + \eta) = \mathcal{F}(\bar{g}\bar{E})(\bar{\xi} + \bar{\eta}), \end{aligned} \tag{4.23}$$

since for all tF we have $\xi + \eta \xrightarrow{tF} \bar{\xi} + \bar{\eta}$ if $\xi \xrightarrow{tF} \bar{\xi}$. Viewed as a multiplication operator on $L^2(\bar{S} \times S)$, (4.22) and (4.23) agree.

Together with Theorem 4.6 we have proved the following

Theorem 4.12 *For every face tF of Λ we have a nondegenerate representation Π_{tF} of the cocrossed product C^*-algebra $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$ on $L^2(\bar{S})$, determined for all $a \in C^*_\Lambda(S)$, $g \in A(S)$ by*

$$\delta(a)(1 \otimes g) \longmapsto \pi_{tF}(a)\mu_{tF}(g). \tag{4.24}$$

5 Structure Theory of Toeplitz C*-Algebras

Consider the *Hardy-Toeplitz algebra*

$$\mathcal{T}(S) := C^*\left(T_S(f) \mid f \in \mathcal{C}_0(S)\right),$$

where l_E is the projection of $L^2(S)$ on the Hardy space $H^2_\Lambda(S)$ and

$$T_S(f) := l_E\, f\, l_E$$

is the Hardy-Toeplitz operator corresponding to $f \in \mathcal{C}_0(S)$. Our aim is to find all irreducible representations of $\mathcal{T}(S)$.

Applying Katayama's isomorphism Φ [LPRS, p. 768] we get for every representation Π_{tF} of $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$ defined by (4.24) a nondegenerate representation ν_{tF} of

$$\mathcal{K}_\Lambda(L^2(S)) := \Phi(C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)) = C^*(l_{fE}\, g\, ;\, f \in A(S),\ g \in \mathcal{C}_0(S)) \tag{5.1}$$

on $L^2(\bar{S})$, given by $\nu_{tF}(\Phi(x)) = \Pi_{tF}(x)$ for all x in $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$. Therefore

$$\nu_{tF}(l_{fE}\, g) = \pi_{tF}(l_{fE})\mu_{tF}(g). \tag{5.2}$$

Lemma 5.1 *Let $f_1, ..., f_n, g_1, ..., g_n, h_0, ..., h_n \in \mathcal{C}_c^\infty(S)$ fulfill the condition*

$$\prod_{j=1}^{n} f_j(t_{j-1}t_j^{-1})g_j(t_j) = \prod_{i=0}^{n} h_i(t_i) \tag{5.3}$$

for all $t_0, ..., t_n \in S$. Then we have the operator identity (on $L^2(S)$)

$$h_0 l_E h_1 l_E \cdots l_E h_n = l_{f_1 E} g_1 \cdots l_{f_n E} g_n. \tag{5.4}$$

Proof. For any $k \in \mathcal{C}_c^\infty(S)$ we calculate the inner product (antilinear in the first variable)

$$\begin{aligned}
&\langle k \mid h_0 l_E \cdots l_E h_n k\rangle \\
= &\int_S \cdots \int_S \overline{k(t_0)} h_0(t_0) h_1(s_1^{-1}t_0) \cdots h_n(s_n^{-1} \cdots s_1^{-1}t_0) \cdot \\
&\qquad \cdot k(s_n^{-1} \cdots s_1^{-1}t_0) E(ds_1) \cdots E(ds_n) dt_0 \\
= &\int_S \cdots \int_S \overline{k(t_0)} g_1(s_1^{-1}t_0) g_2(s_2^{-1}s_1^{-1}t_0) \cdots g_n(s_n^{-1} \cdots s_1^{-1}t_0) \cdot \\
&\qquad \cdot k(s_n^{-1} \cdots s_1^{-1}t_0) f_1(s_1) \cdots f_n(s_n) E(ds_1) \cdots E(ds_n) dt_0 \\
= &\langle k \mid l_{f_1 E} g_1 \cdots l_{f_n E} g_n k\rangle.
\end{aligned}$$

Note that because $l_E \in W^*(S)$ the integral $\int h(st^{-1})E(dt)$ is well defined for all $h \in A(S)$, and therefore for all $h \in \mathcal{C}_c^\infty(S)$. Analogously for the other integrals.

Proposition 5.2

$$l_E\, \mathcal{K}_\Lambda(L^2(S))\, l_E = \mathcal{T}(S). \tag{5.5}$$

Proof. To show the first inclusion it suffices (by density) to show

$$l_E\, (\prod_j l_{f_j E}\, g_j)\, l_E \in \mathcal{T}(S) \tag{5.6}$$

for all $f_1, ..., f_n \in A(S),\ g_1, ..., g_n \in \mathcal{C}_c^\infty(S)$. Since $\mathcal{C}_c^\infty(S)$ is dense in $A(S)$ [E1, Proposition 3.26], we may assume $f_1, ..., f_n \in \mathcal{C}_c^\infty(S)$. To see that

$$F(t_0, ..., t_n) := \prod_{j=1}^{n} f_j(t_{j-1}t_j^{-1})g_j(t_j) \tag{5.7}$$

belongs to $\mathcal{C}_c^\infty(S \times \ldots \times S) = \mathcal{C}_c^\infty(S) \otimes ... \otimes \mathcal{C}_c^\infty(S)$ let $K_{g_j},\ K_{f_j}$ be the compact support of $f_j,\ g_j$ and put

$$K := (K_{f_1} \cdot K_{g_1}) \times K_{g_1} \times \cdots \times K_{g_n}. \tag{5.8}$$

If $(t_0, ..., t_n) \notin K$ then there are three cases. If $t_0 \notin K_{f_1} \cdot K_{g_1}$ and $t_1 \in K_{g_1}$, then $t_0 t_1^{-1} \notin K_{f_1}$ implying $f_1(t_0 t_1^{-1}) = 0$, hence $F(t_0, ..., t_n) = 0$. Now assume $t_0 t_1^{-1} \notin K_{f_1}$ and $t_1 \notin K_{g_1}$. Then $g_1(t_1) = 0$ and the assertion holds. Finally let $t_0 \in K_{f_1} \cdot K_{g_1}$. Then there is a $j \geq 1$ with $t_j \notin K_{g_j}$ implying $g_j(t_j) = 0$ and hence $F(t_0, ...t_n) = 0$. Thus $\mathrm{supp}(F) \subset K$ is compact. Hence F is a uniform limit of finite sums of the form $\prod_{i=0}^n h_i(t_i)$ where $h_i \in \mathcal{C}_c^\infty(S)$. Therefore Lemma 5.1 implies (5.6). Conversely, let $A \in \mathcal{T}(S)$. Since $\mathcal{T}(S) \cdot \mathcal{T}(S)$ is dense in $\mathcal{T}(S)$, we may assume

$$A = \prod_{i=0}^{n} l_E \, h_i \, l_E, \tag{5.9}$$

where $h_i \in \mathcal{C}_0(S)$ and $n \geq 1$. Since $\mathcal{C}_0(S) \cdot \mathcal{C}_0(S)$ is dense in $\mathcal{C}_0(S)$ we may assume $h_0, ..., h_n \in \mathcal{C}_c^\infty(S)$ and $h_i = h_i^1 \cdot h_i^2$ for $i = 1, ..., n-1$, where $h_i^1, \ h_i^2 \in \mathcal{C}_c^\infty(S)$. Then

$$G(s_1, ..., s_n, t_1, ..., t_n) := h_0(s_1 t_1) \Big(\prod_{i=1}^{n-1} h_i^1(t_i) h_i^2(s_{i+1} t_{i+1}) \Big) h_n(t_n) \tag{5.10}$$

belongs to $\mathcal{C}_c^\infty(S \times ... \times S)$ ($2n$ times). Hence G is a uniform limit of finite sums of the form $\prod_{j=1}^n f_j(s_j) g_j(t_j)$ where $f_j, \ g_j \in \mathcal{C}_c^\infty(S)$. By putting $s_i := t_{i-1} t_i^{-1}, \ i = 1, ..., n$ we get $\prod_{i=0}^n h_i(t_i)$ as a uniform limit of finite sums of the form $\prod_{j=1}^n f_j(t_{j-1} t_j^{-1}) g_j(t_j)$. Applying Lemma 5.1 again, we obtain $A \in l_E \, \mathcal{K}_\Lambda(L^2(S)) \, l_E$.

Lemma 5.3 *For all $a, b \in \mathcal{K}_\Lambda(L^2(S))$ we have $a \, l_E \, b \in \mathcal{K}_\Lambda(L^2(S))$ and*

$$\nu_{tF}(a \, l_E \, b) = \nu_{tF}(a) \, l_{\bar{E}} \, \nu_{tF}(b). \tag{5.11}$$

Proof. Let $f, h \in A(S)$ and $g, k \in \mathcal{C}_0(S)$. As in Proposition 5.2 and Lemma 5.1 one shows that $l_{fE} \, g \cdot l_E \cdot l_{hE} \, k$ is a limit of finite sums of the form $(h_0 \, l_E \, h_1) l_E (h_2 \, l_E \, h_3)$, $h_i \in \mathcal{C}_c^\infty(S)$, and $(h_0 \, l_E \, h_1) l_E (h_2 \, l_E \, h_3)$ is a limit of finite sums of the form $l_{f_1 E} \, g_1 \cdot l_{f_2 E} \, g_2 \cdot l_{f_3 E} \, g_3$, $f_i, g_i \in \mathcal{C}_c^\infty(S)$. Therefore $l_{fE} \, g \cdot l_E \cdot l_{hE} \, k \in \mathcal{K}_\Lambda(L^2(S))$. Since

$$\begin{aligned} &\nu_{tF}(l_{f_1 E} g_1 \cdot l_{f_2 E} g_2 \cdot l_{f_3 E} g_3) \\ &= \ \nu_{tF}(l_{f_1 E} \, g_1) \, \nu_{tF}(l_{f_2 E} \, g_2) \, \nu_{tF}(l_{f_3 E} \, g_3) = l_{\bar{f}_1 \bar{E}} \, \bar{g}_1 \cdot l_{\bar{f}_2 \bar{E}} \, \bar{g}_2 \cdot l_{\bar{f}_3 \bar{E}} \, \bar{g}_3, \end{aligned}$$

we see by the same calculations as above, only in a lower dimension, that $\nu_{tF}(l_{fE} \, g \cdot l_E \cdot l_{hE} \, k) = l_{\bar{f}\bar{E}} \, \bar{g} \cdot l_{\bar{E}} \cdot l_{\bar{h}\bar{E}} \, \bar{k} = \nu_{tF}(l_{fE} \, g) \, l_{\bar{E}} \, \nu_{tF}(l_{hE} \, k)$.

Theorem 5.4 *For each face tF of Λ there exists a representation ρ_{tF} of $\mathcal{T}(S) = l_E\, \mathcal{K}_\Lambda(L^2(S))\, l_E$ such that $\rho_{tF}(l_E\, a\, l_E) := l_{\bar{E}}\, \nu_{tF}(a)\, l_{\bar{E}}$ and*

$$\nu_{tF}(b\, l_E\, a\, l_E\, c) = \nu_{tF}(b)\rho_{tF}(l_E\, a\, l_E)\nu_{tF}(c) \tag{5.12}$$

for all a, b, c in $\mathcal{K}_\Lambda(L^2(S))$.

Proof. To show (5.12) consider a sequence of sums of products $\sum_i \prod_j (l_{f_{ij}E}\, g_{ij})$ of typical generators of $\mathcal{K}_\Lambda(L^2(S))$ converging to a. For each of them we have by Lemma 5.3

$$b\, l_E \sum_i \prod_j (l_{f_{ij}E}\, g_{ij})\, l_E\, c = \sum_i b\, l_E \prod_p (l_{f_{ip}E}\, g_{ip}) \prod_q (l_{f_{iq}E}\, g_{iq})\, l_E\, c \in \mathcal{K}_\Lambda(L^2(S)).$$

Here the j-indices are split into $p's$ and $q's$, with both products non-empty. Hence

$$\begin{aligned}
\nu_{tF}(b\, l_E\, a\, l_E\, c) &= \lim \sum_i \nu_{tF}(b\, l_E \prod_j (l_{f_{ij}E}\, g_{ij})\, l_E\, c) \\
&= \lim \sum_i \nu_{tF}(b\, l_E \prod_p (l_{f_{ip}E}\, g_{ip}))\nu_{tF}(\prod_q (l_{f_{iq}E}\, g_{iq})\, l_E\, c) \\
&\overset{(5.11)}{=} \lim \sum_i \nu_{tF}(b)\, l_{\bar{E}}\, \nu_{tF}(\prod_p (l_{f_{ip}E}\, g_{ip}))\nu_{tF}(\prod_q (l_{f_{iq}E}\, g_{iq}))\, l_{\bar{E}}\, \nu_{tF}(c) \\
&= \lim \sum_i \nu_{tF}(b)\, l_{\bar{E}}\, \nu_{tF}(\prod_j (l_{f_{ij}E}\, g_{ij}))\, l_{\bar{E}}\, \nu_{tF}(c) \\
&= \lim \sum_i \nu_{tF}(b)\rho_{tF}(l_E \prod_j (l_{f_{ij}E}\, g_{ij})\, l_E)\nu_{tF}(c) = \nu_{tF}(b)\rho_{tF}(l_E\, a\, l_E)\nu_{tF}(c).
\end{aligned}$$

This implies that the map is well defined, because for an a with $l_E\, a\, l_E = 0$ we get for all b, c in $\mathcal{K}_\Lambda(L^2(S))$

$$0 = \nu_{tF}(b\, l_E\, a\, l_E\, c) = \nu_{tF}(b)\rho_{tF}(l_E\, a\, l_E)\nu_{tF}(c) \tag{5.13}$$

implying $\rho_{tF}(l_E\, a\, l_E) = 0$ by the irreducibility of ν_{tF} (we will show later that the compact operators $\mathcal{K}(L^2(S)) \subset \mathcal{K}_\Lambda(L^2(S))$). To see the homomorphism property calculate for a, b in $\mathcal{K}_\Lambda(L^2(S))$:

$$\begin{aligned}
&\rho_{tF}(l_E\, a\, l_E \cdot l_E\, b\, l_E) = \rho_{tF}(l_E\, a\, l_E\, b\, l_E), \quad \text{because } l_E \text{ is a projection,} \\
&= \; l_{\bar{E}}\, \rho_{tF}(a\, l_E\, b)\, l_{\bar{E}} \overset{(5.11)}{=} l_{\bar{E}}\, \rho_{tF}(a)\, l_{\bar{E}}\, \rho_{tF}(b)\, l_{\bar{E}} \\
&= \; l_{\bar{E}}\, \rho_{tF}(a)\, l_{\bar{E}} \cdot l_{\bar{E}}\, \rho_{tF}(b)\, l_{\bar{E}} = \rho_{tF}(l_E\, a\, l_E) \cdot \rho_{tF}(l_E\, b\, l_E).
\end{aligned}$$

The representations ρ_{tF} just defined give rise to "translated" representations. For $s \in S$ the right translation γ_s maps $f \in \mathcal{C}_0(S)$ to f_s defined by $f_s(t) = f(ts)$.

The translated representation ρ_{tF}^s corresponding to s and ρ_{tF} is defined by the following diagram

$$\begin{array}{ccccc} & \mathcal{T}(S) & \xrightarrow{\rho_{tF}} & \mathcal{L}(l_{\bar{E}}\, L^2(\bar{S})) & \\ Ad(\gamma_s) & \downarrow & \cdot/\cdot & \downarrow & Id \\ & \mathcal{T}(S) & \xrightarrow{\rho_{tF}^s} & \mathcal{L}(l_{\bar{E}}\, L^2(\bar{S})) & . \end{array} \tag{5.14}$$

The fact that $\mathcal{T}(S) = l_E\, \mathcal{K}_\Lambda(S)\, l_E$ is $Ad(\gamma_s)$-invariant follows from the formula

$$\gamma_s\, l_E\, l_{fE}\, g\, l_E\, \gamma_{-s} = l_E\, l_{fE}\, \gamma_s g \gamma_{-s}\, l_E = l_E\, l_{fE}\, g_s\, l_E \tag{5.15}$$

applied to a typical generator $l_E\, l_{fE}\, g\, l_E$ of the Toeplitz algebra, since $g_s \in \mathcal{C}_0(S)$.

Lemma 5.5 *Two representations $\rho_{t_1F_1}^{s_1}$ and $\rho_{t_2F_2}^{s_2}$ are pairwise inequivalent, whenever $F_1 \neq F_2$ or $F_1 = F_2 = F$ and $t_1 t_2^{-1} \notin T_F$ or $s_1 - s_2 \notin S_F$. All representations are irreducible.*

Proof. For the trivial face $F = \Lambda_0$ we have only $\rho_{tF}^s = \rho_{tF}$, the identity representation of $\mathcal{T}(S)$ on $H_\Lambda^2(S)$. It is irreducible as a consequence of Corollary 5.7 below. For a 1-dimensional face tF, the classical Toeplitz extension

$$0 \to \mathcal{K}(H_{tF}^2(tS_F)) \to \mathcal{T}(tS_F) \to \mathcal{C}_0(tS_F) \to 0 \tag{5.16}$$

on the half-line is an exact sequence, so that the representation ρ_{tF} is irreducible and

$$||T_{tS_F}(\tilde{\varphi})||_{\mathcal{T}/\mathcal{K}} = ||\tilde{\varphi}||_\infty \tag{5.17}$$

for $\tilde{\varphi} \in \mathcal{C}_0(tS_F)$. Now consider distinct $\rho_{t_1F_1}^{s_1}, \rho_{t_2F_2}^{s_2}$, where $dim(F_i) = 1$. Then we have to discuss two cases. First let $s_1 + t_1 S_{F_1} \neq s_2 + t_2 S_{F_2}$. Then there exists $\varphi \in \mathcal{C}_0(S)$ such that $\varphi_1 := \gamma_{s_1^{-1}}\varphi\, |_{t_1 S_{F_1}} \neq 0$ and $\varphi_2 := \gamma_{s_2^{-1}}\varphi\, |_{t_2 S_{F_2}} = 0$. Then $\rho_{t_1F_1}^{s_1}(l_E\, \varphi\, l_E) = l_{E_{t_1F_1}}\, \varphi_1\, l_{E_{t_1F_1}} = T_{t_1 S_{F_1}}(\varphi_1) \notin \mathcal{K}(H_{t_1F_1}^2(t_1 S_{F_1}))$ since $||\varphi_1|| \neq 0$, hence $T_{t_1S_{F_1}}(\varphi_1) \neq 0$, but $\rho_{t_2F_2}^{s_2}(l_E\, \varphi\, l_E) = l_{E_{t_2F_2}}\, \varphi_2\, l_{E_{t_2F_2}} = 0$. Now assume $s_1 + t_1 S_{F_1} = s_1 + t_2 S_{F_2}$. Then $F_1 \neq F_2$ and we can assume $t_1 = 1$ and $t_2 = -1$. Then there exists a ψ in $\mathcal{C}_0^\Lambda(X^\#)$ in $Ker\hat{\pi}_{F_1} \backslash Ker\hat{\pi}_{F_2}$ (e.g. constant in the direction of F_2 and converging to 0 in the direction of F_1) such that by taking a suitable $f \in A(S)$ not in $Ker\mu_{t_2F_2}^{s_2}$ we have

$$l_{E_{t_2F_2}} \pi_{t_2F_2}(\hat{\psi}) \mu_{t_2F_2}^{s_2}(f) l_{E_{t_2F_2}} \neq 0. \tag{5.18}$$

Therefore $Ker\, \rho_{t_1F_1}^{s_1} \neq Ker\, \rho_{t_2F_2}^{s_2}$ and the representations are inequivalent. The same reasoning shows that all translated vertex representations are inequivalent. For pairs F_1 and F_2 of different dimension the assertion is clear.

Lemma 5.6

$$C^*(S) \subset C^*_\Lambda(S).$$

Proof. Let Λ_{++} (resp. Λ_{--}) denote the forward (resp. backward) lightcone (c.f. Example 1.2). Then

$$\hat{\chi}_{\Lambda_{++}}(ix) = (Det(ix+0))^{-\frac{3}{2}} := \lim_{y \searrow 0} (Det(ix+y))^{-\frac{3}{2}} \tag{5.19}$$

and

$$\hat{\chi}_{\Lambda_{--}}(ix) = (Det(-ix+0))^{-\frac{3}{2}} := \lim_{y \searrow 0} (Det(-ix+y))^{-\frac{3}{2}} \tag{5.20}$$

as follows from the general theory of euclidean Jordan algebras [FK]. Since $Det(ix+y)$ and $Det(-ix+y)$ are not in $\mathbf{R}^+ \cup \{0\}$ for $y > 0$ we can define (5.19) and (5.20) using the same branch of the logarithm. Now

$$\begin{aligned} E(ix) &= \mathcal{F}^{-1}(1 - \chi_{\Lambda_{--}} - \chi_{\Lambda_{++}})(ix) \\ &= \quad c\delta_0(ix) - (Det(ix+0))^{-\frac{3}{2}} - (Det(-ix+0))^{-\frac{3}{2}}, \end{aligned} \tag{5.21}$$

where c is a constant. Let $f(ix) := -(Det(ix+0))^{\frac{3}{2}} \cdot (Det(-ix+0))^{\frac{3}{2}} \in \mathcal{C}(iX)$, $u \in \mathcal{C}_c(iX)$. Thus $ufE = uf\delta_0 + u\tilde{f}$, where $\tilde{f}(ix) := (Det(ix+0))^{\frac{3}{2}} + (Det(-ix+0))^{\frac{3}{2}}$ is a continuous function on iX. Since $l_{uf\delta_0}(g) = u(0)f(0)g = 0$ for all g in $L^2(iX)$ we get for all $u \in \mathcal{C}_c(iX)$

$$l_{ufE} = l_{u\tilde{f}}. \tag{5.22}$$

Since $N := \{ix \in iX;\ \tilde{f}(ix) = 0\}$ has measure 0, one may choose functions $\varphi_\epsilon \in \mathcal{C}_c^\infty(iX)$, $\epsilon > 0$, with $\varphi_\epsilon = 0$ in the $\frac{\epsilon}{2}$-neighborhood of N and $\varphi_\epsilon = 1$ outside the ϵ-neighborhood of N. For each element $v \in \mathcal{C}_c^\infty(iX)$ the function

$$u_\epsilon := \begin{cases} v\varphi_\epsilon/\tilde{f} & \text{on } iX \backslash N \\ 0 & \text{on } N \end{cases} \tag{5.23}$$

is well defined and we get

$$u_\epsilon \cdot \tilde{f} = v \cdot \varphi_\epsilon \in \mathcal{C}_c^\infty(iX). \tag{5.24}$$

Since $v \cdot \varphi_\epsilon$ converges to v in the L^1-norm and $\mathcal{C}_c^\infty(iX) \subset A(iX)$ by [E1, Proposition 3.26] the proof is complete.

Corollary 5.7

$$\mathcal{K}(H^2_\Lambda(S)) \subset \mathcal{T}(S).$$

Proof. The C^*-algebra $\mathcal{K}(L^2(S))$ of compact operators on $L^2(S)$ is given by the image of $C^*(S) \otimes_\delta \mathcal{C}_0(S)$ under Katayama's isomorphism Φ [LPRS, p. 768]. By Lemma 5.13 $\mathcal{K}(L^2(S))$ is contained in $\Phi(C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)) = \mathcal{K}_\Lambda(L^2(S))$. Since every $x \in \mathcal{K}(H^2_\Lambda(S))$ can be extended trivially to a compact operator on $L^2(S)$, we get $\mathcal{K}(H^2_\Lambda(S)) = l_E\,\mathcal{K}(L^2(S))l_E \subset l_E\,\mathcal{K}_\Lambda(L^2(S))\,l_E = \mathcal{T}(S)$.

As a consequence of Lemma 5.6 the Fourier transform $\hat{f}$ of every $f \in L^1(iX)$ is contained in $\mathcal{C}_0^\Lambda(X^\#)$. Thus $\hat{\pi}_{tF}(\hat{f})(\bar{\xi}) = \lim_{\xi_n \xrightarrow{tF} \bar{\xi}} \hat{f}(\xi_n) = 0$ for all $\bar{\xi}$, implying $\pi_{tF}(f) = 0$ whenever $tF \neq (0)$. It follows that $\nu_{tF}(l_f\, g) = 0$ for all generators of $\mathcal{K}(L^2(S))$, and so

$$\nu_{tF}\left(\mathcal{K}(L^2(S))\right) = 0. \tag{5.25}$$

For the translated representations one has the dual action $Ad(\gamma_s)(\mathcal{K}(L^2(S))) = \mathcal{K}(L^2(S))$ and so for all $s \in S$ $\nu^s_{tF}(\mathcal{K}(L^2(S))) = \nu_{tF}(Ad(\gamma_s))(\mathcal{K}(L^2(S))) = (0)$ and hence

$$\mathcal{K}(L^2(S)) \subset \bigcap_{dim(F)\geq 1} \bigcap_{t\in T/T_F} \bigcap_{s\in S/S_F} Ker\, \nu^s_{tF} \tag{5.26}$$

and (5.26) implies

$$\mathcal{K}(H^2_\Lambda(S)) \subset \bigcap_{dim(F)\geq 1} \bigcap_{t\in T/T_F} \bigcap_{s\in S/S_F} Ker\, \rho^s_{tF}. \tag{5.27}$$

Our aim is to prove equality, i.e.

Theorem 5.8

$$\mathcal{K}(H^2_\Lambda(S)) = \bigcap_{dim(F)\geq 1} \bigcap_{t\in T/T_F} \bigcap_{s\in S/S_F} Ker\, \rho^s_{tF}. \tag{5.28}$$

The proof uses techniques from C*-duality [LPRS]. Let π, μ and W denote the direct sum of π_{tF}, μ_{tF} and $W_{tF} = W_{\bar{S}}$ (respectively), where $tF = t\Lambda'_\pm$ or $tF = \Lambda_0$. Then (π, W) defines a covariant pair of representations of $(C^*_\Lambda(S), S, \delta)$.

Proposition 5.9 *For all* $x \in C^*_\Lambda(S)\otimes_\delta \mathcal{C}_0(S)$, $h = (h_{tF}) \in \sum_{tF=t\Lambda'_\pm,\Lambda_0} L^2(tS_F)$, $\phi = (\phi_{tF}) \in \sum_{tF=t\Lambda'_\pm,\Lambda_0} L^2(S)$, $s = (s_{tF}) \in \prod_{tF=t\Lambda'_\pm,\Lambda_0} tS_F$ *and* $u = (u_{tF}) \in \prod_{tF=t\Lambda'_\pm,\Lambda_0} S$ *we have*

$$(\pi\otimes I)^-(x)(h\otimes\phi)(s,u) = (\pi\otimes\mu^{s^{-1}u})(x)(m(h\otimes\phi^{s^{-1}u}))(s), \tag{5.29}$$

where

$$s^{-1}u = (s^{-1}_{tF}u_{tF}) \in \prod_{tF=t\Lambda'_\pm,\Lambda_0} S,\ \mu^{s^{-1}u} = \sum_{tF=t\Lambda'_\pm,\Lambda_0} \mu_{tF}^{s^{-1}_{tF}u_{tF}},\ I = \sum_{tF=t\Lambda'_\pm,\Lambda_0} i$$

and

$$m(h\otimes\phi)(s) = (m(h_{tF}\otimes\phi_{tF})(s_{tF})) = (h_{tF}(s_{tF})\cdot\phi_{tF}(s_{tF}))\,.$$

Proof. First we show 5.29 for generators $x = \delta(l_{fE})(1\otimes g)$ of $C^*_\Lambda(S)\otimes_\delta \mathcal{C}_0(S)$, and then prove that it respects the C^*-structure. With the above notations

$$\begin{aligned}
&(\pi\otimes I)(x)(h\otimes\phi)(s,u) = (\pi\otimes I)(\delta(l_{fE})(1\otimes g))(h\otimes\phi)(s,u)\\
&= W(\pi(l_{fE})\otimes 1)W^*(h\otimes g\,\phi)(s,u) = (\pi(l_{fE})\otimes 1)W^*(h\otimes g\,\phi)(s,s^{-1}u)\\
&= \pi(l_{fE})\mu^{s^{-1}u}(g)(m(h\otimes\phi^{s^{-1}u}))(s) = \pi\otimes\mu^{s^{-1}u}(x)(m(h\otimes\phi^{s^{-1}u}))(s).
\end{aligned}$$

To see that 5.29 is multiplicative for $x,y\in C^*_\Lambda(S)\otimes_\delta\mathcal{C}_0(S)$, we know that $(\pi\otimes I)(y)(h\otimes\phi)$ is an element of $\sum\limits_{tF=t\Lambda'_\pm,\Lambda_0} L^2(tS_F)\otimes \sum\limits_{tF=t\Lambda'_\pm,\Lambda_0} L^2(S)$, thus can be approximated by functions $f_1\otimes f_2$. Hence

$$\begin{aligned}
&(\pi\otimes I)(x\cdot y)(h\otimes\phi)(s,u)\\
&= (\pi\otimes I)(f_1\otimes f_2)(s,u) = (\pi\otimes\mu^{s^{-1}u})(x)(m(f_1\otimes f_2^{s^{-1}u}))(s)\\
&= (\pi\otimes\mu^{s^{-1}u})(x)(u\longmapsto(\pi\otimes\mu^{r^{-1}rs^{-1}u})(y)(m(h\otimes\phi^{r^{-1}rs^{-1}u})))(s)\\
&= (\pi\otimes\mu^{s^{-1}u})(x)(\pi\otimes\mu^{s^{-1}u})(y)(m(h\otimes\phi^{s^{-1}u}))(s)\\
&= (\pi\otimes\mu^{s^{-1}u})(x\cdot y)(m(h\otimes\phi^{s^{-1}u}))(s).
\end{aligned}$$

Similar arguments give the *-property:

$$\begin{aligned}
&(\pi\otimes I)(x^*)(h\otimes\phi)(s,u) = (\pi\otimes I)(1\otimes g^*)(\pi\otimes I)(\delta(l^*_{fE}))(h\otimes\phi)(s,u)\\
&= g^*(t)(\pi(l^*_{fE})(m(h\otimes\phi^{s^{-1}u}))(s) = (g^*)^{s^{-1}u}(s)\pi(l^*_{fE})(m(h\otimes\phi^{s^{-1}u}))(s)\\
&= (\pi\otimes\mu^{s^{-1}u})(x^*)(m(h\otimes\phi^{s^{-1}u}))(s).
\end{aligned}$$

Corollary 5.10

$$\bigcap_{tF=t\Lambda'_\pm,\Lambda_0}\bigcap_{r\in S} Ker(\pi_{tF}\otimes\mu^r_{tF}) = Ker(\pi\otimes I)^- \ \textit{(strict extension)}.$$

Proof. Let $x\in \bigcap\limits_{tF=t\Lambda'_\pm,\Lambda_0}\bigcap\limits_{r\in S} Ker(\pi_{tF}\otimes\mu^r_{tF})$, then $(\pi_{tF}\otimes\mu_{tF}^{s_{tF}^{-1}u_{tF}})(x) = 0\in\mathcal{L}(L^2(tS_F))$ for all $(s_{tF},u_{tF})\in tS_F\times S$, hence $(\pi\otimes\mu^{s^{-1}u})(x)=0$ on $\sum\limits_{tF=t\Lambda'_\pm,\Lambda_0} L^2(tS_F)$. Therefore $(\pi\otimes\mu^{s^{-1}u})(x)(m(h\otimes\phi^{s^{-1}u}))=0$ for all h and ϕ, thus for all s, u

$$(\pi\otimes I)^-(x)(h\otimes\phi)(s,u) = (\pi\otimes\mu^{s^{-1}u})(x)(m(h\otimes\phi^{s^{-1}u})) = 0,$$

implying $(\pi\otimes I)^-(x)=0$ and we have proved

$$\bigcap_{tF=t\Lambda'_\pm,\Lambda_0}\bigcap_{r\in S} Ker(\pi_{tF}\otimes\mu^r_{tF})\subset Ker(\pi\otimes I)^-. \tag{5.30}$$

The converse inclusion is trivial.

Proposition 5.11 *The induced coaction δ on $Ker\pi$ satisfies*

$$Ker(\pi \otimes I)^{-} \subset Ker\pi \otimes_{\delta} \mathcal{C}_0(S).$$

Proof. By [LPRS, Proposition 4.9] $Ker\pi$ is an ideal invariant under δ [LPRS, Definition 4.1] and $\delta^{Ker\pi}$, given by [LPRS, Lemma 4.6], defines a coaction of S on $C^*_\Lambda(S)/Ker\pi$. Define the quotient map $\Theta : C^*_\Lambda(S) \to C^*_\Lambda(S)/Ker\pi$ and let $(\Theta \otimes_\delta i)$ be the surjection of $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$ onto $C^*_\Lambda(S)/Ker\pi \otimes_{\delta^{Ker\pi}} \mathcal{C}_0(S)$ induced by $(\Theta \otimes i)^{-}$ on the multiplier algebra $M(C^*_\Lambda(S) \otimes \mathcal{K}(L^2(S)))$. The proofs of [LPRS, Lemma 4.6, Proposition 4.9] imply

$$Ker(\Theta \otimes i) = Ker\Theta \otimes C^*(S) = Ker\pi \otimes C^*(S) = Ker(\pi \otimes i), \tag{5.31}$$

so we get by [LPRS, Lemma 1.1] that $Ker(\pi \otimes i)^{-} = Ker(\Theta \otimes i)^{-}$. Therefore the kernel of $(\pi \otimes i)^{-}$ restricted to the cocrossed product $C^*_\Lambda(S) \otimes_\delta \mathcal{C}_0(S)$ agrees with $Ker(\Theta \otimes_\delta i)$. The proof of [LPRS, Proposition 4.8] based on properties of the quotient coaction $\delta^{Ker\pi}$ and Katayama's Theorem, yields the equation

$$Ker\pi \otimes_\delta \mathcal{C}_0(S) = Ker(\Theta \otimes_\delta i), \tag{5.32}$$

which completes the proof.

Lemma 5.12

$$\hat{a} \mid_\Lambda \in \mathcal{C}_0(\Lambda) \text{ for all } a \in Ker\pi. \tag{5.33}$$

Proof. For $a \in Ker\,\pi$ we get $\hat{a} \in Ker\hat{\pi}_{tF}$ whenever $tF \neq (0)$. Suppose $\xi_n = t_n \cdot (\xi_n^1, \xi_n^2)$, $(\xi_n^1, \xi_n^2) \in X_0$ is a sequence in Λ converging to infinity with the property $|\hat{a}(\xi_n)| > \epsilon$ for all n. Since $\mathbf{T}$ is compact we find a subsequence (t_{n_l}) converging to a $t_0 \in \mathbf{T}$. Write l for n_l.

<u>1. Case:</u> One of the sequences $\mathrm{pr}_{\langle t_0 \Lambda'_\pm \rangle}(\xi_l)$ is bounded.
Without loss of generality $\mathrm{pr}_{\langle t_0 \Lambda'_+ \rangle}(\xi_l)$ is bounded, hence we find a subsequence (ξ_{l_m}) such that $\mathrm{pr}_{\langle t_0 \Lambda'_+ \rangle}(\xi_{l_m}) \to \xi' \in \langle t_0(\Lambda'_+) \rangle$. Now $t_{l_m}(\xi_{l_m})$ is admissible for $\hat{\pi}_{t_0\Lambda'_-}$ and the definition of $\hat{\pi}_{t_0\Lambda'_-}$ yields $0 = |\hat{\pi}_{t_0\Lambda'_-}(\hat{a})(\bar{\xi})| = \lim_{m\to\infty} |\hat{a}(\xi_{l_m})|$. Since $\mathbf{T}$ acts continuously on $\mathcal{C}_0^\Lambda(X^\#)$ we get the contradiction

$$\epsilon < |\hat{a}(t_{l_m}\xi_{l_m})| \leq |\hat{a}(t_{l_m}\xi_{l_m}) - \hat{a}(\xi_{l_m})| + |\hat{a}(\xi_{l_m})| \to 0. \tag{5.34}$$

<u>2. Case:</u> None of the sequences $\mathrm{pr}_{\langle t_0 \Lambda'_\pm \rangle}(\xi_l)$ is bounded.
Now we find a subsequence (ξ_{l_m}) admissible for $\hat{\pi}_\Lambda$ and get in view of Remark 4.9 the contradiction

$$0 = |\bar{\psi}(0)| = \lim_{m\to\infty} |\psi(\xi_{l_m})| = \lim_{m\to\infty} |\psi(t_{l_m}\xi_{l_m})| > \epsilon. \tag{5.35}$$

Lemma 5.13

$$l_E \left(\bigcap_{tF=t\Lambda'_{\pm},\Lambda_0} \bigcap_{s\in S} Ker\, \nu^s_{tF} \right) l_E \subset \mathcal{K}(H^2_\Lambda(S)).$$

Proof. Corollary 5.10 and Proposition 5.11 yield together with Katayama's Theorem

$$\bigcap_{tF=t\Lambda'_{\pm},\Lambda_0} \bigcap_{s\in S} Ker\, \nu^s_{tF} \subset \Phi\left(Ker\pi \otimes_\delta \mathcal{C}_0(S)\right). \tag{5.36}$$

Each element a of the right hand side is a limit of sums of products $(\sum_i \prod_j l_{a_{ij}}\, g_{ij})$, with $a_{ij} \in Ker\pi,\ g_{ij} \in \mathcal{C}_0(S)$. Let $u_i \in \mathcal{C}_0(X^{\#})$ be a function with $u_i\mid_\Lambda = \widehat{a_{i1}}$ and $(u_{i_m}) \subset A(S)$ be a sequence converging uniformly to u_i. Thus $\lim_{m\to\infty} \chi_\Lambda u_{i_m} = \chi_\Lambda u_i = \chi_\Lambda \widehat{a_{i1}}$ and applying $Ad(\mathcal{F})$ yields

$$\lim_{m\to\infty} l_E\, l_{\widehat{u_{i_m}}}\, g_{i1} \prod_{i>1} a_{ij}\, g_{ij} = l_E\, a_{i1} g_{i1} \prod_{i>1} a_{ij}\, g_{ij}. \tag{5.37}$$

Since $l_{\widehat{u_{i_m}}} g_{i1} \in \mathcal{K}(L^2(S))$ and this is an ideal we see that $l_E\, l_{\widehat{u_{i_m}}} g_{ij} \prod_{i>1} a_{ij}\, g_{ij}$ is an element of $l_E \mathcal{K}(L^2(S))$ for all m. Therefore

$$l_E \prod_j a_{ij}\, g_{ij} \in l_E\, \mathcal{K}(L^2(S)) \tag{5.38}$$

for all of the above products and hence $l_E\, a \in l_E\, \mathcal{K}(L^2(S))$, thus $l_E\, a\, l_E \in l_E\, \mathcal{K}(L^2(S))\, l_E = \mathcal{K}(H^2_\Lambda(S))$.

In order to complete the proof of Theorem 5.8 let $l_E\, a\, l_E$ be an element of the right side of (5.28) and suppose without loss of generality $a^* = a$. As in Theorem 5.4 one shows for all $tF = t\Lambda'_{\pm}$

$$\nu^s_{tF}(a\, l_E\, a\, l_E\, a) = \nu^s_{tF}(a) \rho^s_{tF}(l_E\, a\, l_E) \nu^s_{tF}(a) = 0, \tag{5.39}$$

implying by Lemma 5.13 that

$$l_E\, a\, l_E\, a\, l_E\, a\, l_E = (l_E\, a\, l_E)^3 \in \mathcal{K}(H^2_\Lambda(S)). \tag{5.40}$$

Let $\mathcal{L}(H^2_\Lambda(S))/\mathcal{K}(H^2_\Lambda(S))$ be the Calkin C^*-algebra, then the Gelfand-Naimark Theorem yields

$$||l_E\, a\, l_E||^3_{\mathcal{L}/\mathcal{K}} = ||(l_E\, a\, l_E)^3||_{\mathcal{L}/\mathcal{K}} = 0 \tag{5.41}$$

and Theorem 5.8 is proved.

Remark 5.14 Applying the same reasoning to each 1-dimensional face $t\Lambda'_\pm$ one shows

$$l_{E_{tF}}\,\mathcal{K}(L^2(tS_F))\,l_{E_{tF}} = \bigcap_{\sigma\in tS_F} Ker\rho^\sigma_{tF} \tag{5.42}$$

where ρ^σ_{tF} is the vertex representation of $\mathcal{L}(L^2(tS_F))$ (translated by σ) and E_{tF} is the corresponding Szegö projection on the half-line.

The next Theorem is our main result:

Theorem 5.15 *For the Toeplitz algebra $\mathcal{T}(S)$ (over the exterior of the double lightcone) define a filtration*

$$I_0 = (0) \triangleleft I_1 = \mathcal{K}(H^2_\Lambda(S)) \triangleleft I_2 \triangleleft I_3 = T(S) \tag{5.43}$$

by

$$I_k = \bigcap_{dim(F)=k}\ \bigcap_{s\in S/S_F}\ \bigcap_{t\in T/T_F} Ker\rho^s_{tF}. \tag{5.44}$$

Then the graduation is as follows

$$I_3/I_2 \cong \mathcal{C}_0(S) \tag{5.45}$$

$$I_2/I_1 \cong \int^\oplus_{dim(F)=1}\int^\oplus_{s\in S/S_F}\int^\oplus_{t\in T/T_F} \mathcal{K}(l_{E_{tF}}\,L^2(tS_F)) \tag{5.46}$$

(continuous field of compact operators)

$$I_1/I_0 \cong \mathcal{K}(H^2_\Lambda(S)). \tag{5.47}$$

Proof. Define

$$\rho_k := \int^\oplus_{dim(F)=k}\int^\oplus_{s\in S/S_F}\int^\oplus_{t\in T/T_F} \rho^s_{tF}. \tag{5.48}$$

Then $Ker\rho_k = I_k$ by definition and we have to show that

$$\rho_k(I_{k+1}) = \int^\oplus_{dim(F)=k}\int^\oplus_{s\in S/S_F}\int^\oplus_{t\in T/T_F} \mathcal{K}(l_{E_{tF}}\,L^2(tS_F)). \tag{5.49}$$

For $k = 0$ we have $\rho_0(I_1) = I_1 = \mathcal{K}(H^2_\Lambda(S))$, as shown in Theorem 5.8, since $F = (0), T_F = T$ and $S_F = S$. For $k = 1$ let $x \in I_2 = Ker\rho_2$ (given by the translated vertex representations). Then $\rho^s_{tF}(x)$ is compact for all $t \in T/T_F = T$ and $s \in S/S_F$, since for each $\sigma \in tS_F$ we have $\rho^\sigma_{tF}(\rho^s_{tF}(x)) = \rho^{\sigma+s}_\Lambda(x) = 0$, where ρ^σ_{tF} is as in Remark 5.14. It is easy to see that $\rho^s_{tF}(x)$ depends continuously on $t \in T$ and $s \in S_F$ (recall $F = \Lambda'_\pm$). Since

$$\rho^s_{tF}(x)(l_E\,(l_{gE}\,f)l_E\,) = l_{E_{tF}}\,(l_{\bar{g}E_{tF}}\,\bar{f}_s)l_{E_{tF}} \tag{5.50}$$

(with the bar denoting restriction to tS_F) it follows that (5.50) vanishes as $s \to \infty$ in S/S_F since $f \in \mathcal{C}_0(S)$. Thus $\rho_1(x)$ belongs to the continuous field algebra (5.49). Also, (5.50) and Lemma 5.6, applied to the half-line case tS_F, imply that $\rho^s_{tF}(I_2)$ gives all compact operators. Now suppose $(F_1, s_1, t_1), (F_2, s_2, t_2)$ are distinct in the sense that either $F_1 \neq F_2$ or $F_1 = F_2$ and $s_1 - s_2 \notin S_F$ or $t_1 \neq t_2$. If $s_1 + t_1 S_{F_1} \neq s_2 + t_2 S_{F_2}$ there exists $x = l_E\, f\, l_E$ with $f_{s_1}\mid_{t_1F_1} = 0, f_{s_2}\mid_{t_2F_2} \neq 0$. Then for any $y = l_E\, g l_E$ we have $z := xy - yx \in I_2$ and $\rho^{s_1}_{t_1F_1}(z) = 0$ but $\rho^{s_2}_{t_2F_2}(z) \neq 0$ since g is still arbitrary. If $s_1 + t_1 S_{F_1} = s_2 + t_2 S_{F_2}$ we construct z as in Lemma 5.5 by taking a suitable function ψ, now even fulfilling $\lim \psi(\xi) = 0$ if $\xi \to \infty$ relative Λ to ensure that $z \in I_2$. Again $\rho^{s_1}_{t_1F_1}(z) = 0, \rho^{s_2}_{t_2F_2}(z) \neq 0$, so that $\rho^{s_2}_{t_2F_2}(\mathcal{T}(S) z \mathcal{T}(S)) = \mathcal{K}(l_{E_{t_2F_2}} L^2(t_2 S_{F_2}))$. The Stone-Weierstrass Theorem for operator fields [D1, 11.5.3] completes the proof. For $k = 2$, the same reasoning shows the assertion using the scalar Stone-Weierstrass Theorem.

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