Sp(3) STRUCTURES ON 14-DIMENSIONAL MANIFOLDS

ILKA AGRICOLA, THOMAS FRIEDRICH, AND JOS HÖLL

Abstract. The present article investigates Sp(3) structures on 14-dimensional Riemannian manifolds, a continuation of the recent study of manifolds modeled on rank two symmetric spaces (here: SU(6)/Sp(3)). We derive topological criteria for the existence of such a structure and construct large families of homogeneous examples. As a by-product, we prove a general uniqueness criterion for characteristic connections of G structures and that the notions of biinvariant, canonical, and characteristic connections coincide on Lie groups with biinvariant metric.

1. Introduction

1.1. **Background.** The present article is a contribution to the investigation of Riemannian manifolds modeled on rank two symmetric spaces, carried out by different authors in recent years (for example, [BN07], [CF07], [N08], [ABBF11], [CM12]). They constitute an interesting new class of special geometries that goes back to Cartan's classical study of isoparametric hypersurfaces ([Ca38], [Ca39]), as we shall now explain.

A Riemannian manifold immersed in a space form with codimension one is called an isoparametric hypersurface if its principal curvatures are constant; the main case of interest are immersions into spheres $S^{n-1} \subset \mathbb{R}^n$, the case we shall be interested in henceforth. If one denotes by p the number of different principal curvatures, Cartan proved that for $p = 1, 2$ only certain spheres are possible, while for $p = 3$, tubes of constant radius over an embedding of $\mathbb{KP}^2$ into S^{n-1} are possible for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$: Hence, for $p = 3$, the dimension n must be $5, 8, 14$, or 26. The main key of the construction are the so called Cartan-Münzner polynomials, homogeneous harmonic polynomials F of degree p satisfying $\|\mathrm{grad} F\|^2 = p^2\|x\|^{2p-2}$. The level sets of $F\big|_{S^{n-1}}$ define an isoparametric hypersurface family. Geometrically, F can be understood as a symmetric rank p tensor Υ, and each level set M will be invariant under the stabilizer G_Υ of Υ. Hence, isoparametric hypersurfaces lead to Euclidean spaces $\mathbb{R}^n$ admitting a symmetric rank p tensor Υ and a G_Υ structure, and, for $p = 3$, this leads us in a natural way to manifolds of dimension $5, 8, 14$, and 26.

The relation to rank two symmetric spaces is as follows: If $M^{n-2} \subset S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is orbit of some Lie group $G \subset \mathrm{SO}(n)$, then it is automatically isoparametric. Hence, the classification of homogeneous isoparametric hypersurfaces can be deduced from the classification of all subgroups $G \subset \mathrm{SO}(n)$ such that the codimension in S^{n-1} (resp. $\mathbb{R}^n$) of its principal G-orbit is one (resp. two). By results of Hsiang and Lawson, this is exactly the case for the isotropy representations of rank 2 symmetric spaces [HL71], [HH80]. From the root data of the symmetric space, one deduces that for $p = 3$, only 4 symmetric spaces are possible, namely, $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathrm{SU}(3)$, $\mathrm{SU}(6)/\mathrm{Sp}(3)$, and E_6/F_4. Their relation to the division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ (see Cartan's result) is through their isotropy representations, they are realized on trace free symmetric endomorphisms (see Table 1).

We are interested in Riemannian manifolds in these 4 dimensions admitting a symmetric, trace free, 3-tensor Υ [N08]; its stabilizer is then resp. $\mathrm{SO}(3)$, $\mathrm{SU}(3), \mathrm{Sp}(3)$, or F_4. The 5-dimensional case and the corresponding $SO(3)$ structures were studied by several authors in [ABBF11], [BN07], [CF07]. For the 8-dimensional case and the corresponding SU(3) structures, we refer to [H01], [W08], and [P11]. The present paper will be the first dealing with $n = 14$. As far as we

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dimension	5	8	14	26
symmetric model	SU(3)/SO(3)	SU(3)	SU(6)/Sp(3)	E_6/F_4
isotropy rep.	SO(3) on $S^2_0(\mathbb{R}^3)$	SU(3) on $S^2_0(\mathbb{C}^3)$	Sp(3) on $S^2_0(\mathbb{H}^3)$	E_6 on $S^2_0(\mathbb{O}^3)$

Table 1. Rank two symmetric spaces and their isotropy representations

know, nothing is known for manifolds modeled on the exceptional symmetric space E_6/F_4. From the experience of the present work, one can expect the computations to be challenging, but this case has the charm that it is the first occurrence of the exceptional Lie group F_4 in differential geometry.

1.2. **Outline.** By definition, an Sp(3) structure on a 14-dimensional Riemannian manifold will be a reduction of the frame bundle to an Sp(3)-bundle. We take a closer look at Sp(3) structures, and classify the different types through their intrinsic torsion. This is the first occurrence where the high dimension implies the failure of standard techniques: we were not able to prove the uniqueness of the so-called characteristic connection of an Sp(3) structures in the usual way, and therefore proved a general uniqueness criterion which is valuable in its own (Theorem 2.1), based on the skew holonomy theorem from [AF04] and [OR12].

We then derive some topological conditions for a 14-dimensional manifold to carry an Sp(3) structure. They are a consequence of the computation of the cohomology ring $H^*(BSp(3)\,;\mathbb{Z}) = \mathbb{Z}[q_4, q_8, q_{12}]$ for some $q_i \in H^i(BSp(3))$, see [MT91]. In particular, for a compact oriented Riemannian manifold with Sp(3) structure the Euler characteristic as well as the i-th Stiefel-Whitney classes ($i \neq 4, 8, 12$) must vanish. Any Sp(3) structure on a 14-dimensional manifold induces a unique spin structure. Besides SU(6)/Sp(3), we will construct large families of manifolds admitting an Sp(3) reduction.

The next section is devoted to the existence problem of Sp(3) structures (and other G structures) on Lie groups—for example, whether G_2 carries an Sp(3) structure. For Lie groups equipped with a biinvariant metric, we prove that the notions of characteristic, canonical, and biinvariant connections coincide, and that these are precisely the connections induced by the commutator (Theorem 3.1). The link to Sp(3) structure is subtle: Firstly, this result treats the case excluded in Theorem 2.1; secondly, the result is intricately linked to previous work by Laquer on biinvariant connections [L92a], [L92b], in which the rank two symmetric spaces and the Lie groups U(n), SU(n) play an exceptional role.

The longest part of the paper is devoted to the explicit construction and investigation of 14-dimensional homogeneous manifolds with Sp(3) structure, hence proving that such manifolds exist and that they carry a rich geometry. The manifolds are a higher dimensional analogue of the Aloff-Wallach space, SU(4)/SO(2), the related quotients U(4)/SO(2) × SO(2), U(4) × U(1)/SO(2) × SO(2) × SO(2), and finally SU(5)/Sp(2) (this is the same manifold as the symmetric space SU(6)/Sp(3), but the homogeneous structure is different). In all situations, there are large families of metrics admitting an Sp(3) structure with characteristic connection. For the first three spaces, the qualitative result is the following: the Sp(3) structure is of mixed type, the characteristic torsion is parallel, and its holonomy is contained in the maximal torus of Sp(3). For the last example, the picture is different: It is a 3-parameter deformation of the integrable Sp(3) structure (i. e. the structure corresponding to the symmetric space), it is of mixed type for most metrics, but of pure type for some, the characteristic connection has parallel torsion for a 2-parameter subfamily, and its holonomy lies between Sp(2) and Sp(3). The Appendix contains the explicit realizations of representations needed for performing the calculations.

1.3. **Acknowledgments.** Some of the results presented in this article rely on involved representation theoretic computations. These were obtained with help of the computer algebra system

LiE [vL00]; hence, we thank Marc van Leeuwen (as representative of the whole LiE team) for making such a nice tool available to the scientific community. The computer algebra system Maple was also intensively used. Some preliminary results of this article appeared in the last author's diploma thesis [H11]. The first author acknowledges financial support by the DFG within the priority programme 1388 "Representation theory". The last author is funded through a Ph. D. grant of Philipps-Universität Marburg.

2. Definition and properties of Sp(3) structures

2.1. **Basic set-up.** The 14-dimensional irreducible representation V^{14} of the Lie group Sp(3) gives rise to an embedding $\mathrm{Sp}(3) \subset \mathrm{SO}(14)$. One possible realization of this representation is by conjugation on trace free hermitian quaternionic endomorphisms of $\mathbb{H}^3$, denoted by $S^2_0(\mathbb{H}^3)$. Therefore, it is natural to realize the Lie Group $Sp(3)$ as quaternionic, hermitian endomorphisms of $\mathbb{H}^3$:

$$\mathrm{Sp}(3) = \{g \in \mathrm{SU}(6) \mid g^t J g = J\} = \{g \in \mathrm{GL}(3, \mathbb{H}) \mid g g^t = \mathbf{I}_3\}, \text{ where } J = \begin{bmatrix} 0 & \mathbf{I}_3 \\ -\mathbf{I}_3 & 0 \end{bmatrix}$$

and $\mathbf{I}_3$ denotes the identity of $\mathbb{C}^3$ (respectively $\mathbb{H}^3$). The second equality is established by $g = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix} \mapsto A + jB$, $\{1, i, j, k\}$ being the usual quaternionic units. Thus we get the Sp(3)-representation as

$$\varrho(g)X \ := \ gXg^{-1} \text{ for } g \in \mathrm{Sp}(3),\ X \in S^2_0(\mathbb{H}^3) \ \cong \ V^{14}.$$

We give a precise description of this representation in Appendix A. The space $S^2(\mathbb{H}^3)$ of symmetric quaternionic endomorphisms of $\mathbb{H}^3$ is a classical Jordan algebra with respect to the product $X \circ Y := \frac{1}{2}(XY + YX)$. We define a symmetric $(3,0)$-tensor Υ by polarization from the trace,

$$\Upsilon(X,Y,Z) \ := \ 2\sqrt{3}[\mathrm{tr}X^3 + \mathrm{tr}Y^3 + \mathrm{tr}Z^3] - \mathrm{tr}(X+Y)^3 - \mathrm{tr}(X+Z)^3 - \mathrm{tr}(Y+Z)^3 + \mathrm{tr}(X+Y+Z)^3.$$

A second tensor is obtained as $\tilde{\Upsilon}(X,Y,Z) := \Upsilon(\bar{X}, \bar{Y}, \bar{Z})$. Because of the non-commutativity of $\mathbb{H}$, the symmetric $(3,0)$-tensors Υ and $\tilde{\Upsilon}$ are not conjugate under the action of SO(14), but they both have stabilizer Sp(3). Alternatively, one may use the Jordan determinant for defining a symmetric tensor; again, the non-commutativity implies the existence of two determinants $\det_1, \det_2$. However, $\det_1(X) = \mathrm{tr}X^3$, hence polarization and hermitian conjugation yields again the same tensors Υ and $\tilde{\Upsilon}$. We observe that, in this special situation, there exists an alternative object realizing the reduction from SO(14) to Sp(3): Sp(3) is the stabilizer of a generic 5-form ω^5 in 14 dimensions. Thus, Sp(3) geometry continues in a natural way the investigation of 3-forms ($n = 7$ and $G = G_2$), and 4-forms ($n = 8$ and $G = \mathrm{Spin}(7)$ as well as all quaternionic Kähler geometries in dimensions $4n$).

By definition, an Sp(3) structure on a 14-dimensional Riemannian manifold (M, g) is a reduction of its frame bundle to a Sp(3) subbundle. This is equivalent to the existence of a $(3,0)$-tensor Υ, which is to be associated with the linear map $TM \to \mathrm{End}(TM)$, $v \mapsto \Upsilon_v$ defined by $(\Upsilon_v)_{ij} = \Upsilon_{ijk} v_k$ with the following properties [N08]

1. it is totally symmetric: $g(u, \Upsilon_v w) = g(w, \Upsilon_v u) = g(u, \Upsilon_w v)$,
2. it is trace-free: $\mathrm{tr}\Upsilon_v = 0$,
3. it reconstructs the metric: $\Upsilon_v^2 v = g(v,v)v$.

A first example of such a manifold is the symmetric space SU(6)/Sp(3). As Kerr shows in [K96, Section 4], this is the space of quaternionic structures on $\mathbb{R}^{12} \cong \mathbb{C}^6$ for a fixed complex structure. Further non symmetric examples will be given in Section 4.

2.2. **Types and general properties of** Sp(3) **structures.** The different geometric types of G structures on a Riemannian manifold (M, g), i. e. of reductions $\mathcal{R}$ of the frame bundle $\mathcal{F}(M)$ to the subgroup $G \subset \mathrm{O}(n)$, are classified via the *intrinsic torsion* ([F03], see also [Sal89], [Fin98]). Given a Riemannian 14-manifold M^{14} with an Sp(3) structure, we consider the Levi-Civita connection Z^g as a $\mathfrak{so}(14)$-valued 1-form on the frame bundle $\mathcal{F}(M^{14})$. If unique, we shall denote the irreducible $\mathfrak{sp}(3)$-representation of dimension n by V^n (in particular, we shall write sometimes $\mathfrak{sp}(3) = V^{21}$). To start with, the complement of the Lie algebra $\mathfrak{sp}(3)$ inside $\mathfrak{so}(14)$ is an irreducible $\mathfrak{sp}(3)$-module V^{70}. Hence, the restriction of Z^g to $\mathcal{R}$ can be split into

$$Z^g\big|_{T\mathcal{R}} \;=\; Z^* \oplus \Gamma \in \mathfrak{so}(14) \;=\; \mathfrak{sp}(3) \oplus V^{70},$$

where Γ is called the *intrinsic torsion*. In every point x, $\Gamma_x \in V^{14} \otimes V^{70}$. The following Lemma may be checked directly with LiE:

Lemma 2.1. $\Lambda^3(V^{14})$ *splits into four irreducible components,*

$$\Lambda^3(V^{14}) \;=\; \mathfrak{sp}(3) \oplus V^{70} \oplus V^{84} \oplus V^{189}\ ,$$

and $V^{14} \otimes V^{70}$ *splits into seven irreducible components,*

$$V^{14} \otimes V^{70} \;=\; \Lambda^3(V^{14}) \oplus V^{14} \oplus V^{90} \oplus V^{512}\ .$$

Thus, there are 7 basic types of Sp(3) structures, classified by the irreducible submodules of $V^{14} \otimes V^{70}$; we call a structure *of type* V^i if Γ is contained in V^i and we call it *of mixed type* if Γ is not contained in one irreducible representation. Recall that a given Sp(3) structures will admit an invariant metric connection with skew symmetric torsion ('a' characteristic connection) if and only if Γ lies in the image of the Sp(3)-equivariant map [F03]

$$\Theta := \mathrm{id} \otimes \mathrm{pr}_{V^{70}} : \quad \Lambda^3(V^{14}) \longrightarrow V^{14} \otimes V^{70}.$$

In this definition, we understand $\Lambda^3(V^{14})$ as a subspace of $V^{14} \otimes \Lambda^2(V^{14})$ and identify $\Lambda^2(V^{14})$ with $\mathfrak{so}(14)$. This shows that Sp(3) structures with $\Gamma \in V^{14} \oplus V^{90} \oplus V^{512}$ cannot admit a characteristic connection. The connection will be unique—and thus will deserve to be called *characteristic connection*—if and only if Θ is injective. For small groups and dimensions, injectivity can often be checked directly, and this is a well-known result for almost Hermitian or G_2 structures. In our case, a direct verification fails for the first time; we will thus prove a general criterion that follows from the skew holonomy Theorem of Olmos and Reggiani [OR12], based on preliminary work from our article [AF04]. Our result generalizes in some sense [OR12, Thm 1.2], stating that the canonical connection of an irreducible naturally reductive space ($\neq S^n$, $\mathbb{RP}^n$ or a Lie group) is unique (i. e. different realizations as a naturally reductive space induce the same canonical connection). The case of an adjoint representation (excluded below) will be treated separately in Section 3.

Theorem 2.1. *Let* $G \subsetneq \mathrm{SO}(n)$ *be a connected Lie subgroup acting irreducibly on* $\mathbb{R}^n$*, and assume that* G *does not act on* $\mathbb{R}^n$ *by its adjoint representation. Let* $\mathfrak{m}$ *be a reductive complement of* $\mathfrak{g}$ *inside* $\mathfrak{so}(n)$, $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$*. Consider the* G*-equivariant map*

$$\Theta := \mathrm{id} \otimes \mathrm{pr}_{\mathfrak{m}} : \quad \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta(T) \;=\; \sum_i e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T).$$

Then $\ker \Theta = \{0\}$*, and hence the characteristic connection of a* G*-structure on a Riemannian manifold* (M, g) *is, if existent, unique.*

Proof. An element $T \in \Lambda^3(\mathbb{R}^n)$ will be in $\ker \Theta$ if an only if all $X \lrcorner T$, identified with elements of $\mathfrak{so}(n)$, lie in $\mathfrak{g}$. In the notation of [AF04], any 3-form $T \in \Lambda^3(\mathbb{R}^n)$ generates a Lie algebra

$$\mathfrak{g}_T^* \;:=\; \mathrm{Lie}\langle X \lrcorner T \,|\, X \in \mathbb{R}^n\rangle,$$

and

$$\ker \Theta \;=\; T(\mathfrak{g}, \mathbb{R}^n) \;:=\; \{T \in \Lambda^3(\mathbb{R}^n) \,|\, \mathfrak{g}_T^* \subset \mathfrak{g}\}.$$

In [OR12], a triple (V, θ, G) is called a *skew holonomy system* if V is an Euclidian vector space, G is a connected Lie subgroup of $\mathrm{SO}(V)$, and $\theta : V \to \mathfrak{g}$ is a totally skew 1-form with values in $\mathfrak{g}$, i.e. $\theta(X) \in \mathfrak{g} \subset \mathfrak{so}(V)$ and $\langle \theta(X)Y, Z\rangle$ defines a 3-form on V. Hence, we see that any $T \in \ker \Theta$ defines a skew holonomy system with $V = \mathbb{R}^n$ and the given G representation, and $\theta(X) = X \lrcorner T$. Furthermore, this skew holonomy system will be *irreducible*, by assumption on the G-representation on $\mathbb{R}^n$. By [AF04], [OR12, Thm 4.1], G cannot act transitively on the unit sphere of $\mathbb{R}^n$, for then $G = \mathrm{SO}(V)$ would hold, and this case was excluded by assumption. Thus, any $T \in \ker \Theta$ defines a non-transitive irreducible skew holonomy system. By the skew holonomy Theorem [OR12, Thm 1.4], $\mathbb{R}^n$ will then itself be a Lie algebra, with the bracket induced by T ($[X, Y] = T(X, Y, -)$), and $G = \mathrm{Ad}\, H$, where H is the connected Lie group associated to the Lie algebra $\mathbb{R}^n$. This case having been excluded by assumption, it follows that any $T \in \ker \Theta$ has to vanish. □

Let us look back at all G-structures modeled on the four rank two symmetric spaces $SU(3)/SO(3)$, $\mathrm{SU}(3)$, $SU(6)/\mathrm{Sp}(3)$, and E_6/E_4. For the 5-dimensional $\mathrm{SO}(3)$-representation, the injectivity of Θ can be established by elementary methods [F03], [ABBF11]. For $\mathrm{SU}(3)$, viewed as a symmetric space, we are dealing with the adjoint representation excluded in Theorem 2.1, and indeed the one-dimensional kernel of Θ was observed by Puhle in [P11]. For the irreducible representations of $\mathrm{Sp}(3)$ on $\mathbb{R}^{14} \cong V^{14}$ and F_4 on $\mathbb{R}^{26}$, Theorem 2.1 is applicable, hence $\ker \Theta = \{0\}$ and the characteristic connection is unique in all situations where at least one such connection exists. Together with the explicit decompositions from Lemma 2.1, we can summarize the result for $\mathrm{Sp}(3)$-structures as follows:

Corollary 2.1. *An* $\mathrm{Sp}(3)$ *structure on a* 14*-dimensional Riemannian manifolds admits a characteristic connection* ∇^c *if and only if the* 14*- ,* 90*- and* 512*-dimensional parts of its intrinsic torsion vanish, and then it is unique.*

Remark 2.1. Even in cases where the G action on $\mathbb{R}^n$ is not irreducible, a modification of the proof of Theorem 2.1 might work. We leave it to the reader to check this for example for the action of $\mathrm{U}(n)$ on $\mathbb{R}^{2n+1}$, thus yielding the uniqueness (if existent) of a characteristic connection for almost metric contact manifolds in all dimensions. Of course, this was shown explicitely before in [FrI02].

Remark 2.2. If the $\mathrm{Sp}(3)$-manifold (M^{14}, g) admits a characteristic connection ∇^c with torsion $T \in \Lambda^3(M^{14})$, it satisfies $\nabla^c \Upsilon = 0$ by the general holonomy principle. But for any $(3, 0)$-tensor field Υ,

$$\nabla^c_V \Upsilon(X, Y, Z) \;=\; \nabla^g_V \Upsilon(X, Y, Z) - \frac{1}{2}[\Upsilon\big(T(V, X), Y, Z\big) + \Upsilon\big(X, T(V, Y), Z\big) + \Upsilon\big(X, Y, T(V, Z)\big)],$$

hence one concludes at once that $\nabla^c \Upsilon = 0$ implies

$$\nabla^g_V \Upsilon(V, V, V) \;=\; 0. \tag{1}$$

Such $\mathrm{Sp}(3)$-manifolds were called *nearly integrable* by Nurowski [N08], in analogy to nearly Kähler manifolds. However, one sees that condition (1) is not a restriction for the $\mathrm{Sp}(3)$ structure, making some of the computations [N08, p. 11] unnecessary. In this paper, we shall just speak of $\mathrm{Sp}(3)$ structures admitting a characteristic connection.

Lemma 2.2. *Suppose that* (M^{14}, g) *is a Riemannian manifold with* $\mathrm{Sp}(3)$*-structure admitting a characteristic connection* ∇ *with torsion* $T \in \Lambda^3(M^{14})$*, and that the torsion is* ∇*-parallel,* $\nabla T = 0$*. Then there exists a* ∇*-parallel* 2*-form* Ω*.*

Proof. The symmetric $(3, 0)$-tensor Υ induces by contraction a ∇-parallel vector field ξ, hence the 2-form $\Omega := \xi \lrcorner T$ will be ∇-parallel as well. □

However, the 2-form Ω will be very degenerate ($\mathrm{Sp}(7)$ is much larger than $\mathrm{Sp}(3)$) and thus it will not induce a symplectic structure. Observe that Ric^∇ will have vanishing eigenvalue in direction ξ.

2.3. **Topological constraints.** Let BSO(14) and BSp(3) be the classifying spaces of SO(14) and Sp(3) respectively. For a 14-dimensional oriented Riemannian manifold M^{14} we consider the classifying map of the frame bundle

$$f : M^{14} \longrightarrow \mathrm{BSO}(14).$$

The existence of a topological Sp(3) structure is equivalent to the existence of a lift $\tilde{f}$,

$$\begin{array}{ccc} & & \mathrm{BSp}(3) \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \iota \\ M^{14} & \xrightarrow{f} & \mathrm{BSO}(14) \end{array}$$

Since the cohomology algebra of the space BSp(3) is generated by three elements in H^4, H^8 and H^{12} (see Theorem 5.6., Chapter III, [MT91]) we immediately obtain the following

Theorem 2.2. *If M^{14} is a compact oriented Riemannian manifold with* Sp(3) *structure, then*

(1) *the Euler characteristic vanishes, $\chi(M^{14}) = 0$,*
(2) *$w_i(M^{14}) = 0$ for $i \neq 4, 8, 12$, where w_i are the Stiefel-Whitney classes.*

Remark 2.3. For example, S^{14} and any product of spheres $S^n \times S^m$ with $m + n = 14$ and m, n both even cannot carry an Sp(3) structure, since the Euler characteristic does not vanish. The requirement $w_1(M^{14}) = w_2(M^{14}) = 0$ means that any manifold with Sp(3) structure is orientable and admits a spin structure. Since Sp(3) is simply connected, the inclusion $\mathrm{Sp}(3) \subset \mathrm{SO}(14)$ admits a unique lift to Spin(14). Thus a Sp(3) structure defines a *unique* spin structure.

Now we are going to construct some examples. In general, let M^n be a manifold with a fixed G-reduction $\mathcal{R}$ of the frame bundle. For any G-representation κ on E_o we consider the associated bundle

$$E \;=\; \mathcal{R} \times_\kappa E_0 \xrightarrow{\pi} M^n \;.$$

The tangent bundle of the manifold E splits into a horizontal and vertical part,

$$T(E) \;=\; T^v(E) \,\oplus\, T^h(E) \;.$$

Since

$$T^h(E) \;=\; \pi^*(T(M^n)) \;=\; \pi^*(\mathcal{R}) \times_G \mathbb{R}^n \,, \quad \text{and} \quad T^v(E) \;=\; \pi^*(E) \;=\; \pi^*(\mathcal{R}) \times_G E_o$$

we obtain the following

Lemma 2.3. *As a manifold, E admits a G structure $TE = \pi^*(R) \times_G (E_0 \oplus \mathbb{R}^n)$.*

With Theorem A.1 in Appendix A.5 we get $V^{14} \;=\; \Delta_5 \,\oplus\, \mathfrak{p}^5 \,\oplus\, \mathfrak{p}^1$ and thus receive

Corollary 2.2. *Any oriented* 1*-dimensional bundle over a* 13*-dimensional Riemannian manifold with an* Sp(2) *structure of type $\Delta_5 \oplus \mathfrak{p}^5$ admits a* $\mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ *structure.*

Let us consider a 5-dimensional manifold M^5 with $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ structure as well as the corresponding spinor bundle. It is a 13-dimensional manifold and Lemma 2.3 gives us the needed Sp(2) structure on it. We summarize the result,

Example 2.1. Any oriented 1-dimensional bundle over the spinor bundle of a 5-dimensional spin manifold admits a $\mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ structure.

Taking a 8-dimensional manifold with a $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ structure $\mathcal{R}$ we consider $M^{13} = \mathcal{R} \times_{\mathrm{Sp}(2)} \mathfrak{p}^5$. Again we have $T(M^{13}) = \mathcal{R} \times_{\mathrm{Sp}(2)} (\Delta_5 \oplus \mathfrak{p}^5)$, leading to a Sp(3) structure on any S^1 bundle over M^{13}.

Example 2.2. Any oriented 1-dimensional bundle over the associated bundle $M^{13} = \mathcal{R} \times_{\mathrm{Sp}(2)} \mathfrak{p}^5$ of a 8-dimensional manifold X^8 with Sp(2) structure $\mathcal{R} \to X^8$ admits a $\mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ structure.

The above examples of Sp(3) spaces being fibrations over special smaller dimensional manifolds used the subgroup $G = \mathrm{Sp}(2) \subset \mathrm{Sp}(3)$ as well as its decomposition of V^{14}. We list the maximal connected subgroups of Sp(3) and their decompositions of V^{14} in Appendix A.5. Particularly interesting are $G = \mathrm{U}(3)$ and $G = SO(3)$.

Example 2.3. A special 9-dimensional real vector bundle over a 5-dimensional manifold equipped with an irreducible SO(3) structure admits a $\mathrm{SO}(3) \subset \mathrm{Sp}(3)$ structure (see [ABBF11]).

Example 2.4. A special 8-dimensional real vector bundle over a 6-dimensional hermitian manifold admits a $\mathrm{U}(3) \subset \mathrm{Sp}(3)$ structure.

3. Sp(3) structures and other G structures on Lie groups

The first 14-dimensional homogeneous space that comes to mind (besides S^{14}) is presumably the Lie group G_2. We will devote this section to the question whether G_2 carries a natural Sp(3) structure. Since it seems that the topic has not been treated before, we shall start with some general comments on G structures on Lie groups.

Let G be a connected compact Lie group with a biinvariant metric g, and $K \subset G$ a connected subgroup of G whose Lie algebra $\mathfrak{k}$ decomposes into center $\mathfrak{z}$ and simple ideals $\mathfrak{k}_i$, i. e. $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_0 \oplus \ldots \oplus \mathfrak{k}_r$. Set $\mathfrak{a} := \mathfrak{k}^\perp$, hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}$. We view G as the homogeneous space $G \times K/\Delta(K)$, where $\Delta K := \{(k,k) : k \in K\}$. D'Atri and Ziller proved in [DZ79, p. 9] that the family of left invariant metrics on G defined by ($\alpha, \alpha_1, \ldots, \alpha_r > 0$, h any scalar product on $\mathfrak{z}$)

$$\langle\,,\,\rangle \;:=\; \alpha \cdot g\big|_{\mathfrak{a}} \oplus h\big|_{\mathfrak{z}} \oplus \alpha_1 \cdot g\big|_{\mathfrak{k}_1} \oplus \ldots \oplus \alpha_r \cdot g\big|_{\mathfrak{k}_r} \tag{2}$$

is naturally reductive for the homogeneous space $G \times K/\Delta(K)$ in the following sense: Write $\mathfrak{g} \oplus \mathfrak{k} = \Delta\mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{p}$ is isomorphic (as a vector space) to $T_e(G \times K/\Delta(K)) \cong \mathfrak{g}$, but with an isomorphism (and thus a commutator) depending on the parameters α, α_i. The metric then satisfies

$$\langle [X,Y]_{\mathfrak{p}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{p}}\rangle \;=\; 0 \quad \forall\; X,Y,Z \in \mathfrak{p}.$$

In the special case that K is chosen to be G, $\mathfrak{a} = 0$ and the metric $\langle\,,\,\rangle$ is precisely a biinvariant metric on G.

By a theorem of Wang [KNI], invariant metric connections ∇ on G (still with respect to its realization as a naturally reductive space) are in bijective correspondence with linear maps $\Lambda : \mathfrak{p} \to \mathfrak{so}(\mathfrak{p})$ that are equivariant under the isotropy representation. As described in [A03, Lemma 2.1, Dfn 2.1], the torsion T of ∇ will be totally skew-symmetric if and only if $\Lambda(X)X = 0$ for all $X \in \mathfrak{p}$ (and this condition is well-known to be equivalent to the fact that the geodesics of ∇ coincide with the geodesics of the canonical connection, [KNII, Prop. 2.9, Ch.X]). Thus, one can give immediately a one-parameter family of invariant metric connections ∇^t with skew torsion, namely the one defined by $\Lambda(X)Y = t[X,Y]_{\mathfrak{p}}$ that was investigated in detail in [A03]. For $t = 0$, ∇^t has holonomy K, thus we can summarize:

Proposition 3.1. *Let G be a connected compact Lie group, $K \subset G$ a connected subgroup, $G \cong G \times K/\Delta K$ as a reductive homogeneous space. For any parameters $\alpha, \alpha_1, \ldots, \alpha_r > 0$, the left invariant metric $\langle\,,\,\rangle$ on G defined by* (2) *is naturally reductive and admits an invariant metric connection with skew torsion and holonomy K.*

In general, this is all we can say; in particular, we do not know about other systematic constructions of interesting K structures on a Lie group G, for example, if K is not a subgroup of G. We shall now investigate further the case $K = G$. First, we can answer the question on G_2 we started with:

Remark 3.1. Since G_2 is simple, there are no center nor non-trivial ideals that would allow for a deformation of the metric, hence $\langle\,,\,\rangle$ has to be a multiple of the negative of the Killing form of G_2. Sp(3) is not a subgroup of G_2, but its maximal *simple* subgroup SU(3) (see Appendix A.5) is also a maximal subgroup of G_2. However, they are not conjugate inside SO(14); this is easiest seen by computing the branching of the 14-dimensional representation of Sp(3) resp. G_2 to their resp. subgroups SU(3); It turns out that these do not coincide. The smaller subgroups do not seem to be very interesting. We conclude that G_2 *does not carry an* $\mathrm{SU}(3) \subset \mathrm{Sp}(3)$ *structure of the type described before.*

Going back to the general case $K = G$, we are now in the situation that $\langle\, ,\, \rangle$ is a biinvariant metric on G; an affine connection ∇ is a bilinear map $\Lambda :\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, with $\nabla_X Y = \Lambda(X)Y$. Alternatively, it is sometimes more useful to formulate the properties of Λ in terms of its dual bilinear map $\lambda :\ \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $\lambda(X,Y) := \Lambda(X)Y$.

Characteristic vs. canonical vs. biinvariant connections on Lie groups. We begin by clarifying the different notions of 'interesting' connections on compact connected Lie groups (still with a biinvariant metric) and their relations.
In [L92a], Laquer defined a *biinvariant connection* on a Lie group G as any $(G\times G)$-invariant connection on the symmetric space $G\times G/\Delta G$, as described through Wang's Theorem in [KNII]. The connection ∇^λ defined by a bilinear map $\lambda :\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ will be biinvariant if and only if [L92a, Thm 6.1]

$$\lambda(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) \ = \ \mathrm{Ad}_g \lambda(X,Y) \quad \forall g\in G.$$

Alternatively, one often uses the adjoint map $\Lambda : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$, $\Lambda_X(Y) = \lambda(X,Y)$, for which the property then reads $\Lambda_{\mathrm{Ad}_g X} = \mathrm{Ad}_g \Lambda_X \mathrm{Ad}_g^{-1}$; of course, Λ is just the map used in Wang' Theorem. Observe that the set of biinvariant connections forms a vector space, so uniqueness is to be expected at best up to a scalar, and that the notion does not depend on the metric. Evidently, $\lambda(X,Y) = c[X,Y]$ is always a biinvariant connection.

Lemma 3.1. *The following conditions for a biinvariant connection* ∇^λ*, on a Lie group* (G,g) *with biinvariant metric are equivalent:*

1. $\Lambda_V \in \mathfrak{so}(\mathfrak{g})$ *for any* $V\in\mathfrak{g}$*, i. e.* $g(\Lambda_V X, Y) + g(X, \Lambda_V Y) \ = \ 0$*;*
2. ∇^λ *is metric;*
3. *The torsion* $T^\lambda(X,Y,Z)$ *of* ∇^λ *is skew symmetric, i. e.* $T\in\Lambda^3(\mathfrak{g})$.

Proof. The equivalence of (1) and (2) is immediate (for any metric). One checks that ∇^λ has torsion and curvature transformation

$$T^\lambda(X,Y) \ = \ \lambda(X,Y) - \lambda(Y,X) - [X,Y], \quad R^\lambda(X,Y) \ = \ [\Lambda_X, \Lambda_Y] - \Lambda_{[X,Y]},$$

which shows the equivalence of (1) and (3) for biinvariant metrics. Observe that Λ_V will not, in general, be a representation; rather, the second formula shows that this is equivalent to ∇^λ being flat. □

On the other hand, the *canonical connection* ∇^c of a reductive homogeneous space $M = \tilde G/\tilde K$ is, by definition, the unique connection induced from the $\tilde K$-principal fibre bundle $\tilde G \to \tilde G/\tilde K$ (alternatively: the ∇^c-parallel tensors are exactly the $\tilde G$-invariant ones). A priori, it depends on the choice of a reductive complement $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{k}}$ in $\tilde{\mathfrak{g}}$, since it has torsion $T^c(X,Y) = -[X,Y]_{\tilde{\mathfrak{m}}}$; for a Lie group (i. e. $\tilde G = G\times G$, $\tilde K = \Delta G$), it turns out that, unlike for naturally reductive spaces (cf. Section 2.2 and the comments to Thm 2.1), each choice of a complement of $\Delta\mathfrak{g} \subset \mathfrak{g}\oplus\mathfrak{g}$ induces a different canonical connection. The easiest way to see this is to construct (some of) them explicitely. One checks that every space ($t\in\mathbb{R}$)

$$\mathfrak{m}_t \ := \ \{X_t := (tX, (t-1)X) \in \mathfrak{g}\oplus\mathfrak{g} \ : \ X\in\mathfrak{g}\} \ \cong \ \mathfrak{g}$$

defines a reductive complement of $\Delta\mathfrak{g}$. One then computes the decomposition of any commutator $[X_t, Y_t]$ in its $\Delta\mathfrak{g}$- and $\mathfrak{m}_t$-part,

$$[X_t,Y_t] \ = \ (t^2[X,Y], (t-1)^2[X,Y]) \ = \ (t^2-t)([X,Y],[X,Y]) + (2t-1)(t[X,Y],(t-1)[X,Y]),$$

Thus, the torsion $T^c(X,Y)$ becomes, after identifying $\mathfrak{m}_t$ with $\mathfrak{g}$ in the obvious way,

$$T^c(X,Y) \ = \ -[X_t,Y_t]_{\mathfrak{m}_t} \ = \ (1-2t)[X,Y]. \tag{3}$$

For a biinvariant metric g, $T^c \in \Lambda^3(\mathfrak{g})$ (and this is equivalent to the property that ∇^c is metric); $t = 1/2$ corresponds to the Levi-Civita connection, while $t = 0, 1$ are the flat $\pm$-connections introduced by Cartan and Schouten (see [AF10] and [R10]). The holonomy of these connections is either trivial ($t = 0, 1$) or G ($t \neq 0, 1$). The corresponding map $\lambda^c: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is $\lambda^c(X,Y) = (1-t)[X,Y]$.

Finally, let us describe characteristic connections on Lie groups—i.e. we take $M = G$ with a biinvariant metric and consider $G \subset \mathrm{SO}(\mathfrak{g})$ through the adjoint representation. Again, we identify $\mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g}$ and decompose it under the action of G into the representations $\mathfrak{so}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{m}$ where, in general, $\mathfrak{m}$ will not be irreducible. The crucial observation is that the intrinsic torsion Γ (Section 2.2 and [F03]) vanishes, $\Gamma = 0$, because the Levi-Civita connection is a $(G \times G)$-invariant connection on G. Hence, $\ker \Theta \subset \Lambda^3(\mathfrak{g})$ parameterizes the space of characteristic connections (recall that a characteristic connection is metric with skew torsion by construction).

Suppose G is a compact connected Lie group. Its Lie algebra splits into $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_q$, where $\mathfrak{z}$ is the center and the $\mathfrak{g}_i$ are the simple ideals in $\mathfrak{g}$. The one-parameter family of connections with torsion given by (3) has then an obvious generalization to a q-parameter family by rescaling the commutator separately in all simple ideals,

$$T(X,Y) \;=\; \sum_{i=1}^{q} \alpha_i [X,Y]\big|_{\mathfrak{g}_i}, \quad \alpha_i \in \mathbb{R}. \tag{4}$$

This connection is certainly biinvariant; it is also a canonical connection for the reductive space $G \times G/\Delta G$, because the complement of $\Delta\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$ can be chosen with a different parameter in each simple ideal $\mathfrak{g}_i$. Finally, it also lies in $\ker \Theta$.

Theorem 3.1. *For a compact connected Lie group G with a biinvariant metric g, the following families of connections coincide:*

1. *metric biinvariant connections with skew torsion,*
2. *metric canonical connections with skew torsion of the reductive spaces $G \times G/\Delta G$,*
3. *characteristic connections,*

and there is exactly one family of connections with these properties, namely the one defined by eq. (4).

Proof. Consider a linear map $0 \neq \lambda: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defining a biinvariant connection. We interpret λ as an intertwining map $\mu : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ for $\mathrm{Ad}\, G$ by setting $\mu(X \otimes Y) := \lambda(X,Y)$; the intertwining property is exactly the biinvariance condition. Hence, the interesting question is to find copies of $\mathfrak{g}$ inside $\mathfrak{g} \otimes \mathfrak{g}$. It is well-known that $\mathfrak{g} \otimes \mathfrak{g}$ splits into the G-modules $\mathfrak{g} \otimes \mathfrak{g} = S^2\mathfrak{g} \oplus \Lambda^2\mathfrak{g}$, and that $\mathfrak{g}$ will always be a submodule of $\Lambda^2\mathfrak{g}$ (however, there are also compact Lie groups for which $\mathfrak{g}$ appears in $S^2\mathfrak{g}$ as we will discuss later). Decompose μ into its symmetric and antisymmetric part, $\mu = \mu^s + \mu^a$, $\mu^s: S^2\mathfrak{g} \to \mathfrak{g}$, $\mu^a: \Lambda^2\mathfrak{g} \to \mathfrak{g}$.

1st case: $\mu^s = 0$, i.e. $\mu = \mu^a$ is antisymmetric. This is the generic case that one would expect, as it includes the family of connections that we already constructed. Since we're only interested in $\mu \neq 0$, its dual map $\tilde{\mu} : \mathfrak{g} \to \mathfrak{g} \subset \Lambda^2\mathfrak{g} \cong \mathfrak{so}(\mathfrak{g})$ exists. The torsion of the connection defined by μ is $T(X,Y,Z) = 2g(\mu(X \otimes Y), Z) - g([X,Y], Z)$; by assumption, it is a 3-form in $\ker \Theta$, and since we knew before that $g([X,Y],Z) \in \ker \Theta$, we can conclude that $g(\mu(X \otimes Y), Z) \in \ker \Theta$ as well. This proves that any biinvariant connection is characteristic in the sense described before. One checks that the argument can be inverted, hence the sets of antisymmetric biinvariant connections and characteristic connection coincide.

Consider now a canonical connection, i.e. the connection induced by the choice of a reductive complement $\mathfrak{m}$ of $\Delta\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$. It is tedious to describe all possible spaces $\mathfrak{m}$; happily it turns out not be necessary (as a remark, we note that different complements will not necessarily induce different connections, for example, the embedding of the center has no influence on the connection). Whatever $\mathfrak{m}$, the torsion $T^c_{\mathfrak{m}}(X,Y) = -[X,Y]_{\mathfrak{m}}$ of its canonical connection is an $\mathrm{Ad}\, G$-equivariant antisymmetric map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$ and hence defines a biinvariant connection with

$\mu^s = 0$. It is a priori not clear whether one can find to any biinvariant connection satisfying $\mu^s = 0$ a reductive complement $\mathfrak{m}$ such that it coincides with its canonical connection, but we will not need this.

2nd case: $\mu^s \neq 0$. We wish to exclude this case. Unfortunately, we have to apply a 'brute force' argument. Recall that we showed that the connections (2) and (3) are (special) biinvariant connections; hence it suffices to prove that a metric biinvariant connection with skew torsion has necessarily $\mu^s = 0$. We will use the classification of biinvariant connections of compact Lie groups given by Laquer. In [L92a, Table I], he decomposed the G-representations $S^2\mathfrak{g}$ and $\Lambda^2\mathfrak{g}$ for all compact simple Lie groups. He confirmed that $\mathfrak{g}$ appears with multiplicity one in $\Lambda^2\mathfrak{g} \subset \mathfrak{g}\otimes\mathfrak{g}$ for all of them, but furthermore, he obtained the surprising result that $\mathfrak{g}$ does not occur in $S^2\mathfrak{g}$ for all of them – except for $G = \mathrm{SU}(n)$, $n \geq 3$ (which includes SO(6), since $\mathfrak{so}(6) \cong \mathfrak{su}(4)$). The Lie group $G = \mathrm{SU}(n)$ has a copy of $\mathfrak{su}(n)$ in $S^2\mathfrak{g}$ as well, corresponding to a symmetric $\mathrm{Ad}\,\mathrm{SU}(n)$-equivariant map $\eta:\ \mathfrak{su}(n)\times\mathfrak{su}(n)\to\mathfrak{su}(n)$ given by [L92a, p.550]

$$\eta(X,Y) \;=\; i\alpha\left[XY+YX-\frac{2}{n}\mathrm{tr}(XY)\cdot I\right],\quad \alpha\in\mathbb{R}. \tag{5}$$

A biinvariant metric on $G = \mathrm{SU}(n)$ is necessarily a multiple of the negative of the Killing form, hence we can take $g(X,Y) = -2n\,\mathrm{tr}(XY)$. An elementary computation shows that, for general $X, Y, Z \in \mathfrak{su}(n)$, the quantity $g(\eta(X,Y),Z)+g(\eta(X,Z),Y) \neq 0$, hence the biinvariant connection defined by η is *not* metric and thus not of relevance for us. In fact, Laquer himself extended his result to *arbitrary* compact Lie groups [L92a, Thm 10.1]. The result is similar, if slightly more involved. Besides $\mathrm{SU}(n)$, only $\mathrm{U}(n)$ admits symmetric maps $\eta:\ \mathfrak{u}(n)\times\mathfrak{u}(n)\to\mathfrak{u}(n)$; they span a 3-dimensional space for $n = 2$ and a 4-dimensional space for $n \geq 3$, and there is an additional antisymmetric map (besides the obvious one $[X,Y]$), namely $\nu(X,Y) = i(X\mathrm{tr}Y - Y\mathrm{tr}X)$. Using that a biinvariant metric on $\mathrm{U}(n)$ is just any positive definite extension to the center of the metric of $\mathrm{SU}(n)$, one checks that none of them yields a metric connection.

All in all, only connections corresponding to the embedding of $\mathfrak{g}$ inside $\Lambda^2\mathfrak{g}$ are candidates for all three types of connections, and these are of course the ones described by eq. (4). This finishes the proof.

Although not necessary, we give an alternative proof of the claim for $\mathfrak{g}$ semisimple (assuming that one already established that the connections (1) and (3) coincide and that they include the connections (2)) — it has the charm that it does not need the classification results of Laquer. However, we were not able to extend it to the compact case without using the classification, so it does not improve the situation much.

We begin with the case that G is simple, hence the adjoint representation is irreducible. As explained in the proof of Theorem 2.1, any $T \in \ker\Theta$ defines then an irreducible skew holonomy system $(V = \mathfrak{g}, \theta, G)$, and for dimensional reasons, $G \neq \mathrm{SO}(\mathfrak{g})$, so the system is non-transitive. By [OR12, Thm 2.4], an irreducible non-transitive skew holonomy system is symmetric, and the map $\theta:\mathfrak{g}\to\mathfrak{g}\subset\mathfrak{so}(\mathfrak{g})\cong\Lambda^2(\mathfrak{g})$ is unique up to a scalar multiple [OR12, Prop. 2.5]—namely, it is given by $\theta(X) = [X,-]\in\Lambda^2(\mathfrak{g})$, and $\theta(X) = X \lrcorner T$. We conclude that for G simple, the space of characteristic connections is indeed given by (3). By the splitting theorem (see [AF04, Section 4] or [OR12, Lemma 2.2]), the conclusion still holds for G semisimple. □

Remark 3.2. The three rank two symmetric spaces SU(3)/SO(3), SU(6)/Sp(3), and E_6/E_4 have a remarkable connection property very similar to the one described for the Lie groups $\mathrm{SU}(n)$, $\mathrm{U}(n)$ in the proof above, again due to Laquer. In [L92b], it was observed that the symmetric spaces $\mathrm{SU}(n)/\mathrm{SO}(n)$, $\mathrm{SU}(2n)/\mathrm{Sp}(n)$, and E_6/E_4 admit invariant affine connections that are *not* induced from the commutator. However, a closer inspection of the three symmetric spaces yields that these connections are not metric with skew-symmetric torsion, and hence do not yield further candidates for a characteristic connection, in agreement with the uniqueness statement from Theorem 2.1. We were not able to relate the existence of these 'exotic' connections to the characteristic connection or any other deeper geometric property of geometries modeled on rank 2 symmetric spaces; but it may be worth mentioning that their existence is linked to the existence of a Jordan product.

4. The geometry of some homogeneous Sp(3) structures

Since there are no Lie groups carrying any reasonable Sp(3) structure, it is natural to ask for homogeneous spaces with such a structure. This section is devoted to the explicit construction of some 14-dimensional homogeneous spaces carrying Sp(3)-structures and their geometric properties.
We choose a reductive complement $\mathfrak{m}$ of $\mathfrak{sp}(3)$ inside $\mathfrak{su}(6)$, $\mathfrak{su}(6) \cong \mathfrak{m} \oplus \mathfrak{sp}(3)$; an explicit realization as well as a description of the 14-dimensional isotropy representation $\varrho(\mathfrak{sp}(3)) \subset \mathfrak{so}(\mathfrak{m}) \cong \mathfrak{so}(14)$ of Sp(3) is being given in Appendix A. The notation will be as follows: The homogeneous spaces will be realized as quotients $M_i = K_i/H_i$ for a running index i, yielding at Lie algebra level the reductive decompositions

$$\mathfrak{k}_i \;\cong\; \mathfrak{m}_i \;\oplus\; \mathfrak{h}_i \;\cong\; \langle K^i_j \mid j = 1..14\rangle \;\oplus\; \langle H^i_j \mid j = 1..r_i\rangle.$$

Again, the explicit elements K^i_j and H^i_j will be listed in the Appendix for each example. We will show that we can identify the subspaces $\mathfrak{m} \cong \mathfrak{m}_i$ inducing $\varrho_i(\mathfrak{h}_i) \subset \varrho(\mathfrak{sp}(3))$ and, consequently, the H_i structure is a reduction of an $Sp(3)$ structure.

4.1. The higher dimensional Aloff Wallach manifold SU(4)/SO(2).

We embed $H_1 = \mathrm{SO}(2)$ in the Lie group $K_1 = \mathrm{SU}(4)$ as

$$SO(2) \ni \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \longmapsto \mathrm{diag}(e^{-it}, e^{-it}, e^{it}, e^{it}) \in \mathrm{SU}(4).$$

The action of $\mathfrak{h}_1 = \mathfrak{so}(2)$ on the 14-dimensional Sp(3)-representation V^{14} splits into four 2-dimensional representations and six trivial ones. For an invariant metric, we choose multiples of the Killing form on the invariant spaces parameterized by coefficients $\alpha, \alpha_2, \ldots > 0$,

$$g^{\alpha,..,\gamma} \;=\; \mathrm{diag}(\alpha, \alpha, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_4, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \beta, \gamma)$$

with respect to the orthonormal basis $K^1_1, \ldots, K^1_{14}$ of $\mathfrak{m}_2$ described in Appendix A.1. The justification for this choice of notation stems from the following result.

Theorem 4.1. *Consider the manifold $M_1 = \mathrm{SU}(4)/\mathrm{SO}(2)$ equipped with the metric $g^{\alpha,..,\gamma}$. For any parameters $\alpha, \alpha_i, \beta, \gamma > 0$, it carries an 98-dimensional space of invariant $Sp(3)$-connections, and for $\alpha = \alpha_2 = .. = \alpha_8$, the $Sp(3)$ structure admits a characteristic connection with torsion $T^{\alpha\beta\gamma} \in \Lambda^3(M_1)$. These $Sp(3)$ structures with characteristic connection have the following properties:*

1. *The characteristic connection has alwas parallel torsion, $\nabla^{\alpha\beta\gamma} T^{\alpha\beta\gamma} = 0$.*
2. *The structure is of mixed type, $T^{\alpha\beta\gamma} \notin V^i$ for $i = 21, 70, 84, 189$.*
3. *The structure is never integrable, i. e. there are no parameters $\alpha, \beta, \gamma > 0$ with vanishing torsion.*
4. *For the characteristic connection $\nabla^{\alpha\beta\gamma}$, the Lie algebra of the holonomy group is a subalgebra of the maximal torus of $\mathfrak{sp}(3)$ and it is*
   - *one-dimensional if $\beta = \alpha = \gamma$,*
   - *two-dimensional if ($\beta = \alpha$ and $\gamma \neq \alpha$) or ($\beta \neq \alpha$ and $\gamma = \alpha$),*
   - *three-dimensional if $\beta \neq \alpha \neq \gamma$.*

Proof. In Appendix A.1 we construct a decomposition of the relevant Lie algebras. By a theorem of Wang [KNI], invariant metric connections $\nabla^{\alpha,..,\gamma}$ are in bijective correspondence with linear maps $\Lambda_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{so}(\mathfrak{m}_1)$ that are equivariant under the isotropy representation ϱ_1,

$$\Lambda_{\mathfrak{m}_1}(\varrho_1(h)X) \;=\; \varrho_1(h)\Lambda_{\mathfrak{m}_1}(X)\varrho_1(h)^{-1} \quad \forall h \in \mathrm{SO}(2),\; X \in \mathfrak{m}_1.$$

A connection is an $Sp(3)$ connection if the image of $\Lambda_{\mathfrak{m}_1}$ is inside $\mathfrak{sp}(3)$, $\Lambda_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{sp}(3)$. One calculates all such maps $\Lambda_{\mathfrak{m}_1}$. They are given by the two following conditions

- $\Lambda_{\mathfrak{m}_1}$ maps the space $\langle K^1_i \mid i = 9..14\rangle$ into the space $\langle \varrho(A_i) \mid i = 1..10, 21\rangle$. This part of $\Lambda_{\mathfrak{m}_1}$ depends on 66 parameters.
- $\Lambda_{\mathfrak{m}_1}$ maps the space $\langle K^1_i \mid i = 1..8\rangle$ into the space $\langle \varrho(A_i) \mid i = 11..18\rangle$. The corresponding (8×8) matrix depends on 32 parameters a_i, $i = 1..32$, via the formulas

$$\begin{bmatrix} M^{1,2} & M^{3,4} & M^{5,6} & M^{7,8} \\ M^{9,10} & M^{11,12} & M^{13,14} & M^{15,16} \\ M^{17,18} & M^{19,20} & M^{21,22} & M^{23,24} \\ M^{25,26} & M^{27,28} & M^{29,30} & M^{31,32} \end{bmatrix}, \quad M^{i,j} := \begin{bmatrix} a_i & -a_j \\ a_j & a_i \end{bmatrix}.$$

Since the torsion of the connection defined by $\Lambda_{\mathfrak{m}}$ is given by [KNI, X.2.3]

$$T(X,Y)_o = \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m} \tag{6}$$

we can calculate that the torsion $T^{\alpha,..,\gamma} \in \Lambda^3(\mathrm{SU}(4)/\mathrm{SO}(2))$ if and only if $\alpha = \alpha_2 = \ .. \ = \alpha_8$ and

$$\Lambda_{m_1}(K^1_{13}) = \frac{\sqrt{2}\,(\alpha - \beta)}{\alpha\sqrt{\beta}}\varrho(A_9), \quad \Lambda_{m_1}(K^1_{14}) = \frac{\sqrt{2}\,(\alpha - \gamma)}{\alpha\sqrt{\gamma}}\varrho(A_{10}),$$

as well as $\Lambda_{m_1}(K^1_i) = 0$ for $i \neq 13, 14$. A closer look at the torsion shows that it never vanishes. For the invariant torsion tensor and $X, Y, V \in \mathfrak{m}$ we have

$$(\nabla_V T)_o(X,Y) = \Lambda(V)T(X,Y)_o - T(\Lambda(V)X,Y)_o - T(X,\Lambda(V)Y)_o \tag{7}$$

and derive that $\nabla T^{\alpha\beta\gamma} = 0$ for all $\alpha, \beta, \gamma > 0$. For $\gamma_{i,j,k} \in \Lambda^3(V^{14})$ and $s = 1..21$ the standard representation ν of $Sp(3)$ on $\Lambda^3(V^{14})$ is given by:

$$\nu(A_s)(\gamma_{i,j,k}) = \sum_l (\gamma_{l,j,k} \cdot \varrho(A_s)_{l,i} + \gamma_{i,j,l} \cdot \varrho(A_s)_{l,j} + \gamma_{i,j,l} \cdot \varrho(A_s)_{l,k}).$$

We calculate the corresponding Casimir operator $C = \sum_{i=1}^{21} \nu(A_i)^2$ of this representation, which commutes with $\nu(A_i)$ for $i = 1..21$. Therefore C is given as a multiple of the identity on the irreducible components $\mathfrak{sp}(3)$, V^{70}, V^{84} and V^{189}. Its eigenvalues are -8, -12, -18 and -16. Applying the operator C to the torsion, for any eigenvalue we obtain a system of equations without solutions.

As stated in Corollary 4.2, Chapter 10 of [KNII], the Lie algebra of the holonomy group is given by

$$\widetilde{\mathfrak{m}_1} + [\Lambda_{\mathfrak{m}_1}(\mathfrak{m}_1), \widetilde{\mathfrak{m}_1}] + [\Lambda_{m_1}(\mathfrak{m}_1), [\Lambda_{\mathfrak{m}_1}(\mathfrak{m}_1), \widetilde{\mathfrak{m}_1}]] + \ldots \tag{8}$$

where $\widetilde{\mathfrak{m}_1}$ is spanned by all elements

$$[\Lambda_{\mathfrak{m}_1}(X), \Lambda_{\mathfrak{m}_1}(Y)] - \Lambda_{\mathfrak{m}_1}(proj_{\mathfrak{m}_1}([X,Y])) - \varrho_1([X,Y]). \tag{9}$$

for $X, Y \in \mathfrak{m}_1$. With $T^3 = \langle \varrho(A_9), \varrho(A_{10}), \varrho(A_{21})\rangle$ being the maximal torus in $\mathfrak{sp}(3) \subset \mathfrak{so}(\mathfrak{m}) \cong \mathfrak{so}(\mathfrak{m}_1)$ we have $\Lambda_{\mathfrak{m}_1}(\mathfrak{m}_1) = \langle (\alpha - \beta)\varrho(A_9), (\alpha - \gamma)\varrho(A_{10})\rangle \subset T^3$. Thus the first term in (9) vanishes and with $\varrho_1(\mathfrak{m}_1) = \langle \varrho(A_{21})\rangle$ one easily gets

$$\widetilde{\mathfrak{m}_1} = \langle \varrho(A_{21}), (\alpha - \beta)\varrho(A_9), (\alpha - \gamma)\varrho(A_{10})\rangle.$$

With $\Lambda_{\mathfrak{m}_1}(\mathfrak{m}_1) \subset T^3$ and $\widetilde{\mathfrak{m}_1} \subset T^3$ all except the first term of (8) vanish and we get the algebra of the holonomy group equal to $\widetilde{\mathfrak{m}_1}$. □

Lemma 4.1 (Curvature properties)**.** *For any characteristic connection $\nabla^{\alpha\beta\gamma}$, the Ricci tensor in the constructed basis is given by ($a := 2\alpha - \gamma$, $b := 2\alpha - \beta$, $c := 2\alpha - \beta - \gamma$)*

$$\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}} = \frac{1}{\alpha^2}\,\mathrm{diag}(a, a, a, a, b, b, b, b, c, c, c, c, 0, 0).$$

Thus the scalar curvature is given by

$$\mathrm{Scal}^{\nabla^{\alpha\beta\gamma}} = \frac{8(3\,\alpha-\beta-\gamma)}{\alpha^2}.$$

The Riemannian Ricci tensor is for $a := 6\alpha-\gamma$, $b := 6\alpha-\beta$ *and* $c := 6\alpha-\beta-\gamma$ *given by*

$$\mathrm{Ric}^g \;=\; \frac{1}{2\alpha^2}\mathrm{diag}(a,a,a,a,b,b,b,b,c,c,c,c,4\beta,4\gamma)$$

with scalar curvature

$$\mathrm{Scal}^g \;=\; \frac{2(18\,\alpha-\beta-\gamma)}{\alpha^2}.$$

In particular, such a manifold is never $\nabla^{\alpha\beta\gamma}$*-Einstein nor Einstein in the Riemannian sense.*

Proof. We calculate the Ricci tensor $\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}}$ for the characteristic connection in the constructed basis. Since $\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}}$ is symmetric, with [IP01] we get the identity

$$\mathrm{Ric}^g(X,Y) \;=\; \mathrm{Ric}^{\nabla^{\alpha\beta\gamma}}(X,Y) + \frac{1}{4}\sum_{i=1}^{14} g^{\alpha\beta\gamma}(T^{\alpha\beta\gamma}(X,K_i^1),T^{\alpha\beta\gamma}(Y,K_i^1)) \tag{10}$$

and calculate the Ricci tensor for the Levi Civita connection Ric^g. □

4.2. **The homogeneous space U(4)/SO(2) × SO(2).**
We parametrize $H_2 := \mathrm{SO}(2)\times\mathrm{SO}(2)$ by a pair of real numbers (t_1,t_2)

$$\left(\begin{bmatrix}\cos t_1 & \sin t_1\\ -\sin t_1 & \cos t_1\end{bmatrix},\begin{bmatrix}\cos t_2 & \sin t_2\\ -\sin t_2 & \cos t_2\end{bmatrix}\right)\in \mathrm{SO}(2)\times\mathrm{SO}(2) =: H_2$$

and embed the Lie group H_2 in $\mathrm{U}(4) =: K_2$ by:

$$H_2\to K_2,\;(t_1,t_2)\mapsto \mathrm{diag}\left(e^{\frac{i}{2}(t_1-t_2)},e^{\frac{i}{2}(t_1+t_2)},e^{\frac{i}{2}(-t_1+t_2)},e^{\frac{i}{2}(-t_1-t_2)}\right).$$

The action of $\mathfrak{h}_2=\mathfrak{so}(2)\oplus\mathfrak{so}(2)$ splits the irreducible 14-dimensional Sp(3)-representation V^{14} in six 2-dimensional representations and two trivial ones. We choose an invariant metric

$$g^{\alpha,..,\gamma} = \mathrm{diag}(\alpha,\alpha,\alpha_2,\alpha_2,\alpha_3,\alpha_3,\alpha_4,\alpha_4,\alpha_5,\alpha_5,\alpha_6,\alpha_6,\beta,\gamma)$$

with an orthonormal basis $K_1^2,\dots,K_{14}^2$ of $\mathfrak{m}_1$ as done in Appendix A.2.

Theorem 4.2. *Consider the manifold* $M_2=\mathrm{U}(4)/\mathrm{SO}(2)\times\mathrm{SO}(2)$ *equipped with the metric* $g^{\alpha,..,\gamma}$*. For general parameters* $\alpha,\alpha_i,\beta,\gamma>0$*, it carries a* 30*-dimensional space of invariant Sp*(3) *connections, and for* $\alpha=\alpha_2=..=\alpha_6$*, the Sp*(3) *structure admits a characteristic connection with torsion* $T^{\alpha\beta\gamma}\in\Lambda^3(M_2)$*. These Sp*(3) *structures with characteristic connection have the following properties:*

1. *The characteristic connection has alwas parallel torsion,* $\nabla^{\alpha\beta\gamma}T^{\alpha\beta\gamma}=0$.
2. *The structure is never integrable.*
3. *The structure is of mixed type.*
4. *The Lie algebra of the holonomy group of the characteristic connection is a subalgebra of the maximal torus of* $\mathfrak{sp}(3)$ *and it is*
   - *two-dimensional, if* $\alpha\neq\gamma$ *and*
   - *three-dimensional, if* $\alpha\neq\gamma$.

Proof. We calculate all invariant Sp(3)-connections via their corresponding equivariant maps $\Lambda_{\mathfrak{m}_2}:\mathfrak{m}_2\to\mathfrak{sp}(3)$ and get all connections via maps $\Lambda_{\mathfrak{m}_2}$ satisfying the following 5 conditions with parameters a_i, $i=1,\dots,20$

- $\Lambda_{\mathfrak{m}_2}$ maps the space $\langle K_i^2 \mid i = 1,2\rangle$ into the space $\langle \varrho(A_i) \mid i = 11,12\rangle$ and the corresponding matrix has the form

$$\Lambda_{\mathfrak{m}_2}|_{\langle K_i^2 \mid i=1,2\rangle} = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix},$$

- $\Lambda_{\mathfrak{m}_2}$ maps the space $\langle K_i^2 \mid i = 3,4\rangle$ into the space $\langle \varrho(A_i) \mid i = 13,14\rangle$ and the corresponding matrix has the form

$$\Lambda_{\mathfrak{m}_2}|_{\langle K_i^2 \mid i=3,4\rangle} = \begin{bmatrix} a_3 & -a_4 \\ a_4 & a_3 \end{bmatrix},$$

- $\Lambda_{\mathfrak{m}_2}$ maps the space $\langle K_i^2 \mid i = 5..8\rangle$ into the space $\langle \varrho(A_i) \mid i = 15..18\rangle$ and the corresponding matrix has the form

$$\Lambda_{\mathfrak{m}_2}|_{\langle K_i^2 \mid i=5..8\rangle} = \begin{bmatrix} a_5 & -a_6 & a_7 & -a_8 \\ a_6 & a_5 & a_8 & a_7 \\ a_9 & -a_{10} & a_{11} & -a_{12} \\ a_{10} & a_9 & a_{11} & a_{12} \end{bmatrix},$$

- $\Lambda_{\mathfrak{m}_2}$ maps the space $\langle K_i^2 \mid i = 9..12\rangle$ into the space $\langle \varrho(A_i) \mid i = 1..4\rangle$ and the corresponding matrix has the form

$$\Lambda_{\mathfrak{m}_2}|_{\langle K_i^2 \mid i=9..12\rangle} = \begin{bmatrix} a_{13} & -a_{14} & -a_{15} & a_{16} \\ a_{14} & a_{13} & a_{16} & a_{15} \\ -a_{17} & a_{18} & a_{19} & -a_{20} \\ a_{18} & a_{17} & a_{20} & a_{19} \end{bmatrix}$$

- $\Lambda_{\mathfrak{m}_2}$ maps the space $\langle K_i^2 \mid i = 13,14\rangle$ into the space $\langle \varrho(A_i) \mid i = 5,7,9,10,21\rangle$. This part depends on 10 parameters, other than a_i, $i = 1..20$.

With equation (6) we compute the torsion tensor, which is skew symmetric if and only if $\alpha = \alpha_2 = \ldots = \alpha_6$ and

$$\Lambda_{\mathfrak{m}_2}(K_{13}^2) = \frac{\sqrt{2}(\alpha-\beta)}{\alpha\sqrt{\beta}}\varrho(A_9) \text{ and } \Lambda_{\mathfrak{m}_2}(K_i^2) = 0 \text{ for } i \neq 13.$$

For such connections $\nabla^{\alpha\beta\gamma}$ the torsion never vanishes. Again we compute that the torsion is parallel for all such connections and that none of the torsion tensors lies in any eigenspace of the Casimir operator.

With the formulas (8) and (9) we get for the maximal torus T^3 in $\mathfrak{sp}(3)$ that $\Lambda_{\mathfrak{m}_2}(\mathfrak{m}_2) = \langle(\alpha - \beta)\varrho(A_9)\rangle \subset T^3$. Thus the first term in (9) again vanishes and with $\varrho_2(\mathfrak{m}_2) = \langle\varrho(A_{10}), \varrho(A_{21})\rangle$ one easily gets

$$\widetilde{\mathfrak{m}_2} = \langle\varrho(A_{21}), (\alpha-\beta)\varrho(A_9), \varrho(B_{10})\rangle$$

and again we get the Lie algebra of the holonomy group being $\widetilde{\mathfrak{m}_2}$. □

Lemma 4.2 (Curvature properties)**.** *On M_2, the Ricci tensor for the characteristic connection is given by ($a := 2\alpha - \beta$)*

$$\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}} = \frac{1}{\alpha^2}\mathrm{diag}(2\alpha, 2\alpha, 2\alpha, 2\alpha, a, a, a, a, a, a, a, a, 0, 0)$$

with scalar curvature

$$\mathrm{Scal}^{\nabla^{\alpha\beta\gamma}} = \frac{8(3\,\alpha - \beta)}{\alpha^2}.$$

The Riemannian Ricci tensor for the Levi Civita is for $a := 6\alpha - \beta$ given by

$$\mathrm{Ric}^g = \frac{1}{2\alpha^2}\mathrm{diag}(6\alpha, 6\alpha, 6\alpha, 6\alpha, a, a, a, a, a, a, a, a, 4\beta, 0)$$

and

$$\mathrm{Scal}^g = \frac{2(18\,\alpha - \beta)}{\alpha^2}.$$

Thus this space is never $\nabla^{\alpha\beta\gamma}$-Einstein nor Einstein for the Levi Civita connection.

Proof. The proof follows immediately from the identity (10). □

We will now have a look at invariant spinors on M_2. Since M_2 carries a unique homogeneous spin structure (see Remark 2.3), we can lift the characteristic connection $\nabla^{\alpha\beta\gamma}$ to the spin bundle. To use the map $\Lambda_{\mathfrak{m}}$ for calculations, we look at elements $\psi \in \Delta_{14}$ that are invariant under the lifted action of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ defining global spinors via the constant map $\mathrm{U}(4) \to \Delta_{14},\ g \mapsto \psi$. We get a 16-dimensional space of such invariant spinors.
The Dirac operator we will look at is the Dirac operator $\not{D}$ of the connection with torsion $T^{\alpha\beta\gamma}/3$. With the lifted map $\widetilde{\Lambda_{\mathfrak{m}}}$ we easily compute for an invariant spinor ψ

$$\not{D}\psi = \sum_{i=1}^{14} \widetilde{\Lambda_{\mathfrak{m}}}(K_i^2)\psi - \frac{1}{2}T^{\alpha\beta\gamma} \cdot \psi, \tag{11}$$

where the torsion $T^{\alpha\beta\gamma}$ is considered as a 3-form and acts on a spinor via Clifford multiplication. Since the dimension 14 is even, the spinor bundle splits in two bundles being invariant under the $\mathrm{Spin}(n)$ action and we calculate

Lemma 4.3. *The lift of the action of* $\mathrm{SO}(2) \times \mathrm{SO}(2)$ *on the* 128*-dimensional space* Δ_{14} *admits a* 16*-dimensional space of invariant spinors. The Dirac operator $\not{D}$ has the two eigenvalues* $\pm\sqrt{\frac{\alpha+4\beta}{\alpha\beta}}$ *on this space.*

As it is just a scaling of the metric, we can fix one parameter of the metric and hence choose $\alpha = 1$. We look at the estimates for the first eigenvalue λ of the Dirac operator valid for a connection with parallel torsion. From the results of [AF04], it follows that

$$\lambda^2 \geq \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}||T^{\alpha\beta\gamma}||^2 - \frac{1}{4}\mu^2 \tag{12}$$

whereas the twistorial eigenvalue estimate derived in [ABBK12] states that

$$\lambda^2 \geq \frac{14}{4(14-1)}\mathrm{Scal}^g + \frac{14(14-5)}{8(14-3)^2}||T^{\alpha\beta\gamma}||^2 + \frac{14(4-14)}{4(14-3)^2}\mu^2, \tag{13}$$

where μ is the largest eigenvalue of the operator $T^{\alpha\beta\gamma}$. Typically, it depends on the underlying geometry which of the inequalities is better (see [ABBK12] for a detailed discussion). We calculate the operator $T^{\alpha\beta\gamma}$ for the orthonormal basis K_i^2, $i = 1, \dots, 14$ of $\mathfrak{m}$ for any $v \in \mathfrak{m}$ as

$$T^{\alpha\beta\gamma}v = \sum_{i,j,k=1}^{14} T^{\alpha\beta\gamma}(K_i^2, K_j^2, K_k^2) K_i^2 \cdot K_j^2 \cdot K_k^2 \cdot v.$$

This yields the eigenvalues $\mu = \pm 2\sqrt{4+\beta}$ of $T^{\alpha\beta\gamma}$ on the space of invariant spinors and with

$$||T^{\alpha\beta\gamma}||^2 = \sum_{i,j,k=1}^{14} T^{\alpha\beta\gamma}(K_i^2, K_j^2, K_k^2)^2 = 8 + 4\beta$$

and Lemma 4.2 we obtain: The estimate (12) is equal to the square of the eigenvalue computed in Lemma 4.3 if $\beta = 1$, and indeed in this case one checks that all invariant spinors are parallel.

The estimate (13) is always strict, hence there does not exist a metric for which an invariant spinor becomes a twistor spinor with torsion. The twistorial estimate is stronger then the first one if $\beta < \frac{166}{275}$.
The manifold M_1 considered in Section 4.1 carries a 48-dimensional spaces of invariant spinors and the computer was not able to compute the eigenvalues of the corresponding Dirac operator.

4.3. **The homogeneous space U(4) × U(1)/SO(2) × SO(2) × SO(2).**
In this example, we shall parametrize the Lie group $H_3 := \mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$ as

$$\left(\begin{bmatrix}\cos t_1 & \sin t_1\\ -\sin t_1 & \cos t_1\end{bmatrix},\begin{bmatrix}\cos t_2 & \sin t_2\\ -\sin t_2 & \cos t_2\end{bmatrix},\begin{bmatrix}\cos t_3 & \sin t_3\\ -\sin t_3 & \cos t_3\end{bmatrix}\right)\in \mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2),$$

and embed it into $K_3 = \mathrm{U}(4)\times\mathrm{U}(1)$ by

$$(t_1,t_2,t_3)\mapsto\left(\mathrm{diag}\left(e^{\frac{i}{2}(t_1+t_2-t_3)},e^{\frac{i}{2}(t_1-t_2+t_3)},e^{\frac{i}{2}(-t_1+t_2+t_3)},e^{\frac{i}{2}(-t_1-t_2-t_3)}\right),1\right).$$

The action of $\mathfrak{h}_3=\mathfrak{so}(2)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2)$ splits V^{14} in the same irreducible representations as the representation of $\mathfrak{h}_2=\mathfrak{so}(2)\oplus\mathfrak{so}(2)$, so we can choose the same Ansatz for the metric

$$g^{\alpha,..,\gamma}=\mathrm{diag}(\alpha,\alpha,\alpha_2,\alpha_2,\alpha_3,\alpha_3,\alpha_4,\alpha_4,\alpha_5,\alpha_5,\alpha_6,\alpha_6,\beta,\gamma)$$

with an orthonormal basis $\{K_i^3 \mid i=1..14\}$ of $\mathfrak{m}_3$ as described in Appendix A.3.

Theorem 4.3. *Consider the manifold $M_3=\mathrm{U}(4)\times\mathrm{U}(1)/\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$ equipped with the metric $g^{\alpha,\cdots,\gamma}$. For any parameters $\alpha,\alpha_i,\beta,\gamma>0$, it carries an 18-dimensional space of invariant Sp(3)-connections, and for $\alpha=\alpha_2=..=\alpha_6$, the Sp(3) structure admits a characteristic connection with torsion $T^{\alpha\beta\gamma}\in\Lambda^3(M_3)$. These Sp(3) structures with characteristic connection have the following properties:*

1. *The characteristic connection has alwas parallel torsion, $\nabla^{\alpha\beta\gamma}T^{\alpha\beta\gamma}=0$, and it coincides with the canonical connection.*
2. *The structure is never integrable.*
3. *The structure is of mixed type.*
4. *The Lie algebra of the holonomy group of the characteristic connection is the maximal torus in $\mathfrak{sp}(3)$.*

Proof. We get all possible Sp(3) connections via equivariant maps $\Lambda_{\mathfrak{m}_3}:\mathfrak{m}_3\to\mathfrak{sp}(3)$ with parameters a_i, $i=1..12$ and the conditions

- for pairs $(i,j)\in\{(1,11),(3,13),(5,17),(7,15),(9,1),(11,3)\}$ we have
$$\Lambda_{\mathfrak{m}_3}(K_i^3)=a_i\varrho(A_j)+a_{i+1}\varrho(A_{j+1})\text{ and }\Lambda_{\mathfrak{m}_3}(K_i^3)=-a_{i+1}\varrho(A_j)+a_i\varrho(A_{j+1}),$$
- $\Lambda_{\mathfrak{m}_3}$ maps the space $\langle K_i^3 \mid i=13,14\rangle$ into the space $\langle\varrho(A_i)\mid i=9,10,21\rangle$, which is dependent on 6 parameters.

We again get a skew symmetric torsion if and only if $\alpha=\alpha_2=..=\alpha_6$ with the only possible invariant Sp(3) connection being the canonical connection defined by $\Lambda_{\mathfrak{m}_3}\equiv 0$. For this connections $\nabla^{\alpha\beta\gamma}$ the torsion never vanishes. Again we compute that the torsion is parallel for all such connections and that none of the torsion tensors lie in any eigenspace of the Casimir operator.

Since $\Lambda_{m_3}\equiv 0$ we get the Lie algebra of the holonomy group via the formulas (8) and (9) being equal to

$$\varrho_3(proj_{\mathfrak{h}_3})([\mathfrak{m}_3,\mathfrak{m}_3]) \;=\; \varrho_3(\mathfrak{h}_3)$$

and thus being the maximal torus in $\mathfrak{sp}(3)$ (see Appendix A.3). □

Lemma 4.4 (Curvature properties). *The Ricci tensor for the characteristic connection is given by*

$$\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}} = \frac{2}{\alpha}\mathrm{diag}(1,1,1,1,1,1,1,1,1,1,1,1,0,0)$$

and its scalar curvature is $\mathrm{Scal}^{\nabla^{\alpha\beta\gamma}} = \frac{24}{\alpha}$.
The Riemannian Ricci tensor for the Levi Civita is given by $\mathrm{Ric}^g = \frac{3}{2}\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}}$. *Thus the Riemannian scalar curvature is* $\mathrm{Scal}^g = \frac{36}{\alpha}$*, and the space is never* $\nabla^{\alpha\beta\gamma}$*-Einstein nor Riemannian Einstein.*

The action of $\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$ lifted in Δ_{14} has no trivial parts and thus there are no invariant spinors. Hence it is not possible to make any statements on the spectrum of the Dirac operator.

Remark 4.1. In the first 3 examples, we constructed Sp(3) spaces via embeddings of K_i in the maximal torus T^3 of SU(4). Since $\varrho_3(T^3)\subset \mathrm{Sp}(3)$, one can choose any embedding of K_i into T^3 to get a Sp(3) manifold K_i/H_i. For those examples there are different possible identifications $\mathfrak{m}_i \cong \mathfrak{m}$ giving different identifications $\mathrm{SO}(\mathfrak{m}_i)\cong \mathrm{SO}(\mathfrak{m})\supset \mathrm{Sp}(3)$, such that $\rho_i(SO(2)^i)\subset \mathrm{Sp}(3)$. Those induce different Sp(3) structures on the given manifolds, but their geometry is just the same.

4.4. **The homogeneous space SU(5)/Sp(2).**

We restrict A_i, $i = 1..10$ to the lower 5×5-matrix and get the Lie algebra of $H_4 = \mathrm{Sp}(2)$ in $\mathfrak{k}_4 = \mathfrak{su}(5)$. In [K96], it was shown that with this embedding $\mathrm{SU}(5)\subset \mathrm{SU}(6)$, the Lie group $K_4 = \mathrm{SU}(5)$ already acts transitively on SU(6)/Sp(3) with isotropy group $H_4 = \mathrm{Sp}(2)$. As a manifold, SU(6)/Sp(3) is hence diffeomorphic to SU(5)/Sp(2), but the homogeneous structure is a different one. The adjoint representation ϱ_4 of $\mathrm{Sp}(2) = H_4$ on this space is just a restriction of the action of $\mathrm{Sp}(2)\subset \mathrm{Sp}(3)$ on $\mathfrak{m}_4$ (see Appendix A.4) and we get an Sp(3) structure on SU(5)/Sp(2). The representation ϱ_4 splits $\mathfrak{m}_4 = \Delta_5\oplus \mathfrak{p}^5\oplus\mathfrak{p}^1$ as shown in Theorem A.1 and we get an 3-dimensional family of invariant metrics using multiples $\alpha,\beta,\gamma > 0$ of the negative of the Killing form on each component,

$$g^{\alpha\beta\gamma} = \mathrm{diag}(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\beta,\beta,\beta,\beta,\beta,\gamma).$$

Theorem 4.4. *Consider the manifold* $M_4 = \mathrm{SU}(5)/\mathrm{Sp}(2)$ *equipped with the metric* $g^{\alpha\beta\gamma}$*. For any parameters* $\alpha,\beta,\gamma > 0$*, it carries an 7-dimensional space of invariant* $Sp(3)$ *connections, and to each of these metrics corresponds exactly one characteristic connection with torsion* $T^{\alpha\beta\gamma}\in\Lambda^3(V^{14})$*. These* $Sp(3)$ *structures with characteristic connection have the following properties:*

(1) *The characteristic connection satisfies* $\nabla^{\alpha\beta\gamma}T^{\alpha\beta\gamma} = 0$ *if and only if either*
  - $\beta = \alpha$ *or*
  - $\beta = 2\alpha$ *and* $\gamma = \frac{6}{5}\alpha$.

(2) *The structure is*
  - *integrable if* $\beta = 2\alpha$ *and* $\gamma = \frac{6}{5}\alpha$,
  - *of type* $\mathfrak{sp}(3)$ *if* $\alpha = \frac{1}{4}(\sqrt{15\beta\gamma}-\beta)$,
  - *of type* V^{189} *if* $\alpha = \frac{1}{12}(9\beta-\sqrt{15\beta\gamma})$.

(3) *The Lie algebra of the holonomy group of the characteristic connection is given by*
  - $\mathfrak{sp}(3)$ *if* $\alpha\neq\beta$,
  - $\mathfrak{sp}(2)\oplus W^1$ *if* $\gamma\neq\alpha=\beta$*, where* W^1 *is the one-dimensional subspace in the maximal torus* T^3 *of* $\mathfrak{sp}(3)$ *such that* $T^3\subset\mathfrak{sp}(2)\oplus W^1$ *and*
  - $\mathfrak{sp}(2)$ *if* $\alpha=\beta=\gamma$.

Remark 4.2. In case of an integrable structure, $T^{\alpha\beta\gamma} = 0$, SU(5)/Sp(2) locally isometric to a symmetric space, as mentioned before [N08].

Proof. Again we look at linear maps $\Lambda_{\mathfrak{m}_4} : \mathfrak{m}_4 \to \mathfrak{sp}(3)$ that are equivariant under the representation ϱ_4.

$$\Lambda_{\mathfrak{m}_4}(\varrho_4(h)X) \;=\; \varrho_4(h)\Lambda_{\mathfrak{m}_4}(X)\varrho_4(h)^{-1} \quad \forall h \in Sp(2),\; X \in \mathfrak{m}_4.$$

One calculates that this is the case if and only if $\Lambda_{\mathfrak{m}_4}$ fulfills the following conditions

- $\Lambda_{\mathfrak{m}_4}$ is identically zero on $\mathfrak{p}^5$.
- $\Lambda_{\mathfrak{m}_4}$ maps p^1 into the space $\langle \varrho(A_i) \mid i = 19..21\rangle$. This gives 3 parameters.
- $\Lambda_{\mathfrak{m}_4}$ maps Δ_5 into the space $\langle \varrho(A_i) \mid i = 11..18\rangle$ and the corresponding matrix is for $a, b, c, d \in \mathbb{R}$ given by

$$\Lambda_{\mathfrak{m}_4}|_{\Delta_5} = \begin{bmatrix} b & -a & -d & c & & & & \\ a & b & c & d & & & & \\ d & -c & b & -a & & & & \\ -c & -d & a & b & & & & \\ & & & & d & -c & b & -a \\ & & & & -c & -d & a & b \\ & & & & b & -a & -d & c \\ & & & & a & b & c & d \end{bmatrix}.$$

Since $\mathfrak{sp}(2) = \langle \varrho(A_i) \mid i = 1..10\rangle$ and $Im(\Lambda_{\mathfrak{m}_4}) \cap \langle \varrho(A_i) \mid i = 1..10\rangle = \{0\}$, the only $\mathfrak{sp}(2)$-connection is the canonical connection.
With (6), we calculate the torsion. The condition $T^{\alpha\beta\gamma} \in \Lambda^3(\mathrm{SU}(5)/\mathrm{Sp}(2))$ for the torsion tensor implies $a = c = d = 0,\; b = \frac{\alpha-\beta}{\alpha\sqrt{\beta}}$ and

$$\Lambda_{\mathfrak{m}_4}|_{\mathfrak{p}^1} : \mathfrak{p}^1 \to \langle \varrho(A_{21})\rangle$$

is given by the multiplication with the constant $\frac{1}{\sqrt{2}}\frac{-\gamma\sqrt{5\beta}+\sqrt{3\gamma}\beta-\sqrt{3\gamma}\alpha+\sqrt{5\beta}\alpha}{\alpha\sqrt{\beta\gamma}}$.
Again with equation (7) we derive that $\nabla^{\alpha\beta\gamma}T^{\alpha\beta\gamma} = 0$ if and only if either $(\beta = \alpha)$ or $(\beta = 2\alpha$ and $\gamma = \frac{6}{5}\alpha)$. One computes that $T^{\alpha\beta\gamma} = 0$ iff $\beta = 2\alpha$ and $\gamma = \frac{6}{5}\alpha$. With the above calculated Casimir operator C we get

$$C(T^{\alpha\beta\gamma}) = -8T^{\alpha\beta\gamma} \Leftrightarrow \alpha = \frac{1}{4}(\sqrt{15\beta\gamma} - \beta)$$

and

$$C(T^{\alpha\beta\gamma}) = -16T^{\alpha\beta\gamma} \Leftrightarrow \alpha = \frac{1}{12}(9\beta - \sqrt{15\beta\gamma}).$$

With equations (8), (9) and an appropriate computer algebra program one computes that the Lie algebra of the holonomy group is given by $\mathfrak{sp}(3)$ if $\alpha \neq \beta$ and by $\mathfrak{sp}(2) \oplus \langle(\alpha-\gamma)B_{21}\rangle$ if $\alpha = \beta$. □

We calculate the Ricci tensor for the characteristic connection and the Levi Civita connection from equation (10).

Lemma 4.5 (Curvature properties)**.** *The Ricci tensor for the characteristic connection is for* $a := \frac{2\sqrt{15\beta\gamma}-11\beta-5\gamma}{4\alpha^2} + \frac{21}{2\alpha} - \frac{4}{\beta} - \frac{\sqrt{15\gamma}}{2\alpha\sqrt{\beta}}$, $b := \frac{2(\alpha+\beta)}{\beta\alpha}$, $c := 2(\beta-\alpha)(\frac{3}{\alpha\beta} + \frac{\sqrt{15\gamma}}{\alpha^2\sqrt{\beta}} - \frac{3}{\alpha^2})$ *given by*

$$\mathrm{Ric}^{\nabla^{\alpha\beta\gamma}} = \mathrm{diag}(a, a, a, a, a, a, a, a, b, b, b, b, b, c).$$

The Riemannian Ricci tensor is for $a := 10\alpha - \frac{5}{4}\beta - \frac{5}{4}\gamma$ *and* $b = \frac{8\alpha^2+\beta^2}{\beta}$ *equal to*

$$\mathrm{Ric}^g = \frac{1}{2\alpha^2}\mathrm{diag}(a, a, a, a, a, a, a, a, b, b, b, b, b, 5\gamma).$$

Its scalar curvature is $\mathrm{Scal}^g = \frac{5(16\alpha\beta-\beta\gamma-\beta^2+8\alpha^2)}{2\alpha^2\beta}$*. Thus, this space is a Riemannian Einstein space if* $\sqrt{2}\alpha = \beta = \frac{1}{\sqrt{8}-1}\gamma$ *and in this case we have*

$$\mathrm{Ric}^g = \frac{5}{2\alpha^2} g^{\alpha\beta\gamma}.$$

We lift the representation of Sp(2) in SO(14) to Spin(14) and with the formula (11) we calculate

Lemma 4.6. Δ_{14} *has a 4-dimensional space of* Sp(2) *invariant spinors and the Dirac operator* $\not{D}$ *has eigenvalues*

$$\pm\frac{1}{2}\sqrt{\frac{5\alpha^2\beta + 3\alpha^2\gamma - 6\alpha\beta\gamma + 2\alpha\sqrt{15\beta\gamma}(\beta-\alpha) + 28\beta^2\gamma}{\alpha^2\beta\gamma}}.$$

As in Section 4.2, we restrict the general case, ignoring the possible scaling, to the case $\alpha = 1$. To look at the inequalities (12) and (13) we need the torsion to be parallel. From the two possible cases mentioned in Theorem 4.4, only the first is of interest, since the torsion vanishes in the second and Friedrich's Riemannian estimate from 1980 applies.
So, assume that $\beta = \alpha = 1$. The operator $T^{\alpha\beta\gamma}$ has eigenvalues $\mu = \pm\sqrt{25+5\gamma}$ and its norm is given by $||T^{\alpha\beta\gamma}||^2 = 5 + 5\gamma$. Thus we obtain that the estimate (13) is always strict, and the estimate (12) becomes an equality for $\gamma = \beta = \alpha = 1$. As expected, all invariant spinors are parallel for $\gamma = \beta = \alpha = 1$. The inequality (13) is better than the inequality (12) if $\gamma < \frac{189}{275}$.

Appendix A. Explicit realizations of representations & other geometric data

Let $\{e_i^n\}_{i=1..n}$ be the standard basis of $\mathbb{R}^n$, $E_{i,j}^n \in \mathfrak{su}(n)$ the matrix given by the linear map $e_i^n \mapsto -e_j^n$, $e_j^n \mapsto e_i^n$ and $S_{i,j}^n$ given by $e_i^n \mapsto e_j^n$, $e_j^n \mapsto e_i^n$. We used throughout the following basis $A_1, \ldots, A_{21}$ of the Lie algebra of $\mathrm{Sp}(3) \subset \mathrm{SU}(6)$,

$$A_1 := \frac{1}{2}(E_{2,3}^6 + E_{5,6}^6),\quad A_2 := \frac{i}{2}(S_{2,3}^6 - S_{5,6}^6),\quad A_3 := \frac{1}{2}(E_{2,6}^6 + E_{3,5}^6),\ A_4 := \frac{i}{2}(S_{2,6}^6 + S_{3,5}^6),$$
$$A_5 := \frac{1}{\sqrt{2}}E_{2,5}^6,\quad A_6 := \frac{1}{\sqrt{2}}E_{3,6}^6,\quad A_7 := \frac{i}{\sqrt{2}}S_{2,5}^6,\quad A_8 := \frac{i}{\sqrt{2}}S_{3,6}^6,\quad A_9 := \frac{i}{\sqrt{2}}(S_{2,2}^6 - S_{5,5}^6),$$
$$A_{10} := \frac{i}{\sqrt{2}}(S_{3,3}^6 - S_{6,6}^6),\quad A_{11} := \frac{1}{2}(E_{1,3}^6 + E_{4,6}^6),\quad A_{12} := \frac{i}{2}(S_{1,3}^6 - S_{4,6}^6),\quad A_{13} := \frac{1}{2}(E_{1,6}^6 + E_{3,4}^6),$$
$$A_{14} := \frac{i}{2}(S_{1,6}^6 + S_{3,4}^6),\quad A_{15} := \frac{1}{2}(E_{1,5}^6 + E_{2,4}^6),\quad A_{16} := \frac{i}{2}(S_{1,5}^6 + S_{2,4}^6),\quad A_{17} := \frac{1}{2}(E_{1,2}^6 + E_{4,5}^6),$$
$$A_{18} := \frac{i}{2}(S_{1,2}^6 - S_{4,5}^6),\quad A_{19} := \frac{1}{\sqrt{2}}E_{1,4}^6,\quad A_{20} := \frac{i}{\sqrt{2}}S_{1,4}^6,\quad A_{21} := \frac{i}{\sqrt{2}}(S_{1,1}^6 - S_{4,4}^6).$$

Hence, we get a basis of $\mathfrak{m}$, $\mathfrak{su}(6) = \mathfrak{m} \oplus \mathfrak{sp}(3)$ as

$$B_1 := \frac{1}{2}(E_{1,3}^6 - E_{4,6}^6),\ B_2 := \frac{i}{2}(S_{1,3}^6 + S_{4,6}^6),\ B_3 := \frac{1}{2}(E_{1,6}^6 - E_{3,4}^6),$$
$$B_4 := \frac{i}{2}(S_{1,6}^6 - S_{3,4}^6),\ B_5 := \frac{1}{2}(E_{1,2}^6 - E_{4,5}^6),\ B_6 := \frac{i}{2}(S_{1,2}^6 + S_{4,5}^6),$$
$$B_7 := \frac{1}{2}(E_{1,5}^6 - E_{2,4}^6),\ B_8 := \frac{i}{2}(S_{1,5}^6 - S_{2,4}^6),\ B_9 := \frac{1}{2}(E_{2,3}^6 - E_{5,6}^6),$$
$$B_{10} := \frac{i}{2}(S_{2,3}^6 + S_{5,6}^6),\ B_{11} := \frac{1}{2}(E_{2,6}^6 - E_{3,5}^6),\ B_{12} := \frac{i}{2}(S_{2,6}^6 - S_{3,5}^6),$$
$$B_{13} := \frac{i}{2}(S_{2,2}^6 - S_{3,3}^6 + S_{5,5}^6 - S_{6,6}^6),\ B_{14} := \frac{i}{2\sqrt{3}}(-2S_{1,1}^6 + S_{2,2}^6 + S_{3,3}^6 - 2S_{4,4}^6 + S_{5,5}^6 + S_{6,6}^6).$$

The isotropy representation of $\mathfrak{sp}(3)$ on $\mathfrak{m} \cong V^{14}$ is thus

$$\varrho(A_1) = -\frac{1}{2}E_{1,5}^{14} - \frac{1}{2}E_{2,6}^{14} - \frac{1}{2}E_{3,7}^{14} - \frac{1}{2}E_{4,8}^{14}, -E_{10,13}^{14},\quad \varrho(A_2) = \frac{1}{2}E_{1,6}^{14} - \frac{1}{2}E_{2,5}^{14} - \frac{1}{2}E_{3,8}^{14} + \frac{1}{2}E_{4,7}^{14} + E_{9,13}^{14}$$

$$\varrho(A_3) = \frac{1}{2}E^{14}_{1,7}+\frac{1}{2}E^{14}_{2,8}-\frac{1}{2}E^{14}_{3,5}-\frac{1}{2}E^{14}_{4,6}-E^{14}_{12,13}, \quad \varrho(A_4) = \frac{1}{2}E^{14}_{1,8}-\frac{1}{2}E^{14}_{2,7}+\frac{1}{2}E^{14}_{3,6}-\frac{1}{2}E^{14}_{4,5}+E^{14}_{11,13}$$

$$\varrho(A_5) = \frac{\sqrt{3}}{2}E^{14}_{5,7}+\frac{\sqrt{3}}{2}E^{14}_{6,8}+\frac{\sqrt{3}}{2}E^{14}_{9,11}-\frac{\sqrt{3}}{2}E^{14}_{10,12}, \quad \varrho(A_6) = \frac{\sqrt{3}}{2}E^{14}_{1,3}+\frac{\sqrt{3}}{2}E^{14}_{2,4}+\frac{\sqrt{3}}{2}E^{14}_{9,11}+\frac{\sqrt{3}}{2}E^{14}_{10,12}$$

$$\varrho(A_7) = \frac{\sqrt{3}}{2}E^{14}_{5,8}-\frac{\sqrt{3}}{2}E^{14}_{6,7}+\frac{\sqrt{3}}{2}E^{14}_{9,12}+\frac{\sqrt{3}}{2}E^{14}_{10,11}, \quad \varrho(A_8) = \frac{\sqrt{3}}{2}E^{14}_{1,4}-\frac{\sqrt{3}}{2}E^{14}_{2,3}+\frac{\sqrt{3}}{2}E^{14}_{9,12}-\frac{\sqrt{3}}{2}E^{14}_{10,11}$$

$$\varrho(A_9) = \frac{\sqrt{3}}{2}E^{14}_{5,6}-\frac{\sqrt{3}}{2}E^{14}_{7,8}-\frac{\sqrt{3}}{2}E^{14}_{9,10}-\frac{\sqrt{3}}{2}E^{14}_{11,12}, \quad \varrho(A_{10}) = \frac{\sqrt{3}}{2}E^{14}_{1,2}-\frac{\sqrt{3}}{2}E^{14}_{3,4}+\frac{\sqrt{3}}{2}E^{14}_{9,10}-\frac{\sqrt{3}}{2}E^{14}_{11,12}$$

$$\varrho(A_{11}) = -\frac{1}{2}E^{14}_{2,13} + \frac{\sqrt{3}}{2}E^{14}_{2,14} - \frac{1}{2}E^{14}_{5,9} + \frac{1}{2}E^{14}_{6,10} - \frac{1}{2}E^{14}_{7,11} - \frac{1}{2}E^{14}_{8,12}$$

$$\varrho(A_{12}) = \frac{1}{2}E^{14}_{1,13} - \frac{\sqrt{3}}{2}E^{14}_{1,14} - \frac{1}{2}E^{14}_{5,10} - \frac{1}{2}E^{14}_{6,9} + \frac{1}{2}E^{14}_{7,12} - \frac{1}{2}E^{14}_{8,11}$$

$$\varrho(A_{13}) = -\frac{1}{2}E^{14}_{4,13} + \frac{\sqrt{3}}{2}E^{14}_{4,14} - \frac{1}{2}E^{14}_{5,11} + \frac{1}{2}E^{14}_{6,12} + \frac{1}{2}E^{14}_{7,9} + \frac{1}{2}E^{14}_{8,10}$$

$$\varrho(A_{14}) = \frac{1}{2}E^{14}_{3,13} - \frac{\sqrt{3}}{2}E^{14}_{3,14} - \frac{1}{2}E^{14}_{5,12} - \frac{1}{2}E^{14}_{6,11} - \frac{1}{2}E^{14}_{7,10} + \frac{1}{2}E^{14}_{8,9}$$

$$\varrho(A_{15}) = +\frac{1}{2}E^{14}_{1,11} - \frac{1}{2}E^{14}_{2,12} - \frac{1}{2}E^{14}_{3,9} + \frac{1}{2}E^{14}_{4,10} + \frac{1}{2}E^{14}_{8,13} + \frac{\sqrt{3}}{2}E^{14}_{8,14}$$

$$\varrho(A_{16}) = +\frac{1}{2}E^{14}_{1,12} + \frac{1}{2}E^{14}_{2,11} - \frac{1}{2}E^{14}_{3,10} - \frac{1}{2}E^{14}_{4,9} - \frac{1}{2}E^{14}_{7,13} - \frac{\sqrt{3}}{2}E^{14}_{7,14}$$

$$\varrho(A_{17}) = +\frac{1}{2}E^{14}_{1,9} + \frac{1}{2}E^{14}_{2,10} + \frac{1}{2}E^{14}_{3,11} + \frac{1}{2}E^{14}_{4,12} + \frac{1}{2}E^{14}_{6,13} + \frac{\sqrt{3}}{2}E^{14}_{6,14}$$

$$\varrho(A_{18}) = -\frac{1}{2}E^{14}_{1,10} + \frac{1}{2}E^{14}_{2,9} - \frac{1}{2}E^{14}_{3,12} + \frac{1}{2}E^{14}_{4,11} - \frac{1}{2}E^{14}_{5,13} - \frac{\sqrt{3}}{2}E^{14}_{5,14}$$

$$\varrho(A_{19}) = +\frac{\sqrt{3}}{2}E^{14}_{1,3} - \frac{\sqrt{3}}{2}E^{14}_{2,4} + \frac{\sqrt{3}}{2}E^{14}_{5,7} - \frac{\sqrt{3}}{2}E^{14}_{6,8}$$

$$\varrho(A_{20}) = +\frac{\sqrt{3}}{2}E^{14}_{1,4} + \frac{\sqrt{3}}{2}E^{14}_{2,3} + \frac{\sqrt{3}}{2}E^{14}_{5,8} + \frac{\sqrt{3}}{2}E^{14}_{6,7}$$

$$\varrho(A_{21}) = -\frac{\sqrt{3}}{2}E^{14}_{1,2} - \frac{\sqrt{3}}{2}E^{14}_{3,4} - \frac{\sqrt{3}}{2}E^{14}_{5,6} - \frac{\sqrt{3}}{2}E^{14}_{7,8}$$

A.1. $\boldsymbol{SU(4)/\mathrm{SO}(2)}$.

Looking at the given embedding, we define

$$K^1_1 := \frac{1}{\sqrt{2\alpha}}E^4_{1,3}, \quad K^1_2 := \frac{i}{\sqrt{2\alpha}}S^4_{1,3}, \quad K^1_3 := \frac{1}{\sqrt{2\alpha_2}}E^4_{2,4}, \quad K^1_4 := \frac{i}{\sqrt{2\alpha_2}}S^4_{2,4},$$
$$K^1_5 := \frac{1}{\sqrt{2\alpha_3}}E^4_{2,3}, \quad K^1_6 := \frac{i}{\sqrt{2\alpha_3}}S^4_{2,3}, \quad K^1_7 := \frac{1}{\sqrt{2\alpha_4}}E^4_{1,4}, \quad K^1_8 := \frac{i}{\sqrt{2\alpha_4}}S^4_{1,4},$$
$$K^1_9 := \frac{1}{\sqrt{2\alpha_5}}E^4_{1,2}, \quad K^1_{10} := \frac{i}{\sqrt{2\alpha_6}}S^4_{1,2}, \quad K^1_{11} := \frac{}{\sqrt{2\alpha_7}}E^4_{3,4}, \quad K^1_{12} := \frac{i}{\sqrt{2\alpha_8}}S^4_{3,4},$$
$$K^1_{13} := \frac{i}{2\sqrt{\beta}}(S^4_{1,1} - S^4_{2,2} + S^4_{3,3} - S^4_{4,4}), \quad K^1_{14} := \frac{i}{2\sqrt{\gamma}}(-S^4_{1,1} + S^4_{2,2} + S^4_{3,3} - S^4_{4,4})$$

and

$$H^1 := \frac{i}{2}(S^4_{1,1} + S^4_{2,2} - S^4_{3,3} - S^4_{4,4}).$$

We have $\mathfrak{su}(4) = \mathfrak{so}(2) \oplus \mathfrak{m}_1 = span(\{H^1\} \cup \{K^1_i \mid i = 1..14\})$. We get the representation of SO(2) as

$$\varrho_1(H^1)K^1_i = K^1_{i+1}, \; \varrho_1(H^1)K^1_{i+1} = -K^1_i \text{ for } i = 1,3,5,7$$

and

$$\varrho_1(H^1)K^1_i = 0 \text{ for } i = 9..14.$$

This gives an identification $\mathfrak{m}_1 \to \mathfrak{m}$, $K^1_i \mapsto B_i$ inducing an inclusion $\mathrm{SO}(2) \subset \mathrm{Sp}(3) \subset \mathrm{SO}(\mathfrak{m})$ because of $\varrho_1(H^1) = \sqrt{2}\varrho(A_{21})$, and therefore defines an Sp(3) structure on SU(4)/SO(2). We compute the torsion and get in the basis we just defined

$$\begin{aligned} T = \frac{1}{\sqrt{2\alpha}}&(e_1e_5e_9 - e_1e_6e_{10} + e_1e_7e_{11} + e_1e_8e_{12} + e_2e_5e_{10} + e_2e_6e_9 - e_2e_7e_{12} + e_2e_8e_{11} \\ &- e_3e_5e_{11} + e_3e_6e_{12} - e_3e_7e_9 - e_3e_8e_{10} - e_4e_5e_{12} - e_4e_6e_{11} + e_4e_7e_{10} - e_4e_8e_9) \\ &+ \frac{\sqrt{\beta}}{\alpha}(e_5e_6e_{13} - e_7e_8e_{13} - e_9e_{10}e_{13} - e_{11}e_{12}e_{13}) + \frac{\sqrt{\gamma}}{\alpha}(e_1e_2e_{14} - e_3e_4e_{14} + e_9e_{10}e_{14} - e_{11}e_{12}e_{14}). \end{aligned}$$

Remark A.1. This is not the only possible inclusion $\mathrm{SO}(2) \subset \mathrm{Sp}(3)$. We get other identifications $\mathfrak{m}_1 \cong \mathfrak{m}$ inducing other Sp(3) structures.

A.2. **U(4)/SO(2) × SO(2).**

We define a basis using almost the same matrices as above but taking other normalizers

$$K^2_1 := \frac{1}{\sqrt{2\alpha}}E^4_{1,3}, \quad K^2_2 := \frac{i}{\sqrt{2\alpha}}S^4_{1,3}, \quad K^2_3 := \frac{1}{\sqrt{2\alpha_2}}E^4_{2,4}, \quad K^2_4 := \frac{i}{\sqrt{2\alpha_2}}S^4_{2,4},$$

$$K^2_5 := \frac{1}{\sqrt{2\alpha_3}}E^4_{2,3}, \quad K^2_6 := \frac{i}{\sqrt{2\alpha_3}}S^4_{2,3}, \quad K^2_7 := \frac{1}{\sqrt{2\alpha_4}}E^4_{1,4}, \quad K^2_8 := \frac{i}{\sqrt{2\alpha_4}}S^4_{1,4},$$

$$K^2_9 := \frac{1}{\sqrt{2\alpha_5}}E^4_{1,2}, \quad K^2_{10} := \frac{i}{\sqrt{2\alpha_5}}S^4_{1,2}, \quad K^2_{11} := \frac{}{\sqrt{2\alpha_6}}E^4_{3,4}, \quad K^2_{12} := \frac{i}{\sqrt{2\alpha_6}}S^4_{3,4},$$

$$K^2_{13} := \frac{i}{2\sqrt{\beta}}(S^4_{1,1} - S^4_{2,2} + S^4_{3,3} - S^4_{4,4}), \quad K^2_{14} := \frac{i}{2\sqrt{\gamma}}(S^4_{1,1} + S^4_{2,2} + S^4_{3,3} + S^4_{4,4})$$

and

$$H^2_1 := \frac{i}{2}(S^4_{1,1} + S^4_{2,2} - S^4_{3,3} - S^4_{4,4}), \quad H^2_2 := \frac{i}{2}(-S^4_{1,1} + S^4_{2,2} + S^4_{3,3} - S^4_{4,4}),$$

getting $\mathfrak{u}(4) = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{m}_2 = \mathit{span}(\{H^2_1,\ H^2_2\} \cup \{K^2_i \mid i = 1..14\})$. The representation of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ is given by

$$\varrho_2(H^2_1)K^2_i = K^2_{i+1},\ \varrho_2(H^2_1)K^2_{i+1} = -K^2_i \text{ for } i = 1, 3, 5, 7, \quad \varrho_2(H^1)K^1_i = 0 \text{ for } i = 9..14,$$

and

$$\varrho_2(H^2_2)K^2_i = -K^2_{i+1},\ \varrho_2(H^2_2)K^2_{i+1} = K^2_i \text{ for } i = 1, 9,$$

$$\varrho_2(H^2_2)K^2_i = K^2_{i+1},\ \varrho_2(H^2_2)K^2_{i+1} = -K^2_i \text{ for } i = 3, 11,$$

$$\varrho_2(H^1)K^2_i = 0 \text{ for } i = 5..8, 13, 14.$$

We choose the identification $\mathfrak{m}_2 \to \mathfrak{m}$, $K^2_i \mapsto B_i$ inducing a inclusion $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{Sp}(3) \subset \mathrm{SO}(\mathfrak{m})$ because of $\varrho_2(H^2_1) = \sqrt{2}\varrho(A_{21})$ and $\varrho_2(H^2_2) = \sqrt{2}\varrho(A_{10})$, therefore defining a Sp(3) structure on SU(4)/(SO(2) × SO(2)). In this basis we can compute the torsion and get

$$\begin{aligned} T = \frac{1}{\sqrt{2\alpha}}&(e_1e_5e_9 - e_1e_6e_{10} + e_1e_7e_{11} + e_1e_8e_{12} + e_2e_5e_{10} + e_2e_6e_9 - e_2e_7e_{12} + e_2e_8e_{11} \\ &- e_3e_5e_{11} + e_3e_6e_{12} - e_3e_7e_9 - e_3e_8e_{10} - e_4e_5e_{12} - e_4e_6e_{11} + e_4e_7e_{10} - e_4e_8e_9) \\ &+ \frac{\sqrt{\beta}}{\alpha}(e_5e_6e_{13} - e_7e_8e_{13} - e_9e_{10}e_{13} - e_{11}e_{12}e_{13}) \end{aligned}$$

A.3. **U(4) × U(1)/SO(2) × SO(2) × SO(2).**

We define a basis of $\mathfrak{u}(4) \oplus \mathfrak{u}(1) = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{m}_3$ with K_i^3 for $i = 1..14$ a basis of $\mathfrak{m}_3$ and H_i^3 for $i = 1..3$ a basis of $\mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$:

$$K_i^3 := (K_i^2, 0) \text{ for } i \neq 13 \text{ and } K_{13}^3 := (0, \frac{i}{\sqrt{\beta}}),$$

$$H_i^3 := (H_i^2, 0) \text{ for } i = 1, 2 \text{ and } H_3^3 := (\frac{i}{2}(S_{1,1}^4 - S_{2,2}^4 + S_{3,3}^4 - S_{4,4}^4), 0).$$

Identifying $\mathfrak{m} \cong \mathfrak{m}_1 \cong \mathfrak{m}_2 \cong \mathfrak{m}_3$ with $A_i \mapsto K_i^1 \mapsto K_i^2 \mapsto K_i^3$ we get the representation ϱ_3 of $\mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$ by

$$\sqrt{2}\varrho(A_{21}) = \varrho_1(H^1) = \varrho_2(H_1^2) = \varrho_3(H_1^3), \quad \sqrt{2}\varrho(A_{10}) = \varrho_2(H_2^2) = \varrho_3(H_2^3), \quad \sqrt{2}\varrho(A_9) = \varrho_3(H_3^3)$$

and again we get a Sp(3) structure. In this basis we can compute the torsion and get

$$\begin{aligned} T = &\frac{1}{\sqrt{2\alpha}}(e_1e_5e_9 - e_1e_6e_{10} + e_1e_7e_{11} + e_1e_8e_{12} + e_2e_5e_{10} + e_2e_6e_9 - e_2e_7e_{12} + e_2e_8e_{11} \\ &- e_3e_5e_{11} + e_3e_6e_{12} - e_3e_7e_9 - e_3e_8e_{10} - e_4e_5e_{12} - e_4e_6e_{11} + e_4e_7e_{10} - e_4e_8e_9). \end{aligned}$$

A.4. **SU(5)/Sp(2).**

The Lie algebra of Sp(2) ⊂ Sp(3) and its splitting of V^{14} is given by Theorem A.1. Calculating the torsion tensor we get

$$\begin{aligned} T = &\frac{2\alpha - \beta}{2\alpha\sqrt{\beta}}(e_1e_2e_{13} + e_1e_5e_9 - e_1e_6e_{10} + e_1e_7e_{11} + e_1e_8e_{12} + e_2e_5e_{10} + e_2e_6e_9 - e_2e_7e_{12} \\ &+ e_2e_8e_{11} + e_3e_4e_{13} + e_3e_5e_{11} - e_3e_6e_{12} - e_3e_7e_9 - e_3e_8e_{10} + e_4e_5e_{12} \\ &+ e_4e_6e_{11} + e_4e_7e_{10} - e_4e_8e_9 - e_5e_6e_{13} - e_7e_8e_{13}) \\ &+ \frac{\sqrt{5\beta\gamma} - \sqrt{6}(\alpha + \beta)}{2\alpha\sqrt{\beta}}(e_1e_2e_{14} + e_3e_4e_{14} + e_5e_6e_{14} + e_7e_8e_{14}). \end{aligned}$$

A.5. **Maximal subgroups of Sp(3).** Using Dynkin's results [D57], Gorodski and Podesta listed the maximal connected subgroups of Sp(n) in [GP05]. We restate the result for $G \subset$ Sp(3) and add the decompositions of V^{14} into subrepresentations of $G \subset$ Sp(3), computed easily via an appropriate computer algebra system.
Given a group $G \subset$ Sp(3), we give a basis of its Lie algebra $\mathfrak{g} \subset \mathfrak{sp}(3) = \langle A_i \mid i = 1..21\rangle$, the decomposition of $V^{14} = V_1 \oplus .. \oplus V_r$ in irreducible subspaces, and a basis of each $V_k \subset V^{14} = \langle B_i \mid i = 1..14\rangle$.

Theorem A.1. *All maximal connected subgroups of* Sp(3) *and the decomposition of* V^{14} *into submodules for these subgroups are listed in Table* 2. *Furthermore, the subgroup* Sp(2) ⊂ Sp(2) × Sp(1) ⊂ Sp(3) *with Lie algebra* $\mathfrak{sp}(2) = \langle\{A_i \mid i = 1..10\}\rangle$ *acts irreducibly on* Δ_5*, the irreducible* 8*-dimensional spin representation of* Spin(5) ≅ Sp(2) *and on* $\mathfrak{p}^5$*, its usual vector representation, and thus* V^{14} *has the same decomposition into* Sp(2)*-isotopic summands as under* Sp(2) × Sp(1),

$$V^{14} \overset{\mathrm{Sp}(2)}{=} \Delta_5 \oplus \mathfrak{p}^5 \oplus \mathfrak{p}^1,$$

$\mathfrak{p}^1$ *being the trivial representation.*

$G \subset \mathrm{Sp}(3)$	Basis of $\mathfrak{g} \subset \mathfrak{sp}(3)$	V^{14}	Basis of invariant submodules
U(3)	$A_1, A_2, A_9, A_{10}, A_{11}, A_{12}, A_{17}, A_{18}, A_{21}$	$\mathbb{R}^8$	$B_1, B_2, B_5, B_6, B_9, B_{10}, B_{13}, B_{14}$
		$\mathbb{R}^6$	$B_3, B_4, B_7, B_8, B_{11}, B_{12}$
SO(3)	$\sqrt{10}A_1 + 4A_{17} - 3A_{19}$, $\sqrt{10}A_2 + 4A_{18} + 3A_{20}$, $3A_9 + 5A_{10} + A_{21}$	$\mathbb{R}^9$	$\frac{2}{\sqrt{3}}B_{13} + B_{14}, -\sqrt{\frac{5}{2}}B_6 + B_{10}, -\sqrt{\frac{5}{2}}B_5 + B_9, \frac{3}{\sqrt{5}}B_2 + B_8, -\frac{3}{\sqrt{5}}B_1 + B_7, B_3, B_4, B_{11}, B_{12}$
		$\mathbb{R}^5$	$-\frac{\sqrt{3}}{2}B_{13} + B_{14}, \sqrt{\frac{2}{5}}B_6 + B_{10}, \sqrt{\frac{2}{5}}B_5 + B_9, -\frac{\sqrt{5}}{3}B_2 + B_8, \frac{\sqrt{5}}{3}B_1 + B_7$
$\mathrm{Sp}(2) \times \mathrm{Sp}(1)$	$A_1,\ \ldots\ , A_{10},\ A_{19}, A_{20}, A_{21}$	Δ_5	$B_1, .., B_8$
		$\mathfrak{p}^5$	$B_9, ..B_{13}$
		$\mathfrak{p}^1$	B_{14}
$\mathrm{SO}(3) \times \mathrm{Sp}(1)$	$A_1, A_{11}, A_{17}, A_9 + A_{10} + A_{21}, A_5 + A_6 + A_{19}, A_7 + A_8 + A_{20}$	$\mathbb{R}^3 \otimes \mathbb{R}^3$	$B_1, B_3, B_4, B_5, B_7, B_8, B_9, B_{11}, B_{12}$
		$\mathbb{R}^5 \otimes \mathbb{R}^1$	$B_2, B_6, B_{10}, B_{13}, B_{14}$

TABLE 2. Maximal connected subgroups of Sp(3) and decompositions of V^{14} into submodules.

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