Quarklet Frames in Adaptive Numerical Schemes

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AT15, San Antonio, TX

5-23-2016

Joint work with Stephan Dahlke and Thorsten Raasch
Let $H$ be a Hilbert space with its dual $H'$. We are interested in the solution $u \in H$ of the operator equation

$$\mathcal{L}u = f,$$

with $\mathcal{L} : H \to H'$ a boundedly invertible operator and $f \in H'$.

Example: Poisson equation on $\Omega = [-1, 1]^2 \setminus (0, 1]^2$ with homogeneous Dirichlet boundary conditions,

$$-\Delta u = f \quad \text{in } \Omega,$$

with $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ and $f \in H^{-1}(\Omega)$. 


Numerical treatment

- Adaptive space refinement methods with high convergence rate available (FEM and wavelet methods).
- There is some hope that we can beat them.
- Idea: combine adaptivity and space refinement with polynomial enrichment (hp-method).
- Until now hp-methods exist only in the finite element setting.
- Quarklets are the first hp approach based on wavelets.
- In the long run: provable exponential convergence rates.
Discretization

- Discretization of the operator equation via a frame $\Psi = \{\psi_\lambda\}_{\lambda \in \Lambda}$ of $H$ leads to a matrix-vector equation

$$Au = f,$$

with $A = \{L\psi_\lambda(\psi_\mu)\}_{\lambda, \mu \in \Lambda}$, $f = \{f(\psi_\lambda)\}_{\lambda \in \Lambda}$ and $u = \Psi^T u$.

- Note: for a redundant frame the matrix $A$ is only positive semidefinite.

- Use an adaptive approximation scheme (e.g.: Richardson, steepest descent) to solve this equation.

- Essential for high convergence rates:
  - fast convergence in respect of best N-term approximation.
  - certain compression properties of the stiffness matrix $A$.  

- Background
- Operator equation
- Discretization
- Quarklet frames
- Construction of the quarklets
- Properties
- Outlook
- Literature
Construction of quarklets

- To keep it simple: quarklet frames on the real axis.
- CDF basis. Mother wavelet $\psi$ with

\[ \psi(x) = \sum_{k \in \mathbb{Z}} b_k \varphi(2x - k) \quad \text{for all } x \in \mathbb{R}, \]

of order $d$ with $\tilde{d}$ vanishing moments and generator function $\varphi = N_d(\cdot + \lfloor \frac{d}{2} \rfloor)$. 
Quarks through multiplying the generator function $\varphi$ with monomes:

$$\varphi_p(x) := \left(\frac{x}{\lceil m/2 \rceil}\right)^p \varphi(x), \quad \text{for all } p \geq 0, x \in \mathbb{R}.$$ 

(a) $p = 0$
(b) $p = 1$
(c) $p = 2$

Figure: B-Spline Quarks $\varphi_p$ of order $d = 2$
Quarklets

Quarklets are defined by

$$\psi_p(x) = \sum_{k \in \mathbb{Z}} b_k \varphi_p(2x - k) \quad \text{for all } p \geq 0, x \in \mathbb{R}.$$
For $j_0 \in \mathbb{N}$ we analyze systems of dilated and translated quarklets

$$
\psi_{p,j,k} := 2^{j/2} \psi_p(2^j \cdot -k), \quad \text{for all } p \geq 0, j \geq j_0, k \in \mathbb{Z},
$$

and quarks

$$
\psi_{p,j_0-1,k} := \varphi_{p,j_0,k} := 2^{j_0/2} \varphi_p(2^{j_0} \cdot -k), \quad \text{for all } p \geq 0, k \in \mathbb{Z}.
$$
Properties

- Quarklets have the same amount of vanishing moments as the underlying wavelets.
- Properly scaled versions of the quarklet systems build frames in $L_2(\mathbb{R})$ and $H^1(\mathbb{R})$.
- The corresponding Laplacian stiffness matrix is compressible (i.e. it can be well approximated by matrices with finitely many nontrivial entries).
Compression property

Theorem (Compression of the Laplacian)

Let \( d \geq 3 \). For \( J \in \mathbb{N}_0 \), we define the biinfinite matrix \( A_J \) by dropping the entries \( a_{\lambda, \lambda'} \) from \( A \) when

\[
a \log_2(1 + |p - p'|) + b |j' - j| > J,
\]

with \( a, b > 0 \) fulfilling some technical conditions. Then the number of non-zero entries in each row and column of \( A_J \) is of order \( 2^J \), and

\[
\| A - A_J \|_{\mathcal{L}(\ell_2(\Lambda))} \lesssim 2^{-J(d-2)/b}.
\]
Numerical results

Figure: Compression of the Laplacian $\mathbf{A}$ for $d = \tilde{d} = 3$. 
Outlook

- Convergence rates
- 2d (L-shaped domain)
- Implementation
Literature


Thank you very much!