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Lösungen zum 12. Übungsblatt zur Analysis I

12.1:
Zu (1): $I_n := \int_0^b x^n \exp(-x)dx = \left[-x^n \exp(-x)\right]_0^b + n\int_0^b x^{n-1}\exp(-x)dx$, also

$$I_n = -b^n \exp(-b) + nI_{n-1} \tag{1}$$

$n=1: \quad I_1 = -b\exp(-b) + I_0 = -b\exp(-b) + \left[-\exp(-x)\right]_0^b = 1 - \exp(-b) - b\exp(-b)$

$n=2: \quad I_2 \underset{(1)}{=} -b^2\exp(-b) + 2\big(1-\exp(-b) - b\exp(-b)\big)$

$\quad = 2(1 - \exp(-b) - b(\exp(-b) - \frac{1}{2}b^2\exp(-b))$

$n=3: \quad I_3 \underset{(1)}{=} -b^3\exp(-b) + 3\Big(2(1-\exp(-b) - b(\exp(-b) - \frac{1}{2}b^2\exp(-b))\Big)$

$\quad = 3!\Big((1 - (1 + b + \frac{1}{2}b^2 + \frac{1}{3!}b^3)\exp(-b)\Big)$

Sei $\quad T := \left\{ n \geq 1 | I_n = n!\left(1 - \left(\sum_{k=0}^n \frac{b^k}{k!}\right)\exp(-b)\right)\right\}$ und $n \in T$. Dann folgt

$n+1: \quad I_{n+1} \underset{(1)}{=} -b^{n+1}\exp(-b) + (n+1)n!\left(1 - \left(\sum_{k=0}^n \frac{b^k}{k!}\right)\exp(-b)\right)$

$\quad = (n+1)!\left(1 - \left(\sum_{k=0}^{n+1}\frac{b^k}{k!}\right)\exp(-b)\right)$, d.h. $n+1 \in T$.

Folglich $T = \{n \in \mathbb{N} | n \geq 1\}$.

Zu (2): $I_n := \int_0^\pi \sin^n x dx = \int_0^\pi \sin^{n-1} x \cdot \sin x dx$.

Setze $u(x) := \sin^{n-1} x$ und $v'(x) := \sin x$. Dann ist $u'(x) = (n-1)\sin^{n-2} x \cos x$ und $v(x) = -\cos x$, folglich

$I_n = \left[-\sin^{n-1} x\cos x\right]_0^\pi + \int_0^\pi (n-1)\sin^{n-2} x\cos^2 x dx = (n-1)\int_0^\pi \sin^{n-2} x(1-\sin^2 x)dx$

$\quad = (n-1)I_{n-2} - (n-1)I_n$

Daraus folgt für $n \geq 2$

$$I_n = \frac{n-1}{n} I_{n-2} \tag{2}$$

Mit $I_0 = \int_0^\pi dx = \pi$ und $I_1 = \int_0^\pi \sin(x)dx = \left[-\cos x\right]_0^\pi = 2$ folgt hieraus für n gerade, $n = 2k$, bzw. n ungerade, $n = 2k+1$,

$$I_{2k} = \frac{2k-1}{2k} I_{2(k-1)} = \underset{k \text{ Schritte}}{\ldots\ldots} = \prod_{j=1}^k \frac{2j-1}{2j} I_0 = \pi \prod_{j=1}^k \frac{2j-1}{2j}$$

$$I_{2k+1} = \frac{2k}{2k+1} I_{2(k-1)+1} = \underset{k \text{ Schritte}}{\ldots\ldots} = \prod_{j=1}^k \frac{2j}{2j+1} I_1 = 2\prod_{j=1}^k \frac{2j}{2j+1}$$

Zu (3): $I_{p,q} := \int_0^1 x^p(1-x)^q dx,\ p,q \in \mathbb{N}_+$.

Setze $u(x) := x^p,\ v'(x)(1-x)^q$. Dann ist $u'(x) = px^{p-1}$ und $v(x) = -\frac{1}{q+1}(1-x)^{q+1}$. Folglich $I_{p,q} = \left[-\frac{1}{q+1}x^p(1-x)^{q+1}\right]_0^1 + \int_0^1 px^{p-1}\frac{1}{q+1}(1-x)^{q+1}dx = \frac{p}{q+1}\int_0^1 x^{p-1}(1-x)^{q+1}dx$. Daraus folgt für $p,q \in \mathbb{N}_+$

$$I_{p,q} = \frac{p}{q+1} I_{p-1,q+1} \tag{3}$$

$$\begin{aligned} I_{p,q} &= \frac{p}{q+1} I_{p-1,q+1} = \frac{p\cdot(p-1)}{(q+1)\cdot(q+2)} I_{p-2,q+2} \underset{p \text{ Schritte}}{=\ \cdots\cdots =} \frac{p\cdot(p-1)\cdot\ldots\cdot 1}{(q+1)\cdot(q+2)\cdot\ldots\cdot(q+p)} I_{0,q+p} \\ &= \frac{p!q!}{(p+q)!}\int_0^1 (1-x)^{p+q}dx = \frac{p!q!}{(p+q)!}\left[-\frac{1}{p+q+1}(1-x)^{p+q+1}\right]_0^1 \\ &= \frac{p!q!}{(p+q+1)!} \end{aligned}$$

12.2:
Zu (1): Mit $u := \varphi(x) := 2x+3,\ \varphi'(x) = 2$ folgt

$$\begin{aligned} \int_0^1 (2x+3)^{1/2}dx &= \frac{1}{2}\int_0^1 \big(\varphi(x)\big)^{1/2}\varphi'(x)dx = \frac{1}{2}\int_{\varphi(0)}^{\varphi(1)} u^{1/2}du = \frac{1}{2}\Big[\frac{2}{3}u^{3/2}\Big]_3^5 = \frac{1}{3}\big(5^{3/2} - 3^{3/2}\big) \\ &= \frac{1}{3}\big(\sqrt{125} - \sqrt{27}\big) \end{aligned}$$

Zu (2): Mit $u := \varphi(x) := \log x,\ \varphi'(x) = \frac{1}{x}$ folgt

$$\int_e^{2e} \frac{dx}{x\log x} = \int_e^{2e} \frac{1}{\varphi(x)}\varphi'(x)dx = \int_{\varphi(e)}^{\varphi(2e)} \frac{1}{u}du = \Big[\log u\Big]_1^{\log(2e)} = \log(\log(2e))$$

Zu (3): Nicht lösbar mit Stoff der Vorlesung. Hätte "xdx" im Zähler lauten müssen.
Fall "xdx" im Zähler: Mit $u = \varphi(x) := 1 - 2x^2,\ \varphi'(x) = -4x$ folgt

$$\begin{aligned} \int_0^{\sqrt{2}/8} \frac{xdx}{\sqrt{1-2x^2}} &= -\frac{1}{4}\int_0^{\sqrt{2}/8} \big(\varphi(x)\big)^{-1/2}\varphi'(x)dx = -\frac{1}{4}\int_{\varphi(0)}^{\varphi(\sqrt{2}/8)} u^{-1/2}du = -\frac{1}{2}\big[\sqrt{u}\big]_1^{60/64} \\ &= \frac{1}{2} - \frac{\sqrt{15}}{8} \end{aligned}$$

12.3:
Per Definition ist $x \geq 0$. Aus der Voraussetzung

$$x'(t) = c\cdot t\cdot x(t), \quad x(t_0) > 0,\ c > 0 \tag{4}$$

folgt ($t_0 \geq 0$ vorausgesetzt) $x(t) - x(t_0) = \displaystyle\int_{t_0}^t x'(t)dt = c\int_{t_0}^t tx(t)dt \geq 0$, also $x(t) \geq x(t_0) > 0$ für alle $t > 0$. Division der Gleichung (4) durch das positive $x(t)$ liefert $\frac{x'(t)}{x(t)} = ct$ und damit für alle $T \geq t_0$

$$\begin{aligned}
\int_{t_0}^{T} \frac{x'(t)}{x(t)} dt &= \int_{t_0}^{T} ctdt \\
\Big[\log\big(x(t)\big)\Big]_{t_0}^{T} &= \frac{c}{2}[t^2]_{t_0}^{T} \\
\log\big(x(T)\big) &= \log\big(x(t_0)\big) + \frac{c}{2}\big(T^2 - t_0^2\big) \\
x(T) &= \exp\Big(\log\big(x(t_0)\big) + \frac{c}{2}\big(T^2 - t_0^2\big)\Big) = x(t_0)\exp\Big(\frac{c}{2}\big(T^2 - t_0^2\big)\Big) \\
&= \frac{x(t_0)}{\exp\big(\frac{c}{2}t_o^2\big)} \exp\Big(\frac{c}{2}T^2\Big), \quad T \geq 0.
\end{aligned}$$

12.4:
(1) $\sim$ is an equivalence relation $\iff$ $\sim$ is reflexive, symmetric and transitive.
reflexive: $f \sim f$ holds because of $\lim\limits_{\substack{x\mapsto\overline{x} \\ x\neq\overline{x}}} \left|\frac{f(x)-f(x)}{x-\overline{x}}\right| = 0.$

symmetric: $f \sim g \Rightarrow g \sim f$ follows in a similar way because of $|f(x) - g(x)| = |g(x) - f(x)|$.
reflexive: $f \sim g$, $g \sim h \Rightarrow f \sim h$ is a consequence of the triangular inequality: For all $x \neq \overline{x}$ we have

$$0 \leq \left|\frac{f(x)-h(x)}{x-\overline{x}}\right| \leq \left|\frac{f(x)-g(x)}{x-\overline{x}}\right| + \left|\frac{g(x)-h(x)}{x-\overline{x}}\right| \underset{x\to\overline{x}}{\longrightarrow} 0$$

It follows that $\lim\limits_{\substack{x\mapsto\overline{x} \\ x\neq\overline{x}}} \left|\frac{f(x)-h(x)}{x-\overline{x}}\right|$ exists and equals 0, i.e. $f \sim h$ holds.

(2): uniqueness: Let L, L' be both affine linear and $f \sim L$, $f \sim L'$. By (1) $L \sim L'$. L(x):=ax+b, L'(x)=a'x+b'. For all $x \neq \overline{x}$ we have

$$0 = \lim_{\substack{x\mapsto\overline{x} \\ x\neq\overline{x}}} \left|\frac{L(x)-L'(x)}{x-\overline{x}}\right| \cdot |x-\overline{x}| = \lim_{\substack{x\mapsto\overline{x} \\ x\neq\overline{x}}} |L(x) - L'(x)| = |L(\overline{x}) - L'(\overline{x})| = |(a-a')\overline{x} + (b-b')|,$$

hence

$$\begin{aligned}
& b - b' = -(a-a')\overline{x} \\
\Rightarrow\quad & L(x) - L'(x) = (a-a')x + (b-b') = (a-a')(x-\overline{x}) \\
\Rightarrow\quad & \tfrac{L(x)-L'(x)}{x-\overline{x}} = a - a' \\
\Rightarrow\quad & 0 = \lim_{\substack{x\mapsto\overline{x} \\ x\neq\overline{x}}} \left|\frac{L(x)-L'(x)}{x-\overline{x}}\right| = |a-a'| \\
\Rightarrow\quad & a = a' \\
\Rightarrow\quad & b = b' \\
\Rightarrow\quad & L = L'
\end{aligned}$$

Assume $f \sim L$, $L(x) = ax + b$.

$$
\begin{aligned}
&\Rightarrow \quad 0 = \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \left| \frac{f(x) - L(x)}{x - \overline{x}} \right| \\
&\Rightarrow \quad 0 = \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \left| \frac{f(x) - L(x)}{x - \overline{x}} \right| \cdot |x - \overline{x}| = \lim_{x \to \overline{x}} |f(x) - L(x)| = |f(\overline{x}) - a\overline{x} - b| \\
&\Rightarrow \quad b = f(\overline{x}) - a\overline{x} \\
&\Rightarrow \quad L(x) = ax + b = f(\overline{x}) + a(x - \overline{x}) \\
&\Rightarrow \quad 0 = \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \left| \frac{f(x) - L(x)}{x - \overline{x}} \right| = \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \left| \frac{f(x) - f(\overline{x}) - a(x - \overline{x})}{x - \overline{x}} \right| = \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \left| \frac{f(x) - f(\overline{x})}{x - \overline{x}} - a \right| \\
&\Rightarrow \quad \lim_{\substack{x \mapsto \overline{x} \\ x \neq \overline{x}}} \frac{f(x) - f(\overline{x})}{x - \overline{x}} \text{ exists and equals } a \\
&\Rightarrow \quad f \text{ is differentiable at } \overline{x} \text{ and } Df(\overline{x}) = a \\
&\Rightarrow \quad L(x) = f(\overline{x}) + Df(\overline{x})(x - \overline{x})
\end{aligned}
$$

Now let f be differentiable at $\overline{x}$ and $L : \mathbb{R} \longrightarrow \mathbb{R}$, $L(x) := f(\overline{x}) + Df(\overline{x})(x - \overline{x})$.
By definition of differentiability we have $f \sim L$, with L affine linear. If also $g \sim f$, then, by transitivity of $\sim$, $g \sim L$, therefore, by the above arguments, g is differentiable at $\overline{x}$ and $L(x) = Dg(\overline{x}) + (x - \overline{x})$ for all $x \in \mathbb{R}$. So $0 = \big(f(\overline{x}) - g(\overline{x})\big) + \big(Df(\overline{x}) - Dg(\overline{x})\big)(x - \overline{x})$ for all $x \in \mathbb{R}$, hence

$$f(\overline{x}) = g(\overline{x}) \text{ and } Df(\overline{x}) = Dg(\overline{x}).$$