Two–Dimensional Interpolating Lagrange Wavelets

The construction of interpolating Lagrange wavelets and scaling functions for scaling matrices A, $|\det(A)| = 2$ or $\det(A)$ odd, is based on trigonometric polynomials $m(\omega_1, \omega_2)$ satisfying

$$m(\omega_1,\omega_2) + \sum_{\rho\in\mathcal{D}\setminus\{0\}} m(\omega + A^{-t}\rho) \;=\; 1$$

$$m(\omega_1,\omega_2) \;=\; \frac{1}{|\det(A)|} + \frac{1}{|\det(A)|}\left\{\sum_{\rho\in\mathcal{D}\setminus\{0\}} e^{2\pi i\rho\cdot\omega} \sum_{k\in\mathcal{T}} s_k(-A\rho)e^{2\pi i Ak\cdot\omega}\right\}$$

Integer grid $A\mathbb{Z}^2$, index set $\mathcal{T}\subset\mathbb{Z}^2$, set of representatives $\mathcal{D}\hat{=}A^t\mathbb{Z}^2/\mathbb{Z}^2$,
s_k interpolating Lagrange polynomial: $s_k(k') = \delta_{kk'}$ for all $k, k' \in \mathcal{T}$

Example: quincunx grid $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathcal{T} = \{(l_1, l_2) \mid -L \le l_1 \le L-1\ ,\ -L+1 \le l_2 \le L\ \}$

coefficient mask ($L = 2$):

$$\begin{array}{ccccccc}
 & & & \frac{1}{512} & & & \\
 & & \frac{-9}{512} & & \frac{-9}{512} & & \\
 & \frac{-9}{512} & & \frac{81}{512} & & \frac{-9}{512} & \\
\frac{1}{512} & & \frac{81}{512} & \frac{1}{2} & \frac{81}{512} & & \frac{1}{512} \\
 & \frac{-9}{512} & & \frac{81}{512} & & \frac{-9}{512} & \\
 & & \frac{-9}{512} & & \frac{-9}{512} & & \\
 & & & \frac{1}{512} & & &
\end{array}$$

properties:

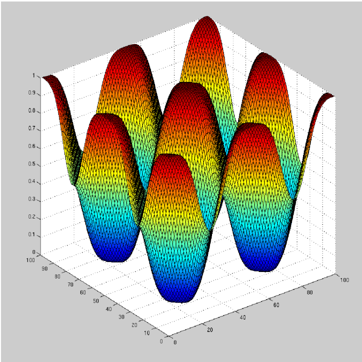
number of coefficients:	17
Strang–Fix conditions:	order 3
regularity in $H^s(\mathbb{R}^2)$:	$s = 1.415$
Hölder exponent α:	$\alpha = 0.415$

regularity function:

$$F(\omega) \;=\; \prod_{j=0}^{2} m((A^t)^j\omega)/C((A^t)^j\omega)$$

$$C(\omega_1,\omega_2) = \left[\frac{\sin^2(\frac{\omega_1+\omega_2}{2})+\sin^2(\frac{\omega_1-\omega_2}{2})}{2\left(\sin^2(\omega_1/2)+\sin^2(\omega_2/2)\right)}\right]^2$$

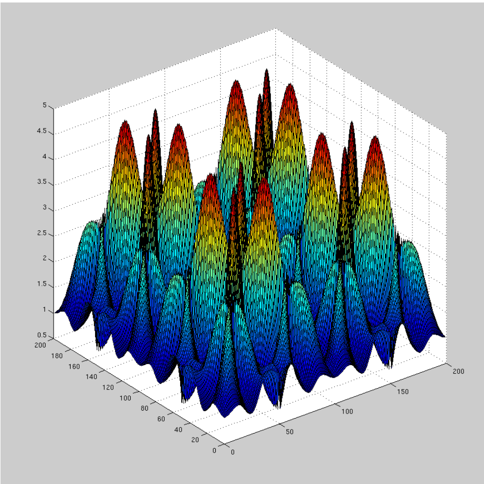
symbol $m(\omega_1,\omega_2)$:



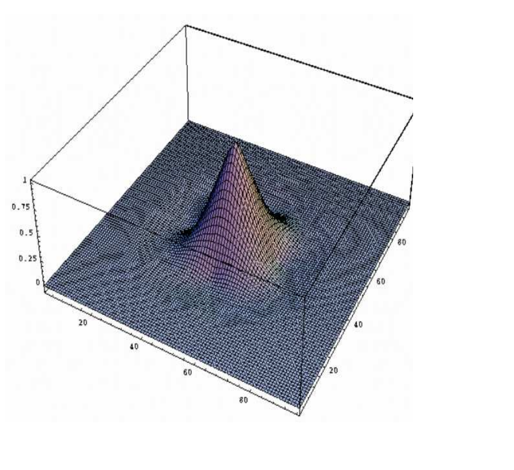
The resulting compactly supported, interpolating scaling function
($\rho = (1,0), h_0 = 1/2, h_{Ak+\rho} = s_k(-1/2, 1/2)/2, k \neq 0:\ h_{Ak} = 0$)

$$x \in \mathbb{R}^2 \;:\; \varphi(x) \;=\; \sum_{k\in\mathbb{Z}^2} h_k\, \varphi(Ax - k)$$

regularity function $F(\omega_1,\omega_2)$:



scaling function $\varphi(x)$:



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