Spatial Besov Regularity for SPDEs on Lipschitz Domains

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Motivation

Goal: Theoretical foundation of adaptive (wavelet) methods for Stochastic Partial Differential Equations (SPDEs) on bounded Lipschitz domains $\mathcal{O} \subseteq \mathbb{R}^d$.
General setting ([3]):

nonadaptive methods $\curvearrowright$ linear approximation
adaptive methods $\curvearrowright$ nonlinear approximation

Target function $u \in L_2(\mathcal{O})$:

$u \in W_2^s(\mathcal{O}) \quad \curvearrowright \quad \|u - u_N\|_{L_2(\mathcal{O})} = O(N^{-s/d})$ for linear approximation

$u \in B^\alpha_{\tau,\tau}(\mathcal{O}),\ \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2} \quad \curvearrowright \quad \|u - u_N\|_{L_2(\mathcal{O})} = O(N^{-\alpha/d})$ for nonlinear approximation

Resulting Question: How high is the Besov Regularity of the solutions of SPDEs?

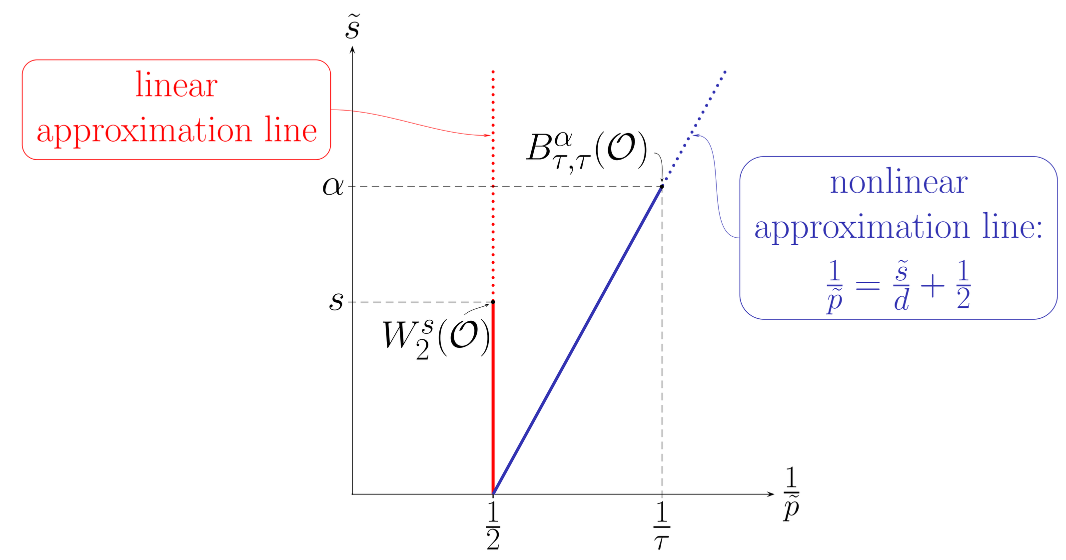

Figure 1: Linear vs. nonlinear appoximation illustrated in a so called DeVore-Triebel-diagram.

SPDEs on Lipschitz Domains and Weighted Sobolev Spaces

The model equation:

$$\mathrm{d}u = \sum_{i,j=1}^d a^{ij} u_{x^i x^j}\, \mathrm{d}t + \sum_{k=1}^\infty g^k\, \mathrm{d}w_t^k \quad \text{on} \quad (0,T] \times \mathcal{O}, \quad u(0,.) = u_0 \quad \text{on } \mathcal{O}, \tag{1}$$

- $\mathcal{O} \subseteq \mathbb{R}^d$ bounded Lipschitz domain and $T \in (0,\infty)$;
- $\{(w_t^k)_{t\in[0,T]},\ k \in \mathbb{N}\}$ independent, real-valued Brownian motions w.r.t. a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $(a^{ij})_{1\le i,j\le d} \subseteq \mathbb{R}^{d\times d}$ strictly positive symmetric matrix;
- $p \in [2,\infty)$; $\gamma \in \mathbb{R}$; $\theta \in \mathbb{R}$;
- $(g^k)_{k\in\mathbb{N}} \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$.

A solution of equation (1) in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ is a stochastic process $u \in \mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ which fulfills equation (4) (see Background) with f replaced by $\sum_{i,j=1}^d a^{ij} u_{x^i x^j} \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)$.

Theorem 1 ([4]). *Let $p \in [2,\infty)$ and $\gamma \in \mathbb{R}$. There exists a constant $\kappa = \kappa(d,p,(a^{ij}),\mathcal{O}) \in (0,1)$ such that if $\theta \in (d-\kappa, d-2+p+\kappa)$, then for any $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$ equation (1) has a unique solution in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. For this solution*

$$\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)} \le C\Big(\|g\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|^p_{U^\gamma_{p,\theta}(\mathcal{O})}\Big), \tag{2}$$

where the constant C depends only on d, p, γ, θ, (a^{ij}), T and $\mathcal{O}$.

Main Result

Theorem 2 ([1]). *Let $p \in [2,\infty)$ and let $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$, $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$ for some $\gamma \in \mathbb{N}$ and $\theta \in (d-\kappa, d-2+p+\kappa)$ with $\kappa = \kappa(d,p,(a^{ij}),\mathcal{O}) \in (0,1)$ as in the Theorem 1.*
Let u be the unique solution in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ of equation (1) and assume furthermore that $u \in L_p(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\ B^s_{p,p}(\mathcal{O}))$ for some $0 < s \le \gamma \wedge (1 + \frac{d-\theta}{p})$.
Then,

$$u \in L_\tau(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad 0 < \alpha < \gamma \wedge \frac{sd}{d-1},$$

and

$$\|u\|_{L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))} \le C\Big(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})} + \|u\|_{L_p(\Omega_T;B^s_{p,p}(\mathcal{O}))}\Big).$$

Proof: Wavelet Characterization of Besov Spaces (Idea of the proof from [2]) & Inequality (2)

Example

Here $d = p = \gamma = \theta = 2$ and $s = 1$. Let $g = (g^k)_{k\in\mathbb{N}}$ be given by

$$g^k(\omega, t, \cdot) := \lambda_k e_k, \qquad k \in \mathbb{N},\ (\omega,t) \in \Omega_T,$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $W_2^1(\mathcal{O})$ (alternatively of $H^1_{2,2}(\mathcal{O})$) and $(\lambda_k)_{k\in\mathbb{N}} \in \ell_1(\mathbb{N})$. Then g is an element of $\mathbb{H}^1_{2,d}(\mathcal{O},T;\ell_2)$. For every initial condition $u_0 \in U^2_{2,2}(\mathcal{O}) = L_2(\Omega, \mathcal{F}_0, \mathbb{P}; H^1_{2,2}(\mathcal{O}))$ equation (1) has a unique solution u in the class $\mathfrak{H}^2_{2,2}(\mathcal{O},T) \subset \mathbb{H}^2_{2,0}(\mathcal{O},T) = L_2(\Omega_T; H^2_{2,0}(\mathcal{O}))$. As a trivial consequence,

$$u \in L_2(\Omega_T; W_2^1(\mathcal{O})) = L_2(\Omega_T; B^1_{22}(\mathcal{O})).$$

The above Theorem states that we have

$$u \in L_2(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O}))$$

for every $0 < \alpha < 2$ and $\tau = \frac{2}{\alpha+1}$. Note that in general u does not belong to $L_2(\Omega_T; W_2^r(\mathcal{O}))$ for $r \in (1,2]$ since $\mathcal{O}$ is an arbitrary bounded Lipschitz domain and thus the higher derivatives might explode near the boundary.

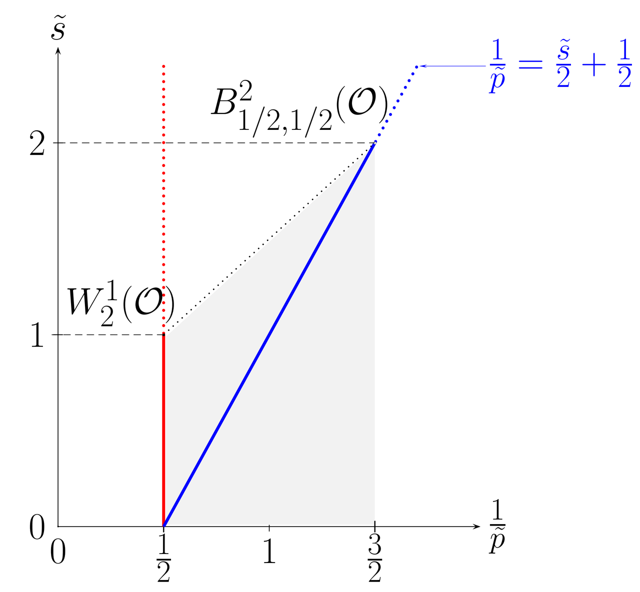

Figure 2: Our results for the paramter constellation $d = p = \gamma = \theta = 2$ and $s = 1$ illustrated in a so called DeVore-Triebel diagram.

General Linear Equations

Our results can be extended to more general linear equations of the type

$$\mathrm{d}u = \sum_{i,j=1}^d \big(a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f\big)\, \mathrm{d}t + \sum_{k=1}^\infty \big(\sigma^{ik} u_{x^i} + \eta^k u + g^k\big)\, \mathrm{d}w_t^k, \quad u(0,\cdot) = u_0, \tag{3}$$

including in particular the case of multiplicative noise. Here the coefficients a^{ij}, b^i, c, σ^{ik}, η^k and the free terms f and g^k are random functions depending on t and x. See [1, Appendix B] for details.

Background Weighted Sobolev Spaces

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\rho(x) := \mathrm{dist}(x, \partial\mathcal{O})$ for $x \in \mathcal{O}$.
Deterministic ([5]). Fix $c > 1$, $k_0 > 0$ and $\zeta_n \in C_0^\infty(\mathcal{O}_n)$ for $n \in \mathbb{Z}$, where

$$\mathcal{O}_n := \{x \in \mathcal{O} : c^{-n-k_0} < \rho(x) < c^{-n+k_0}\} \subseteq \mathcal{O},$$

satisfying $\sum_{n\in\mathbb{Z}} \zeta_n(x) = 1$ and $|D^m \zeta_n(x)| \le N(m) c^{mn}$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}_0$ and $x \in \mathcal{O}$.
For $p \in (1,\infty)$ and $\gamma, \theta \in \mathbb{R}$:

$$H^\gamma_{p,\theta}(\mathcal{O}) := \Big\{u \in \mathcal{D}'(\mathcal{O}) : \|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n\in\mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n\,\cdot)u(c^n\,\cdot)\|^p_{H^\gamma_p(\mathbb{R}^d)} < \infty\Big\},$$

where $H^\gamma_p(\mathbb{R}^d) := (1-\Delta)^{-\gamma/2} L_p(\mathbb{R}^d)$ denotes the Bessel-potential space. If $\gamma \in \mathbb{N}$, we have:

$$H^\gamma_{p,\theta}(\mathcal{O}) = \Big\{u : \rho^{|\alpha|} D^\alpha u \in L_p(\mathcal{O}, \rho(x)^{\theta-d}\mathrm{d}x) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le \gamma\Big\},$$

with the norm equivalence

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} \asymp \sum_{\alpha\in\mathbb{N}_0^d,\, |\alpha|\le\gamma} \int_{\mathcal{O}} \Big|\rho(x)^{|\alpha|} D^\alpha u(x)\Big|^p \rho(x)^{\theta-d}\, \mathrm{d}x.$$

Spaces $H^\gamma_{p,\theta}(\mathcal{O};\ell_2)$ of $\ell_2 = \ell_2(\mathbb{N})$-valued functions are defined similarly.

Stochastic ([4]). Let $\mathcal{P}$ be the predictable σ-algebra w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Set $\Omega_T := \Omega \times [0,T]$. For $\gamma,\ \theta \in \mathbb{R}$ and $p \in [2,\infty)$ we set

$$\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T) := L_p\big(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\ H^\gamma_{p,\theta}(\mathcal{O})\big) \quad \text{and} \quad \mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2) := L_p\big(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\ H^\gamma_{p,\theta}(\mathcal{O};\ell_2)\big),$$

$$U^\gamma_{p,\theta}(\mathcal{O}) := L_p\big(\Omega, \mathcal{F}_0, \mathbb{P};\ H^{\gamma-2/p}_{p,\,\theta+2-p}(\mathcal{O})\big) \quad \text{and}$$

$$\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T) := \Big\{u \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) : u(0,\cdot) \in U^\gamma_{p,\theta}(\mathcal{O}) \text{ and } \mathrm{d}u = f\,\mathrm{d}t + g^k\,\mathrm{d}w_t^k \text{ for some } f \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T),\ g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)\Big\},$$

equipped with the norm

$$\|u\|_{\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)} := \|u\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} + \|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u(0,\cdot)\|_{U^\gamma_{p,\theta}(\mathcal{O})}.$$

The equality $\mathrm{d}u = f\,\mathrm{d}t + g^k\,\mathrm{d}w_t^k$ above is shorthand for

$$\langle u(t,\cdot), \varphi\rangle = \langle u(0,\cdot), \varphi\rangle + \int_0^t \langle f(s,\cdot), \varphi\rangle\, \mathrm{d}s + \sum_{k=1}^\infty \int_0^t \langle g^k(s,\cdot), \varphi\rangle\, \mathrm{d}w_s^k \tag{4}$$

for all $\varphi \in C_0^\infty(\mathcal{O})$, $t \in [0,T]$.

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