Interpolating Scaling Functions with Smooth Duals

Interpolating scaling functions are needed for various applications, e.g. CAGD and collocation methods for operator equations. Moreover, they constitute the basic building blocks for constructing corresponding wavelet bases.

A function $\varphi \in L^2(I\!R^d)$ is called an interpolating scaling function for the scaling matrix A if it is continuous and satisfies a two-scale-relation

$$\varphi(x) \;=\; \sum_{k \in \mathbf{Z}^d} a_k \, \varphi(Ax-k) \quad , \quad \varphi(k) \;=\; \delta_{0,k}, \;\; k \in Z\!\!Z^d \quad .$$

Present research interest centers around the construction of smooth multi-dimensional duals $\tilde{\varphi} \in L^2(I\!R^d)$, which satisfy

$$\langle \varphi(\cdot), \tilde{\varphi}(\cdot - k) \rangle = \delta_{0,k}, \quad k \in \mathbf{Z}^d \quad .$$

So far the best construction for the most important case (A quincunx matrix, $d=2$, φ, $\tilde{\varphi}$ compactly supported) yielded a dual $\tilde{\varphi} \in \mathcal{C}^{\alpha}(\mathbf{R}^2)$ with $\alpha = 0.3132$, see Ji, Riemenschneider and Shen (1999).

primary mask $(L=2)$:

$$\begin{array}{ccccccc}
 & & & \frac{1}{512} & & & \\
 & & \frac{-9}{512} & & \frac{-9}{512} & & \\
 & \frac{-9}{512} & & \frac{81}{512} & & \frac{-9}{512} & \\
\frac{1}{512} & & \frac{81}{512} & \frac{1}{2} & \frac{81}{512} & & \frac{1}{512} \\
 & \frac{-9}{512} & & \frac{81}{512} & & \frac{-9}{512} & \\
 & & \frac{-9}{512} & & \frac{-9}{512} & & \\
 & & & \frac{1}{512} & & &
\end{array}$$

properties of primary function $L=2$:

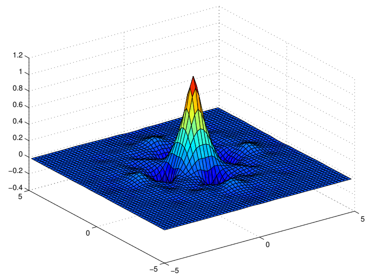
number of coefficients:	17
Strang–Fix conditions:	order 3
Hölder exponent α:	$\alpha = 1.515$

regularity estimates for duals $\tilde{\varphi}$:

Strang-Fix conditions: order p

L	2	2	3	3
K	2	3	1	2
p	7	10	5	11
α_{sup}	0.729	1.823	0.204	1.952

dual $\tilde{\varphi}$ $(L=3,\; K=2)$:



Our construction is based on a three step procedure:

- Choose a primary mask $\{a_k^L\}, L = 2,3$ for a smooth interpolating scaling function, see Dahlke, Gröchenig, Maass (1998).
  This construction utilizes the basic Lagrange interpolation polynomials ℓ_n for $n \in \{-L, -L+1, .., L-1\}$. With $q_L(x) := \sum_{n=-L}^{L-1} \ell_n(-1/2) e^{-inx}$, we obtain the Fourier series of the primary mask $a(\omega)$
  $$a(\omega) = \frac{1}{2} + \frac{1}{2} e^{-i(\omega_1+\omega_2)/2} q_L(\omega_1+\omega_2) e^{-i(\omega_2-\omega_1)/2} q_L(\omega_2-\omega_1).$$
- Compute the mask for the dual function $\tilde{\varphi}$, $K = 1,2,3$:
  $$H_K(z) = a(z)^K T_K(z) = \sum_{\gamma \in G_0} C_{qK}^{\gamma} a(z)^{\gamma_0 - 1} \prod_{\hat{\rho} \in \{0,1\} \setminus \{0\}} b_{\hat{\rho}}^{\gamma}(z) + C_{qK}^{(K,K)} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^{K}(z)$$
  according to the construction procedure in Ji, Riemenschneider, Shen (1999).
- Estimate the Hölder regularity by the invariant subspace theory of Villemoes (1994).

This yields the first examples of continuously differentiable dual scaling functions of this type, e.g. $L=3$, $K=2$, $\alpha = 1.952$, for details see Dahlke, Maass, Teschke (2000).

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