Adaptive Wavelet Frame Methods for Operator Equations: Sparse Grids, Vector-Valued Spaces and Applications to Nonlinear Inverse Parabolic Problems II

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Scope

Parameter identification related to nonlinear parabolic equations

(1) $$g' - \operatorname{div}(D \operatorname{grad} g) + \lambda g = R\Phi(Wg),$$

on bounded Lipschitz domains $U \subset \mathbb{R}^d$ with nonlinear signal response function Φ

Goals

- Reconstruction of reaction parameters $p = (D, \lambda, R, W)$ from noisy data g^δ
- Application to embryogenesis of Drosophila [10]

Ansatz

Ill-posed problem ▹ regularization needed ▹ minimization problem

- Tikhonov regularization with Besov penality term

(2) $$p_\alpha^\delta = \underset{p=(D,\lambda,R,W)}{\operatorname{argmin}} \left\| \mathscr{S}(p) - g^\delta \right\|^2 + \alpha |p|_s,$$

with $p \mapsto \mathscr{S}(p)$:= solution to (1) with parameters p

- Besov penality term promotes sparse solutions

Tasks

- Efficient solver for the forward problem with optimal convergence rates for prescribed tolerance
- Efficient mimimization strategy for (2)

Numerical treatment of the forward problem

I. Parabolic problem

- Solution space for PDE (1)

$$\mathscr{W} = \{u \in L_q(0,T,V_1) : u' \in L_q(0,T,V_2)\},$$

where $V_1 = W_q^1(U, \mathbb{R}^d)$, $V_2 = (W_{q'}^1(U, \mathbb{R}^d))'$

- Implicit discretization by Rothes method: ▹ first in time, then in space ▹ system of elliptic equations per time step:

(3) $$g(t_{n+1}) \approx g^{(n+1)} = g^{(n)} + m_1 g_1$$
$$\left(\frac{1}{h\gamma_{11}} I - J\right) g_1 = \operatorname{div}(D \operatorname{grad} g^{(n)}) - \lambda g^{(n)} + R\Phi(Wg^{(n)})$$

- Strategy: employ adaptive methods (in time/space) to increase efficiency

II. Elliptic problems

Ansatz: adaptive algorithms based on tensor wavelets ▹ dimension-independent convergence rates

- Construction of generalized anisotropic tensor wavelets

Theorem. *Let U be decomposable into cubes $\square_i$. Let Ψ_i be an anisotropic tensor wavelet basis for $L_2(\square_i)$, scaled: $W_2^s(\square_i)$. Let E be a combination of univariate extension operators, bounded on $\prod_{i=1}^m L_2(\square_i)$ and $\prod_{i=1}^m W_2^s(\square_i)$.*
Then
$$\Psi_U := E\{\Psi_i\}_{i=1}^m, \tilde{\Psi}_U := E^{-*}\{\tilde{\Psi}_i\}_{i=1}^m$$
defines a (biorthogonal) wavelet basis for $L_2(U)$, scaled: $W_2^s(U)$.

▹ advantages: constructions retains

- ○ tensor structure
- ○ vanishing moments
- ○ global/local smoothness
- ○ piecewise polynomiality

▹ dimension-independent approximation rates

Theorem. *Given $u \in W_2^m(U)$ there exists an N-term approximation $u_N = \sum_{i=1}^N c_\lambda \psi_\lambda$ from Ψ_U, such that*
$$\|u - u_N\|_{W_2^m(U)} \lesssim N^{-(\tau-m)} \left(\prod_{\square_i \subset U} \|u\|^2_{\mathscr{H}^\tau_{m,\theta}(\square_i)} \right)^{\frac{1}{2}}$$
whenever u is contained in the weighted Sobolev spaces $\mathscr{H}^\tau_{m,\theta}(\square_i)$.

- Design of adaptive method
  - ○ main building block of adaptive methods: approximation of biinfinite matrix/vector-multiplication $\mathbf{APPLY}[\mathbf{A}, \mathbf{x}, \delta] \to v_\delta$ finitly supported with $\|\mathbf{Ax} - v_\delta\| \le \delta$
  - ○ design of $\mathbf{APPLY}$ possible if $\mathbf{A}$ is compressible

Theorem. *Let $\mathbf{A}$ be the stiffness matrix according to (3) with respect to Ψ_U. Then there exist matrices $\mathbf{A}^M$ with $\mathscr{O}(M^{\frac{1-\varepsilon}{2}} d)$ many entries per column, such that*
$$\|\mathbf{A} - \mathbf{A}^M\| \lesssim 2^{-M}.$$

▹ Adaptive method converges with optimal order!

Parameter identification problem

I. Parameter-to-state map

- Parameters $p = (D, \lambda, R, W)$
  - ○ $D \in L_\infty([0,T], L_\infty(U, \mathbb{R}^d))$ ○ $\lambda, R \in L_r([0,T], L_r(U, \mathbb{R}^d))$, $W \in L_r([0,T], L_r(U, \mathbb{R}^{d\times d}))$
  - ○ pointwise global bounds, $r \in (2, \infty)$ chosen respectively for λ, R, W ▹ admissible sets are non-complete metric spaces
- In practice: noisy data measurements $g_\mathbf{t}^\delta \in L(\mathbf{t}, L_2(U, \mathbb{R}^d))$ at discrete time points $\mathbf{t} = \{t_i\}_{i=1}^k \subset [0,T]^k$ ▹ need sampling operator $\mathscr{P} : L_2([0,T], L_2(U, \mathbb{R}^d)) \to L_2(\mathbf{t}, L_2(U, \mathbb{R}^d))$
- Resulting parameter identification problem

(4) $$\mathscr{P}\mathscr{S}(p) = g_\mathbf{t}^\delta$$

Theorem. *The parameter-to-state map $\mathscr{S}$ is continuously differentiable with Hölder continuous derivative $\mathscr{S}'$ on convex, bounded, admissible sets.*

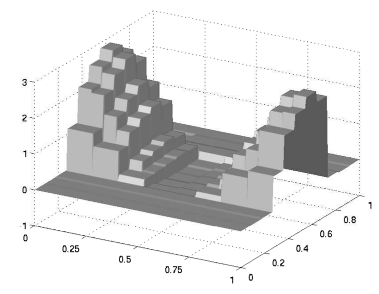
II. Regularization

- Tikhonov approach yields regularization scheme: $p_\alpha^\delta \to p$ for a suitable parameter choice rule for α
- Minimization problem (2) is solved by generalized conditional gradient method (iterative soft-shrinkage)

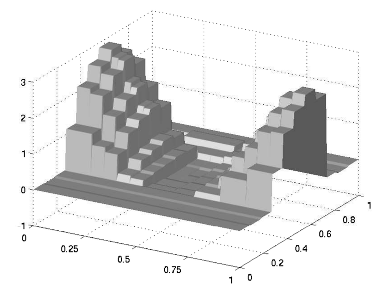
$$p_{n+1} = \operatorname{Shrink}_\alpha(p_n - (\mathscr{S}'(p_n))^* \mathscr{P}^* (\mathscr{P}\mathscr{S}(p_n) - g_\mathbf{t}^\delta)) \to p_\alpha^\delta$$

with Besov ℓ_s penality term on coefficients with respect to wavelet basis/frame

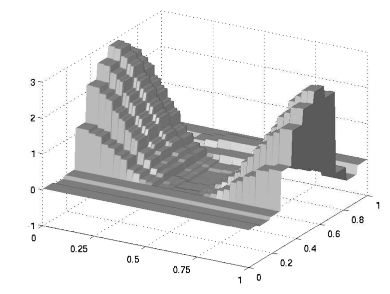
Numerical realization



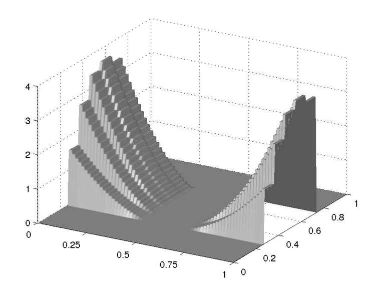
$\alpha = 2.8 \cdot 10^{-6}, \delta = 2 \cdot 10^{-4}$



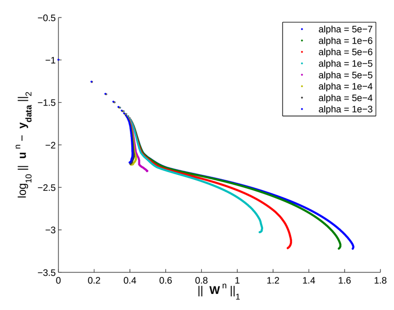
$\alpha = 1.8 \cdot 10^{-6}, \delta = 2 \cdot 10^{-4}$



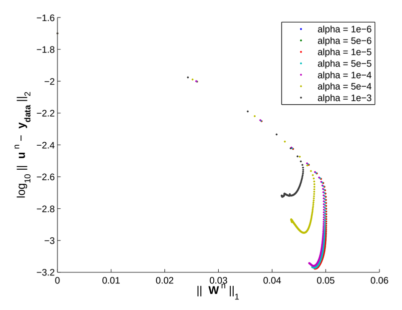
$\alpha = 1.2 \cdot 10^{-6}, \delta = 10^{-7}$



unknown solution



discrepancy vs ℓ_1, $d = 1$



discrepancy vs ℓ_1, $d = 2$

- Linearized model problem, $d = 1, 2$
- Reconstruction of coupling parameter $W(t,x)$
- Adaptive (tensor) wavelet approach
- Based on Marburg software library

Objectives and work schedule

Forward problem: adaptive tensor product wavelet schemes for elliptic/parabolic equations

I. Finetuning of adaptive algorithms

- Vector-valued case
- Parallelization strategies

II. Generalization to nonlinear problems

- Sparse evaluation of nonlinear functionals applied to tensor wavelet expansions
- Generalization to vector-valued case

III. Theoretical justification

- Investigate regularity of solution in generalized weighted Sobolev spaces
- Generalizations to nonlinear problems

Inverse problem: parameter identification with sparsity constraints

I. Convergence aspects

- Convergence theory and rates in case of inexact solution of forward problem
- Exploration of different approaches encompassing positivity constraints

II. Accelaration strategies

- Speed-up for iterated soft shrinkage ▹ decreasing thresholding
- Investigate alternative minimization procedures

Numerical realization and application

I. Fully adaptive solver in time and space

- Efficient sparse evaluation of nonlinearity Φ
- Vector-valued case
- Adaptive discretization in time
- Parallelization
- Based on Marburg software library

II. Application to full embryogenesis model

- Combination of new forward solver with iterated soft shrinkage
- Complex geometry
- High-dimensional parameter space

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