Adaptive Wavelet Methods for Stochastic Partial Differential Equations
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Scope
Numerical treatment of SPDEs of parabolic type

\[ dU_t = \langle a(U_t), W(t) \rangle dt + \sigma(U_t) dW_t, \quad t \in [0, T]. \]

on bounded Lipschitz domains \( D \subset \mathbb{R}^d \), driven by a (cylindrical) stochastic process \( W \).

Goals:
- Adaptive wavelet scheme in time & space
- Abstract Cauchy problem, Rothe method
- Theoretical justification
- Regularity in Besov spaces
- Numerical realization
- Marburg software library

State of the art
Adaptive wavelet methods
- Optimal algorithms for deterministic elliptic operators

Regularity theory of SPDEs
- Usually with \( p > 1 \), convex domains, Hölder or Lipschitz regularity

Numerics of SPDEs
- Typically non-adaptive (uniform) space-time approximations

Adaptive wavelet algorithm for SPDEs: The stationary case

I. Noise representation
- Sparse wavelet expansions

\[ X = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \psi_j(k) \langle \Psi_j(k), X \rangle. \]

\[ 0 \leq \beta \leq 1, \quad \alpha + \beta > 1. \]

where \( \psi_j, \phi_j \) are 

\[ \text{Theorem 1. (Besov regularity of the noise.) Let } p > 0. \]

\[ X \in B_{p, \infty}^\beta(D) \text{ P.-a.s. } \iff \left( \frac{1}{p} - 1 \right) \frac{1}{\beta} - 1 \frac{1}{\beta} = \frac{1}{2} - 1 \frac{1}{\beta} \]

II. Noise approximation
- Error \( E\|X - \hat{X}\|_{B_p^\beta}^{1/2} \)

\[ \text{Linear approximation rate: } \frac{\alpha + \beta}{2} \text{ enhances Besov regularity & sparsity} \]

\[ \text{III. Rothe method} \quad \text{first in time, then in space (stiff problem)} \quad \text{implicit discretization in time} \]

\[ \tau_j \approx \tau \text{ at } j \text{-th level} \]

IV. Approximation rates of solutions to elliptic subproblems
- Uniform vs. adaptive

Regularity of SPDEs in Besov spaces: Spatial regularity

I. Parabolic model equation
- Linear equation, additive noise

\[ dU_t = \langle a(U_t), W(t) \rangle dt + \sigma(U_t) dW_t, \quad U(0) = u_0. \]

A 2nd order elliptic PDE, \( \gamma \in \mathbb{R}, \quad \gamma = \gamma(\xi) \in (D_{1,2}(D), \mathcal{H}^1_0(D)). \]

For a \( \kappa \in (0, 1) \) and \( \theta \in (\kappa + d + k - 2) \), there is a unique \( U \in L^2(D), \mathcal{H}_{2,\theta}^\kappa(D) \) solving (4), i.e.

\[ (U_t, \phi) = (u_0, \phi) + \int_0^t \langle a(U_s), \phi \rangle ds + \int_0^t \langle \sigma(U_s), \phi \rangle \, dW_s, \quad \phi \in \mathcal{H}^\kappa_0(D). \]

Theorem 2. Let \( U \) be the solution of (4) for some \( \gamma \in \mathbb{R} \). If \( U \in L^2(D_{1,2}(D), \mathcal{H}^1_0(D)). \)

\[ \frac{1}{p} \frac{1}{\gamma} = \frac{1}{p} \frac{1}{\gamma} < 0. \]

\[ \text{II. Generalizations} \quad \text{linear equation, multiplicative noise} \]

\[ dU_t = \langle a(U_t - \hat{X})(U_t - \hat{X}) \rangle dt + \sum_{j \in \mathbb{Z}} \left( \sigma_j(U_t) \psi_j \right)^2 \quad \text{on } [0, T]. \]

\[ U(0) = u_0. \]

III. Besov regularity - limited by re-entering cones

Theorem 4. Let \( \gamma_{\text{res}} \text{ be the largest interior angle of a polygonal D. For certain equations of type (4)} \]

\[ \text{IV. Numerical realization} \]

Elliptic subproblem on L-shaped domain
- Singularities in solution due to noise and domain
- Adaptive wavelet frame approach
- Constructed by overlapping domain decomposition, parametric images of unit cube
- Adaptive and multiplicative Schwarz methods
- Based on Marburg software library - MarRG

Objectives and work schedule

Adaptive wavelet algorithms for SPDEs
- Time dependent extension of \( X \) using independent scalar Brownian motions \( \{W^i(\cdot, \cdot)\} \)

\[ W_i = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \psi_j(k) \langle \Psi_j(k), W^i \rangle. \]

\[ \text{Characterization of spaces of negative smoothness} \quad \text{characterization by frames} \]

\[ \text{Approximation in Sobolev norms} \quad \text{anisotropic spaces} \]

II. Setup of fully adaptive scheme
- Regularity estimates in time direction
- Rigorous error estimates for uniform time discretization
- Fully adaptive wavelet scheme in space and time
- Convergence and optimality analysis
- Evaluation of nonlinear functional in the stochastic setting
- Tensor wavelet schemes

Regularity of SPDEs in Besov spaces

I. Nonlinear problems
- As perturbation of linear setting
- Use of variational techniques
- Establishing Besov regularity

II. Regularity in space and time
- Hölder and Sobolev regularity in time
- Weighted Sobolev estimates in time and space
- Full space-time Besov regularity

III. Extensions
- Allow non-local semigroup generators
- Non-Markovian or non-continuous driving noises

Numerical realization

I. Setup of Rothe method
- Uniform time discretization
- Adaptive time-step control
- Comparison to existing schemes

II. Parallel algorithms
- On the level of the SPDE solver
- For Monte Carlo simulation experiments
- Added efficiency

III. Implementation of fully adaptive algorithm in space and time
- Evaluation of nonlinear functional
- Application to problems on polygonal, polyhedral domains
- Based on Marburg software library

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