

# TIME-FREQUENCY SHIFT INVARIANCE AND THE AMALGAM BALIAN LOW THEOREM

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ABSTRACT. We consider smoothness properties of the generator of a principal Gabor space on the real line which is invariant under some additional translation-modulation pair. We prove that if a Gabor system on a lattice with rational density is a Riesz basis for its closed linear span, and if the closed linear span, a Gabor space, has any additional translation-modulation invariance, then its generator cannot decay well in time and in frequency simultaneously.

## 1. INTRODUCTION

The Balian-Low Theorem, a key result in time-frequency analysis, expresses the fact that time-frequency concentration and non redundancy are essentially incompatible. Specifically, if  $\varphi \in L^2(\mathbb{R})$ ,  $\Lambda \subset \mathbb{R}^d$  is a lattice and the system  $(\varphi, \Lambda) = \{e^{2\pi i\eta x} \varphi(x - u) : (u, \eta) \in \Lambda\}$  is a Riesz basis for  $L^2(\mathbb{R})$ , then  $\varphi$  satisfies

$$(1) \quad \left( \int (x - a)^2 |\varphi(x)|^2 dx \right) \cdot \left( \int (\omega - b)^2 |\widehat{\varphi}(\omega)|^2 d\omega \right) = \infty, \quad a, b \in \mathbb{R}.$$

This theorem was originally stated independently by Balian [Bal81] and Low [Low85] for orthogonal systems, but both of their proofs contained a gap, which was later filled by Coifman et. al [Dau90] who also generalized it to Riesz bases. For general references on the Balian-Low Theorem we refer the reader to [BHW95, Hei07]. In [BHW95], the authors also state and prove the so called Amalgam Balian Low Theorem, which states that if  $(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  is a Riesz basis for  $L^2(\mathbb{R})$ , then  $\varphi$  cannot belong to the Feichtinger algebra  $S_0(\mathbb{R})$ , a class of functions decaying well in time and frequency. For a definition of  $S_0(\mathbb{R})$  see (2) below. Note that the Amalgam Balian Low Theorem is seemingly weaker than Balian-Low Theorem, but is not implied by it.

We define the unitary operators, translation  $T_u : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ ,  $T_u f(x) = f(x - u)$ , modulation  $M_\eta : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ ,  $M_\eta f(x) = e^{2\pi i\eta x} f(x)$ , and time-frequency shift  $\pi(u, \eta) = M_\eta T_u$ , where  $u \in \mathbb{R}$  and  $\eta \in \widehat{\mathbb{R}}$ , the dual group of  $\mathbb{R}$  which is isomorphic to  $\mathbb{R}$ . For  $\varphi \in L^2(\mathbb{R})$  and a lattice  $\Lambda = R\mathbb{Z}^2 \subset \mathbb{R} \times \widehat{\mathbb{R}}$ ,  $R \in \mathbb{R}^{2 \times 2}$ , with density  $|\det R|^{-1}$  if  $R$  is full rank and density 0 else, we define Gabor systems as  $(\varphi, \Lambda) = \{\pi(\lambda)\varphi\}_{\lambda \in \Lambda}$  and Gabor spaces as  $\mathcal{G}(\varphi, \Lambda) = \overline{\text{span}\{\pi(\lambda)\varphi\}}$ , where  $\overline{V}$  is the closure of  $V$  in  $L^2(\mathbb{R})$ . For background on Gabor systems we refer to the monograph [Grö01].

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*Key words and phrases.* Balian-Low Theorem, additional shift invariance, Gabor frames, time-frequency analysis, Feichtinger algebra.

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This paper addresses the question whether there may exist a  $\mu \in \mathbb{R} \times \widehat{\mathbb{R}} \setminus \Lambda$  with  $\pi(\mu)\varphi \in \mathcal{G}(\varphi, \Lambda)$ . Equivalently, for  $\Lambda'$  being a subgroup of  $\mathbb{R} \times \widehat{\mathbb{R}}$  containing  $\Lambda$ , under which conditions on  $\varphi$  is it possible that  $\mathcal{G}(\varphi, \Lambda) = \mathcal{G}(\varphi, \Lambda')$ ?

The case that  $\mu, \Lambda \subseteq \mathbb{R} \times \{0\}$  is discussed at length in terms of shift-invariant spaces in the literature, see for example [ACH<sup>+</sup>10, ASW11, AKTW12]. Since the Fourier transform is unitary, analogous results are implied for  $\mu, \Lambda \in \{0\} \times \widehat{\mathbb{R}}$ . As we shall see in Remark 6 in Example 1, the case  $\mu \in \mathbb{R} \times \{0\}$  and  $\pi(\mu)\varphi \in \mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  does not necessitate that  $\pi(\mu)\varphi$  is in the shift-invariant space  $\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \{0\})$ , so even the case with  $\mu \in \mathbb{R} \times \{0\}$  is not covered in the literature.

On the other hand, the existing Balian Low type results for shift-invariant spaces only apply to *principal* shift-invariant spaces, that is, spaces that can be generated by just one generator. Even though Gabor spaces are particular cases of shift-invariant spaces, except for the case  $\Lambda = \alpha\mathbb{Z} \times \{0\}$ , they are not principal shift-invariant spaces, so these results do not apply in the setting considered here.

To state our result, we recall that the *Feichtinger algebra*  $S_0(\mathbb{R})$  is defined by

$$(2) \quad S_0(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : Vf(t, \nu) = \int f(x) e^{-(x-t)^2} e^{2\pi i x \nu} dx \in L^1(t, \nu) \right\}.$$

Note that  $Vf(t, \nu) \in L^2(t, \nu) \cap L^\infty(t, \nu)$  for all  $f \in L^2(\mathbb{R})$  and the requirement  $Vf(t, \nu) \in L^1(t, \nu)$  essentially necessitates  $L^1$  decay of  $f$  and of its Fourier transform  $\widehat{f}$ . For details on the Feichtinger algebra see [Fei81, FZ98, Grö01].

We establish the following theorem.

**Theorem 1.** *If  $(\varphi, \Lambda)$  is a Riesz basis for its closed linear span  $\mathcal{G}(\varphi, \Lambda)$  with  $\varphi \in S_0(\mathbb{R})$  and the density of the lattice  $\Lambda$  is rational, then  $\pi(u, \eta)\varphi \notin \mathcal{G}(\varphi, \Lambda)$  for all  $(u, \eta) \notin \Lambda$ .*

*In the case  $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ , then the condition  $\varphi \in S_0(\mathbb{R})$  can be replaced with the weaker condition that  $Z_\alpha\varphi(x, \omega) = \sum_{n \in \mathbb{Z}} f(x + n\alpha) e^{-2\pi i \omega n \alpha}$  is continuous on  $\mathbb{R} \times \widehat{\mathbb{R}}$ .*

Theorem 1 generalizes the Amalgam Balian Low Theorem stated above. Indeed,  $(\varphi, \Lambda)$  being a Riesz basis for  $L^2(\mathbb{R})$  implies that the density of  $\Lambda$  equals 1, that is,  $(\alpha\beta)^{-1} = 1 \in \mathbb{Q}$  in case  $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ , and  $\mathcal{G}(\varphi, \Lambda) = L^2(\mathbb{R})$  implies that  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$  for all  $(u, \eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$ , so Theorem 1 implies that  $\varphi \notin S_0(\mathbb{R})$ .

**Remark 2.** The question of whether the condition  $\varphi \in S_0(\mathbb{R})$  in Theorem 1 can be replaced with having a finite uncertainty product (1) is left for further exploration. Similarly, we do not discuss the case of  $\Lambda$  having irrational density in this paper.

To generalize our proof of Theorem 1 to a higher dimensional setting, that is,  $\varphi \in L^2(\mathbb{R}^d)$  and  $\Lambda \subset \mathbb{R}^{2d}$ , requires a restriction to  $\Lambda$  being a so-called symplectic lattices in order to use intertwining operators to reduce the general problem to lattices of the form  $\alpha_1\mathbb{Z} \times \dots \times \alpha_d\mathbb{Z} \times \beta_1\mathbb{Z} \times \dots \times \beta_1\mathbb{Z}$  [Grö01].

Our investigation is motivated in part by the following. In orthogonal frequency division multiplexing, short, OFDM, information in form of a coefficient sequence  $\{c_{k,\ell}\}_{k \in \mathbb{Z}, \ell \in I}$  is transmitted through a channel using the signal

$$F\{c_{k,\ell}\} = \sum_{k \in \mathbb{Z}} \sum_{\ell \in I} c_{k,\ell} T_{k\alpha} M_{\ell\beta} \varphi.$$

The index set  $I$  depends on the for transmission available frequency band and is therefore finite in most OFDM applications. For  $F$  to be boundedly invertible,  $\varphi$  is chosen so that  $(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  is a Riesz basis for its closed linear span. Moreover, to utilize a communications

channel efficiently, it is beneficial to choose  $\varphi$  with good decay in time and in frequency, that is,  $\varphi \in S_0(\mathbb{R})$ , or better,  $\varphi$  is a Schwartz class function.

Theorem 1 then implies that under these conditions,  $\pi(u, \eta)\varphi \notin \mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  whenever  $(u, \eta) \notin \alpha\mathbb{Z} \times \beta\mathbb{Z}$ . Unfortunately, distortions that the signal undergoes are time-shifts (delays of the signal) in case of time-invariant channels, or time-frequency shifts in case of mobile, time-varying communications channels. Theorem 1 shows that we cannot choose  $\varphi \in S_0(\mathbb{R})$  so that the transmission space  $\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  is invariant under perturbations  $\pi(u, \eta)$  for  $(u, \eta) \notin \alpha\mathbb{Z} \times \beta\mathbb{Z}$ .

In some cases, the leakage out of  $\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  can be used to identify an unknown channel operator  $H$ , in particular, if  $H$  is well approximated by a single time-frequency shift  $\pi(u, \eta)$  [KP06, PW06, Pfa13]. Unfortunately,  $\pi(u, \eta)\varphi \notin \mathcal{G}(\varphi, \Lambda)$  for all  $(u, \eta) \notin \Lambda$  and  $(\varphi, \Lambda)$  is a Riesz sequence for  $G(\varphi, \Lambda)$  does not imply that there is no  $f \in G(\varphi, \Lambda)$  with  $\pi(u, \eta)f \in G(\varphi, \Lambda)$ , so a receiver would not be able to know whether  $\pi(u, \eta)f$  was transmitted through the identity operator, or  $f$  was transmitted and then perturbed by the operator  $\pi(u, \eta)$ .<sup>1</sup>

*Related work.* Aldroubi, Sun and Wang showed that if a principal shift-invariant space on the real line is also translation-invariant, that is, invariant under *every* translation operator, then any of its Riesz generators are non-integrable. Moreover, if the generator of the shift-invariant space is also invariant under the translate by  $\frac{1}{n}$ ,  $n \in \mathbb{N} \setminus \{1\}$ , then  $\int |x|^{1+\epsilon} |\varphi(x)|^2 dx = \infty$  for all  $\epsilon > 0$  [ASW11].

Gabardo and Han showed that if  $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$  has integer density  $(\alpha\beta)^{-1} \geq 2$  and  $\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z}) \neq L^2(\mathbb{R})$ , then (1) holds. In the reciprocal case, they show that if  $\alpha\beta \in \mathbb{N} \setminus \{1\}$  and  $(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  is not a Riesz system for its closed linear span, then again (1) holds. Note that both cases do not represent the generic case [GH04].

Gröchenig, Han, Heil, and Kutyniok show that if  $(\varphi, \Lambda)$  and  $(\tilde{\varphi}, \Lambda)$  are biorthogonal Riesz basis for  $\mathcal{G}(g, \Lambda)$ , then (1) holds for either  $\varphi$  or  $\tilde{\varphi}$  [GHHK02].

For general Balian Low type results, we refer the reader to [BHW92, BHW95, BHW98, BW94, DLL95, FG97, Jan08].

*Organisation of the paper.* In Section 2 we discuss our main tool, the Zak transform. We then proceed to prove Theorem 1 in Section 3; and in Section 4 we construct functions that generate Gabor spaces containing additional shifts of the generator.

## 2. THE ZAK TRANSFORM

The analysis offered below is based on the Zak transform which is densely defined on  $L^2(\mathbb{R})$  by

$$Z_\alpha f(x, \omega) = \sum_{k \in \mathbb{Z}} f(x + \alpha k) e^{-2\pi i \alpha k \omega}, \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

where  $\alpha > 0$ . We write  $Zf(x, \omega) = Z_1 f(x, \omega)$ .

It is easily observed that

$$Zf(x + n, \omega) = e^{2\pi i n \omega} Zf(x, \omega), \quad Zf(x, \omega + m) = Zf(x, \omega),$$

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<sup>1</sup>For example, if  $g$  is a Gaussian, we have  $(g, \mathbb{Z} \times \frac{3}{2}\mathbb{Z})$  is a Riesz basis for  $\mathcal{G}(g, \mathbb{Z} \times \frac{3}{2}\mathbb{Z})$  since the density of  $\mathbb{Z} \times \frac{3}{2}\mathbb{Z}$  is  $\frac{2}{3} < 1$ , see [Hei07] and references therein. It is then not difficult to construct  $f \neq 0$  such that  $f, \pi(\frac{1}{2}, 0)f \in \mathcal{G}(g, \mathbb{Z} \times \frac{3}{2}\mathbb{Z})$ .

in short,  $Zf$  is quasiperiodic. Not only does  $Zf$  on  $[0, 1] \times [0, 1]$  fully describe  $f$ , but we have  $\|Zf\|_{L^2([0,1] \times [0,1])} = \|f\|_{L^2(\mathbb{R})}$ , that is,  $Z$  is a unitary map onto the space of quasiperiodic functions on  $\mathbb{R} \times \widehat{\mathbb{R}}$  where the latter is equipped with the  $L^2([0, 1] \times [0, 1])$  norm.

We shall utilize the fact that with  $\pi(u, \eta) = M_\eta T_u$  we have

$$\begin{aligned} (Z\pi(u, \eta)f)(x, \omega) &= \sum_{k \in \mathbb{Z}} (\pi(u, \eta)f)(x+k) e^{-2\pi i k \omega} \\ &= \sum_{k \in \mathbb{Z}} e^{2\pi i(x+k)\eta} f(x+k-u) e^{-2\pi i k \omega} \\ &= e^{2\pi i x \eta} \sum_{k \in \mathbb{Z}} f(x-u+k) e^{-2\pi i k(\omega-\eta)} \\ &= e^{2\pi i \eta x} Zf(x-u, \omega-\eta). \end{aligned}$$

In particular, we have for  $k, \ell \in \mathbb{Z}$  that

$$(Z\pi(k, \ell)f)(x, \omega) = e^{2\pi i \ell x} Zf(x-k, \omega-\ell) = e^{2\pi i(\ell x + k\omega)} Zf(x, \omega),$$

where we used the quasiperiodicity of the Zak transform.

Note that  $S_0(\mathbb{R})$  is invariant under the Fourier transform, so  $\varphi \in S_0(\mathbb{R})$  if and only if  $\widehat{\varphi} \in S_0(\mathbb{R})$ . The key property of  $S_0(\mathbb{R})$  that we use is that  $\varphi \in S_0(\mathbb{R})$  implies  $Z\varphi$  continuous. Indeed, if  $\varphi$  is in the Wiener Amalgam space

$$W(C(\mathbb{R}), l^1(\mathbb{Z})) = \{f \in L^2(\mathbb{R}) \text{ continuous with } \sum_{k \in \mathbb{Z}} \|f\|_{L^\infty([k, k+1])} < \infty\} \supset S_0(\mathbb{R}),$$

then the sum defining the Zak transform converges uniformly, so the given continuity of  $\varphi \in S_0(\mathbb{R})$  implies continuity of its Zak transform. Note that  $\varphi \in W(C(\mathbb{R}), l^1(\mathbb{Z}))$  is not necessary for the Zak transform to be continuous. In Theorem 1 we therefore offer the two conditions  $\varphi \in S_0(\mathbb{R})$  and, if  $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ , more generally, the scaled Zak transform  $Z_\alpha\varphi$ , that is, the Zak transform adjusted to the lattice  $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ , is continuous.

### 3. PROOF OF THEOREM 1

The proof is by contradiction. Let  $\Lambda \in \mathbb{R} \times \widehat{\mathbb{R}}$  be a discrete subgroup of rational density. Assume there exists  $\varphi \in S_0(\mathbb{R})$ , such that  $(\varphi, \Lambda)$  is a Riesz basis for its closed linear span  $\mathcal{G}(\varphi, \Lambda)$ , and assume further that there is an element  $(u, \eta) \in \mathbb{R} \times \widehat{\mathbb{R}} \setminus \Lambda$  with  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$ .

*Step 1. Without loss of generality*  $\Lambda = \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$  with  $P, Q \in \mathbb{N}$ . Clearly, any generic full rank lattice  $\Lambda$  of density  $\frac{P}{Q}$  can be written as  $\Lambda = A(\frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  with  $A \in \mathbb{R}^{2 \times 2}$ ,  $\det A = 1$ . Since any  $A \in \mathbb{R}^{2 \times 2}$  with  $\det A = 1$  is element of the symplectic group, there exists a so-called metaplectic operator  $U = U(A)$  with  $U^* \pi(\frac{m}{Q}, nP)U = \pi(A(\frac{m}{Q}, nP)^T)$  [Grö01]. The metaplectic operator  $U$  is unitary, hence,  $(\varphi, \Lambda)$  is a Riesz basis for its closed linear span  $\mathcal{G}(\varphi, \Lambda)$  if and only if  $(U\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  is a Riesz basis for its closed linear span  $\mathcal{G}(U\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$ . Moreover,  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$  implies for some sequence  $\{c_\lambda\} \in \ell^2(\Lambda)$

$$\begin{aligned} U^* \pi(A^{-1}(u, \eta)^T)U\varphi &= \pi(AA^{-1}(u, \eta)^T)\varphi = \pi(u, \eta)\varphi = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)\varphi \\ &= \sum_{m, n \in \mathbb{Z}} c_{m, n} \pi(A(\frac{m}{Q}, nP)^T)\varphi = \sum_{m, n \in \mathbb{Z}} c_{m, n} U^* \pi(\frac{m}{Q}, nP)U\varphi. \end{aligned}$$

We summarize that with  $(\tilde{u}, \tilde{v}) = A^{-1}(u, \eta)^T \notin \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$  since  $(u, \eta) \notin \Lambda$ , and  $\tilde{\varphi} = U\varphi \in S_0(\mathbb{R})$  by invariance of  $S_0(\mathbb{R})$  under metaplectic operators [Grö01], we have  $\pi(\tilde{u}, \tilde{\eta})\tilde{\varphi} \in \mathcal{G}(\tilde{\varphi}, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$ .

If density  $\Lambda = 0$ , then we can increase  $\Lambda$  to a full rank lattice, maintaining the property that  $(\varphi, \Lambda)$  is a Riesz sequence for its closed linear span. The argument above is then applicable.

*Step 2. Without loss of generality, we can choose  $u$  and  $\eta$  to be rational.* Clearly, this is equivalent to the existence of  $R \in \mathbb{N}$  with  $R \cdot (u, \eta) \in \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$ .

We proceed by showing that if there exists  $(u, \eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$  with  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$ , then exists also a rational pair  $(\tilde{u}, \tilde{\eta}) \in \mathbb{R} \times \widehat{\mathbb{R}}$  with  $\pi(\tilde{u}, \tilde{\eta})\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$ .

First, observe that  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  implies that  $\mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  is invariant under both,  $\pi(u, \eta)$  and  $\pi(\frac{m}{Q}, nP)$ ,  $m, n \in \mathbb{Z}$ , and therefore,  $\mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  is invariant under  $\pi(\lambda)$  where  $\lambda$  is in the group  $\tilde{\Lambda}$  generated by  $(u, \eta)$  and  $\frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$ . Moreover, we have  $\pi(\lambda)\varphi \in \mathcal{G}(\varphi, \Lambda)$  for all  $\lambda \in \text{closure } \tilde{\Lambda} \subseteq \mathbb{R} \times \widehat{\mathbb{R}}$ .

If,  $u$  is irrational, then closure  $\tilde{\Lambda}$  contains  $\mathbb{R} \times \{\eta\}$  and we can replace  $(u, \eta) \notin \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$  by  $(\tilde{u}, \eta) \in \text{closure } \tilde{\Lambda} \setminus \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$  with  $\tilde{u} \in \mathbb{Q}$ . With the same argument, we are able to replace an irrational  $\eta$  with a rational number  $\tilde{\eta} \notin P\mathbb{Z}$ .

*Step 3. The case  $Q = 1$ .* Choose  $R \in \mathbb{N}$  with  $(Ru, R\eta) \in \mathbb{Z} \times P\mathbb{Z}$ . Set  $M_2 = Ru$  and  $M_1 = R\eta$ , by increasing  $R$  we can assume that  $M_2\eta/2$  is an integer and  $P$  divides  $M_1$ .

We have  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$  if and only if

$$e^{2\pi i \eta x} Z\varphi(x - u, \omega - \eta) = (Z\pi(u, \eta)\varphi)(x, \omega) \in Z\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z}).$$

But

$$\begin{aligned} Z\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z}) &= \overline{\text{span}}\{Z\pi(\lambda)\varphi, \lambda \in \mathbb{Z} \times P\mathbb{Z}\} \\ &= \overline{\text{span}}\{e^{2\pi i(P\ell x + k\omega)} Z\varphi(x, \omega), (k, \ell) \in \mathbb{Z} \times \mathbb{Z}\}. \end{aligned}$$

So  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$  if and only if there exist a sequence  $c = (c_{k, \ell}) \in \ell^2(\mathbb{Z}^2)$  with

$$\begin{aligned} e^{2\pi i \eta x} Z\varphi(x - u, \omega - \eta) &= \sum_{k, \ell \in \mathbb{Z}} c_{k, \ell} e^{2\pi i(P\ell x + k\omega)} Z\varphi(x, \omega) \\ &= h(x, \omega) Z\varphi(x, \omega), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}, \end{aligned}$$

where

$$h(x, \omega) = \sum_{k, \ell \in \mathbb{Z}} c_{k, \ell} e^{2\pi i(P\ell x + k\omega)}$$

is a locally  $L^2$  function which is  $1/P$  periodic in  $x$  and 1 periodic in  $\omega$ . Note that the construction of  $h$  is based on the assumption that  $(\varphi, \mathbb{Z} \times P\mathbb{Z})$  is a Riesz basis for its closed linear span. Hence

$$(3) \quad Z\varphi(x, \omega) = e^{-2\pi i \eta(x+u)} h(x+u, \omega + \eta) Z\varphi(x+u, \omega + \eta), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}.$$

The above together with the quasiperiodicity of the Zak transform implies

$$\begin{aligned}
Z\varphi(x, \omega) &= e^{-2\pi i\eta(x+u)} h(x+u, \omega+\eta) Z\varphi(x+u, \omega+\eta) \\
&= e^{-2\pi i\eta(x+u)} h(x+u, \omega+\eta) e^{-2\pi i\eta(x+2u)} h(x+2u, \omega+2\eta) Z\varphi(x+2u, \omega+2\eta) \\
&= \dots = Z\varphi(x+Ru, \omega+R\eta) \exp\left(-2\pi i\eta\left(Rx+u\sum_{r=1}^R r\right)\right) \prod_{r=1}^R h(x+ru, \omega+r\eta) \\
&= e^{2\pi i M_2 \omega} Z\varphi(x, \omega) \exp\left(-2\pi i(M_1 x + M_2 \eta(R+1)/2)\right) \prod_{r=1}^R h(x+ru, \omega+r\eta) \\
&= e^{2\pi i(M_2 \omega - M_1 x)} Z\varphi(x, \omega) \prod_{r=1}^R h(x+ru, \omega+r\eta), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},
\end{aligned}$$

where we used that  $M_2\eta/2$  is an integer.

Hence  $h$  satisfies the quasiperiodicity condition

$$(4) \quad \prod_{r=1}^R h(x+ru, \omega+r\eta) = e^{2\pi i(M_1 x - M_2 \omega)}, \quad (x, \omega) \in \text{supp } Z\varphi \subseteq \mathbb{R} \times \widehat{\mathbb{R}}.$$

Equation (4) holds a-priori only on  $\text{supp } Z\varphi$ , we shall now extend it to hold on all of  $\mathbb{R} \times \widehat{\mathbb{R}}$  based on the assumption that  $(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  is a Riesz sequence for its closed linear span.

Indeed, a standard periodization trick gives

$$\begin{aligned}
\int_{\mathbb{R}} \left| \sum_{k, \ell \in \mathbb{Z}} d_{k, \ell} \pi(k, P\ell) \varphi(x) \right|^2 dx &= \int_0^1 \int_0^1 \left| \sum_{k, \ell \in \mathbb{Z}} d_{k, \ell} e^{2\pi i(P\ell t - k\omega)} Z\varphi(t, \nu) \right|^2 dt d\nu \\
&= \int_0^1 \int_0^{1/P} \left| \sum_{k, \ell \in \mathbb{Z}} d_{k, \ell} e^{2\pi i(P\ell t - k\omega)} \right| \left| \sum_{p=0}^{P-1} Z\varphi\left(t - \frac{p}{P}, \nu\right) \right|^2 dt d\nu,
\end{aligned}$$

and, hence, we have that  $(\varphi, \mathbb{Z} \times P\mathbb{Z})$  is a Riesz sequence if and only if

$$A \leq \sum_{p=0}^{P-1} \left| Z\varphi\left(x - \frac{p}{P}, \omega\right) \right|^2 \leq B, \quad \text{a.e. } (x, \omega),$$

for some  $0 < A \leq B < \infty$ . So, for almost every  $x_0, \omega_0$  exists  $p_0 \in \{0, 1, \dots, P-1\}$  so that

$$Z\varphi\left(x_0 - \frac{p_0}{P}, \omega_0\right) \neq 0.$$

Using the computations above, we have

$$\begin{aligned}
Z\varphi\left(x_0 - \frac{p_0}{P}, \omega_0\right) &= Z\varphi\left(x_0 - \frac{p_0}{P}, \omega_0\right) e^{2\pi i(M_2 \omega_0 - M_1(x_0 - \frac{p_0}{P}))} \prod_{r=1}^R h\left(x_0 - \frac{p_0}{P} + ru, \omega_0 + r\eta\right) \\
&= Z\varphi\left(x_0 - \frac{p_0}{P}, \omega_0\right) e^{2\pi i(M_2 \omega_0 - M_1 x_0)} \prod_{r=1}^R h(x_0 + ru, \omega_0 + r\eta),
\end{aligned}$$

where we used the fact that  $h$  is  $\frac{1}{P}$  periodic in  $x$  and  $P$  divides  $M_1$ . As  $Z\varphi\left(x_0 - \frac{p_0}{P}, \omega_0\right) \neq 0$ , we have indeed

$$\prod_{r=1}^R h(x_0 + ru, \omega_0 + r\eta) = e^{2\pi i(M_1 x_0 - M_2 \omega_0)}.$$

As  $(x_0, \omega_0)$  was chosen arbitrarily (a.e.), we conclude (4) holds for almost every  $(x_0, \omega_0)$ .

Moreover, observe that (3) implies that the zero set of  $Z\varphi$  is  $(u, \eta)$  periodic, hence if  $(x_0, \omega_0)$  satisfies  $Z\varphi(x_0 - \frac{p_0}{P}, \omega_0) \neq 0$ , we have

$$\begin{aligned} 0 \neq Z\varphi(x_0 - \frac{p_0}{P}, \omega_0) &= Z\varphi(x_0 - u - \frac{p_0}{P}, \omega_0 - \eta) \\ &= e^{-2\pi i \eta (x_0 - u - \frac{p_0}{P} + u)} h(x_0 - u - \frac{p_0}{P} + u, \omega_0 - \eta + \eta) Z\varphi(x_0 - u - \frac{p_0}{P} + u, \omega_0 - \eta + \eta) \\ &= e^{-2\pi i \eta (x_0 - \frac{p_0}{P})} h(x_0, \omega_0) Z\varphi(x_0 - \frac{p_0}{P}, \omega_0). \end{aligned}$$

Solving for  $h(x_0, \omega_0)$  implies that  $h(x, \omega)$  is continuous on  $\mathbb{R} \times \widehat{\mathbb{R}}$ , and therefore (4) holds on all of  $\mathbb{R} \times \widehat{\mathbb{R}}$ .

The proof of the case  $Q = 1$  is completed, by proving in Step 4 that a function  $h$  as constructed above does not exist.

*Step 4. Periodicity vs quasiperiodicity and conclusion of the case  $Q = 1$ .* Proposition 3 below is an extension of the simple fact that if  $h(x)$  is a function satisfying  $e^{2\pi i M x} = \prod_{r=1}^R h(x + r0) = h(x)^R$ , then  $h(x) \neq 0$  for all  $x$ . If further  $h$  is  $\frac{1}{P}$ -periodic, then

$$h(x) = h(x + \frac{1}{P}) = e^{2\pi i \frac{M}{R}(x + \frac{1}{P})} = e^{2\pi i \frac{M}{RP}} h(x),$$

and, hence,  $RP$  divides  $M$ .

**Proposition 3.** *Let  $P_1, P_2, R \in \mathbb{N}$ ,  $M_1, M_2 \in \mathbb{Z}$ , and  $u, \eta \in \mathbb{R}$ . If  $h(x, \omega)$  is continuous on  $\mathbb{R} \times \widehat{\mathbb{R}}$ ,  $1/P_1$  periodic in  $x$ ,  $1/P_2$  periodic in  $\omega$  and*

$$(5) \quad e^{2\pi i (M_1 x + M_2 \omega)} = \prod_{r=0}^{R-1} h(x + ru, \omega + r\eta), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

*then  $RP_1$  divides  $M_1$  and  $RP_2$  divides  $M_2$ .*

Before giving a proof, let us first use Proposition 3 to conclude the proof of Theorem 1 for  $\Lambda = \mathbb{Z} \times P\mathbb{Z}$ .

Using all assumptions, we have established the existence of a continuous  $h(x, \omega)$  which satisfies (4) and is  $1/P$  periodic in  $x$ , and 1-periodic in  $\omega$ . Therefore (5) is satisfied with

$$M_1 = R\eta, \quad M_2 = -Ru, \quad P_1 = P \quad \text{and} \quad P_2 = 1.$$

Then Proposition 3 implies  $M_1/(RP_1) \in \mathbb{Z}$ , that is,  $\eta = M_1/R \in P\mathbb{Z}$ , and  $u = -M_2/R \in \mathbb{Z}$ . We conclude that  $(u, \eta) \in \Lambda = \mathbb{Z} \times P\mathbb{Z}$ , a contradiction.

*Proof of Proposition 3.*

We have

$$M_1 x + M_2 \omega = \sum_{r=0}^{R-1} \arg h(x + ru, \omega + r\eta) \pmod{1}, \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

where by continuity of  $h$ , we can choose  $\arg h(x, \omega)$  to be continuous as well. (Note that this necessitates the values of  $\arg h$  to be real numbers, not only values in  $[0, 1)$ .)

For  $x = \omega = 0$ , we have  $\sum_{r=0}^{R-1} \arg h(ru, r\eta) = p \in \mathbb{Z}$ .

As  $\arg h(x, \omega)$  is continuous, we have

$$\sum_{r=0}^{R-1} \arg h(x + ru, \omega + r\eta) = p + M_1 x + M_2 \omega, \quad x, \omega \in \mathbb{R} \times \widehat{\mathbb{R}}.$$

Indeed, by varying  $\omega$  (or  $x$ ) by a small value,  $\sum_{r=0}^{R-1} \arg h(x + ru, \omega + r\eta) - M_1x - M_2\omega$  can only vary marginally and not jump by an integer value. We conclude in particular that (for  $x = 1, \omega = 0$  and  $x = 0, \omega = 1$  respectively)

$$\sum_{r=0}^{R-1} \arg h(1 + ru, r\eta) = p + M_1, \quad \sum_{r=0}^{R-1} \arg h(ru, 1 + r\eta) = p + M_2.$$

But, now,  $\arg h(0, 0) - \arg h(1/P_1, 0) = q_1 \in \mathbb{Z}$  by  $1/P_1$  periodicity of  $\arg h(x, \omega)$  in  $x$ . Similarly to before,  $\arg h(x, \omega) - \arg h(x + 1/P_1, \omega)$  is an integer, and, this time by continuity in  $x$  and  $\omega$ , we must have  $\arg h(x, \omega) - \arg h(x + 1/P_1, \omega) = q_1$  for all  $x, \omega \in \mathbb{R} \times \hat{\mathbb{R}}$ . Hence,  $\arg h(x, \omega) - \arg h(x + 1, \omega) = P_1q_1$ . Similarly,  $\arg h(x, \omega) - \arg h(x, \omega + 1) = P_2q_2$  for all  $x, \omega \in \mathbb{R} \times \hat{\mathbb{R}}$  where  $q_2 \in \mathbb{Z}$ .

We conclude

$$p = \sum_{r=0}^{R-1} \arg h(ru, r\eta) = \sum_{r=0}^{R-1} (\arg h(ru + 1, r\eta) + P_1q_1) = p + M_1 + RP_1q_1,$$

and

$$p = \sum_{r=0}^{R-1} \arg h(ru, r\eta) = \sum_{r=0}^{R-1} (\arg h(ru, r\eta + 1) + P_2q_2) = p + M_2 + RP_2q_2,$$

that is,  $RP_1q_1 + M_1 = 0 = RP_2q_2 + M_2$ , and the conclusion follows since  $q_1, q_2 \in \mathbb{Z}$ .  $\square$

**Remark 4.** If we drop the assumption that  $Z\varphi$  is continuous but maintain the assumption that  $(\varphi, \Lambda)$  is a Riesz sequence, then the arguments above allow to construct an  $L^2$  function  $h$  satisfying (4) a.e. on  $\mathbb{R} \times \hat{\mathbb{R}}$ . Then, Proposition 3 implies that  $h$  is discontinuous, so  $h$  is neither a trigonometric polynomial nor an absolutely convergent Fourier series. We conclude that whenever  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$ ,  $(\varphi, \Lambda)$  is a Riesz sequence, and  $(u, \eta) \notin \Lambda$ , then  $\pi(u, \eta)\varphi$  has a slowly convergent series expansion in  $(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ .

*Step 5. The rational case  $\frac{P}{Q} \notin \mathbb{N}$ .* We choose again  $R \in \mathbb{N}$  with  $(Ru, R\eta) \in \mathbb{Z} \times P\mathbb{Z}$ . Set  $M_2 = Ru$  and  $M_1 = R\eta$ , by increasing  $R$  we can assume that  $M_2\eta/2$  is an integer and  $P$  divides  $M_1$ .

We have  $\pi(u, \eta)\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  if and only if

$$e^{2\pi i \eta x} Z\varphi(x - u, \omega - \eta) = (Z\pi(u, \eta)\varphi)(x, \omega) \in Z\mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}).$$

But

$$\begin{aligned} Z\mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}) &= \overline{\text{span}}\{Z\pi(\lambda)\varphi, \lambda \in \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}\} \\ &= \overline{\text{span}}\{e^{2\pi i \ell P x} Z\varphi(x - \frac{k}{Q}, \omega - \ell P), (k, \ell) \in \mathbb{Z} \times \mathbb{Z}\} \\ &= \overline{\text{span}}\{e^{2\pi i \ell P x} Z\varphi(x - \frac{k}{Q}, \omega), (k, \ell) \in \mathbb{Z} \times \mathbb{Z}\}. \end{aligned}$$



That is, if and only if there exist a sequence  $c = (c_{k,\ell}) \in \ell^2(\mathbb{Z}^2)$  with

$$\begin{aligned} e^{2\pi i \eta x} Z\varphi(x-u, \omega-\eta) &= \sum_{k,\ell \in \mathbb{Z}} c_{k,\ell} e^{2\pi i \ell P x} Z\varphi(x - \frac{k}{Q}, \omega) \\ &= \sum_{q=0}^{Q-1} \sum_{k,\ell \in \mathbb{Z}} c_{q+kQ,\ell} e^{2\pi i (\ell P x + k\omega)} Z\varphi(x - \frac{q}{Q}, \omega) \\ &= \sum_{q=0}^{Q-1} h_q(x, \omega) Z\varphi(x - \frac{q}{Q}, \omega), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}, \end{aligned}$$

that is

$$(6) \quad Z\varphi(x, \omega) = e^{-2\pi i \eta(x+u)} \sum_{q=0}^{Q-1} h_q(x+u, \omega+\eta) Z\varphi(x+u - \frac{q}{Q}, \omega+\eta), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

where

$$h_q(x, \omega) = \sum_{k,\ell \in \mathbb{Z}} c_{q+kQ,\ell} e^{2\pi i (P\ell x + k\omega)}$$

are locally  $L^2$  functions which are  $1/P$  periodic in  $x$  and 1 periodic in  $\omega$ . (Note that we can assume that all  $h_q$  are locally in  $L^2$ , since  $(\varphi, \Lambda)$  is a Riesz system.)

Following Zeevi and Zibulski (see [KZZ04, ZZ97, ZZ93]) we set

$$\mathcal{Z}\varphi(x, \omega) = (Z\varphi(x, \omega), Z\varphi(x - \frac{1}{Q}, \omega), Z\varphi(x - \frac{2}{Q}, \omega), \dots, Z\varphi(x - \frac{Q-1}{Q}, \omega))^T,$$

but extend it quasiperiodically to an infinite vector  $\mathcal{Z}^\circ\varphi(x, \omega)$ , that is, for  $p = sQ + r$ ,  $r \in \{0, 1, \dots, Q-1\}$ ,  $s \in \mathbb{Z}$ , we have

$$\mathcal{Z}_p^\circ\varphi(x, \omega) = Z\varphi(x - \frac{p}{Q}, \omega) = e^{-2\pi i s\omega} \mathcal{Z}_r^\circ\varphi(x, \omega) = e^{-2\pi i s\omega} \mathcal{Z}_r(x, \omega).$$

The above translates then into

$$\mathcal{Z}_p^\circ\varphi(x, \omega) = e^{-2\pi i \eta(x - \frac{p}{Q} + u)} \sum_{q=p}^{Q-1+p} h_{q-p}(x - \frac{p}{Q} + u, \omega + \eta) \mathcal{Z}_q^\circ\varphi(x + u, \omega + \eta)$$

which leads to the biinfinite matrix equation

$$(7) \quad \mathcal{Z}^\circ\varphi(x, \omega) = e^{-2\pi i \eta(x+u)} H(x+u, \omega+\eta) \mathcal{Z}^\circ\varphi(x+u, \omega+\eta)$$

where

$$H_{pq}(x, \omega) = e^{2\pi i \eta \frac{p}{Q}} h_{q-p}(x - \frac{p}{Q}, \omega) \quad \text{if } q-p \in \{0, 1, \dots, Q-1\} \text{ and 0 else.}$$

The above and quasiperiodicity of the Zak transform implies similarly as in the case  $Q = 1$  that

$$\begin{aligned} \mathcal{Z}^\circ\varphi(x, \omega) &= \exp\left(-2\pi i \eta(Rx + u \sum_{r=1}^R r)\right) \cdot \prod_{r=1}^R H(x + ru, \omega + r\eta) \mathcal{Z}^\circ\varphi(x + Ru, \omega + R\eta) \\ &= e^{2\pi i (M_2\omega - M_1x)} \prod_{r=1}^R H(x + ru, \omega + r\eta) \mathcal{Z}^\circ\varphi(x, \omega), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}. \end{aligned}$$

where we used as before that  $M_2\eta/2$  is an integer.

Using the fact that  $H(x, \omega)$  is  $1/P$  periodic in  $x$  and that  $P$  divides  $M_1$  we have in addition that

$$\mathcal{Z}^\circ \varphi(x + \frac{p}{P}, \omega) = e^{2\pi i(M_2\omega - M_1x)} \prod_{r=1}^R H(x + ru, \omega + r\eta) \mathcal{Z}^\circ \varphi(x + \frac{p}{P}, \omega), \quad p = 0, \dots, P-1.$$

Hence, for fixed  $(x, \omega)$ , we have

$$(8) \quad e^{2\pi i(M_1x - M_2\omega)} I = \prod_{r=1}^R H(x + ru, \omega + r\eta), \quad \text{a.e. } (x, \omega) \in \mathbb{R} \times \mathbb{R},$$

for every quasiperiodic sequence in the span of  $\mathcal{Z}^\circ \varphi(x + \frac{p}{P}, \omega)$ ,  $p = 0, \dots, P-1$ . The following lemma implies that (8) is an identity of operators on  $Q$ -quasiperiodic sequences for a.e.  $(x, \omega)$ .

**Lemma 5.** *If  $\varphi \in S_0(\mathbb{R})$  and  $(\varphi, \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z})$  is a Riesz basis for its closed linear span, then  $\mathcal{Z}^\circ \varphi(x + \frac{p}{P}, \omega)$ ,  $p = 0, \dots, P-1$ , spans the space of  $Q$ -quasiperiodic sequences for almost every  $(x, \omega) \in \mathbb{R} \times \mathbb{R}$ .*

*Proof.* For any  $d = (d_{k,\ell}) \in \ell^2(\mathbb{Z}^2)$ , we have

$$\|\{d_{k,\ell}\}\|_{\ell^2} \asymp \left\| \sum_{k,\ell \in \mathbb{Z}} d_{k,\ell} \pi\left(\frac{k}{Q}, \ell P\right) \varphi \right\|_{L^2(\mathbb{R})} = \left\| \sum_{k,\ell \in \mathbb{Z}} d_{k,\ell} Z\pi\left(\frac{k}{Q}, \ell P\right) \varphi \right\|_{L^2([0,1] \times [0,1])}.$$

We compute as above

$$\begin{aligned} \sum_{k,\ell \in \mathbb{Z}} d_{k,\ell} Z\pi\left(\frac{k}{Q}, \ell P\right) \varphi(x, \omega) &= \sum_{k,\ell \in \mathbb{Z}} d_{k,\ell} e^{2\pi i \ell P x} Z\varphi\left(x - \frac{k}{Q}, \omega\right) \\ &= \sum_{q=0}^{Q-1} \sum_{k,\ell \in \mathbb{Z}} d_{q+kQ,\ell} e^{2\pi i(\ell P x + k\omega)} Z\varphi\left(x - \frac{q}{Q}, \omega\right) \\ &= \sum_{q=0}^{Q-1} m_q(x, \omega) Z\varphi\left(x - \frac{q}{Q}, \omega\right), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}. \end{aligned}$$

We conclude that for some  $A > 0$  and all  $m_0(x, \omega), \dots, m_{Q-1}(x, \omega)$  that are 1 periodic in  $\omega$  and  $1/P$  periodic in  $x$ , we have

$$\begin{aligned} A \|\{d_{k,\ell}\}\|_{\ell^2}^2 &= A \sum_{q=0}^{Q-1} \|m_q\|_{L^2([0,1])}^2 \leq \left\| \sum_{q=0}^{Q-1} m_q(x, \omega) Z\varphi\left(x - \frac{q}{Q}, \omega\right) \right\|_{L^2([0,1] \times [0,1])}^2 \\ &= \sum_{p=0}^{P-1} \int_0^{\frac{1}{P}} \int_0^1 \left| \sum_{q=0}^{Q-1} m_q\left(x - \frac{p}{P}, \omega\right) Z\varphi\left(x - \frac{p}{P} - \frac{q}{Q}, \omega\right) \right|^2 d\omega dx \\ (9) \quad &= \sum_{p=0}^{P-1} \int_0^{\frac{1}{P}} \int_0^1 \left| \sum_{q=0}^{Q-1} m_q(x, \omega) Z\varphi\left(x - \frac{p}{P} - \frac{q}{Q}, \omega\right) \right|^2 d\omega dx \\ (10) \quad &\leq \int_0^{\frac{1}{P}} \int_0^1 \sum_{q=0}^{Q-1} |m_q(x, \omega)|^2 \sum_{p=0}^{P-1} \left| Z\varphi\left(x - \frac{p}{P} - \frac{q}{Q}, \omega\right) \right|^2 d\omega dx. \end{aligned}$$

From (10) we conclude that for  $q = 0, \dots, Q - 1$  we have

$$A \leq \sum_{p=0}^{P-1} \left| Z\varphi\left(x - \frac{p}{P} - \frac{q}{Q}, \omega\right) \right|^2 \quad \text{a.e. } (x, \omega) \in [0, 1/P] \times [0, 1].$$

As  $\sum_{p=0}^{P-1} \left| Z\varphi\left(x - \frac{p}{P} - \frac{q}{Q}, \omega\right) \right|^2$  is  $1/P$  periodic in  $x$ , this inequality holds in fact for a.e.  $(x, \omega) \in \mathbb{R} \times \mathbb{R}$ . Moreover, (9) implies that for  $q = 0, \dots, Q - 1$ , the  $\mathbb{C}^P$  vectors

$$\left( Z\varphi\left(x - \frac{q}{Q}, \omega\right), Z\varphi\left(x - \frac{q}{Q} - \frac{1}{P}, \omega\right), Z\varphi\left(x - \frac{q}{Q} - \frac{2}{P}, \omega\right), \dots, Z\varphi\left(x - \frac{q}{Q} - \frac{P-1}{P}, \omega\right) \right)$$

are linearly independent for a.e.  $(x, \omega) \in [0, 1/P] \times [0, 1]$ , indeed, else we could find  $L^2(\mathbb{R}^2)$  functions  $m_q(x, \omega)$ , not all  $m_q(x, \omega) = 0$ , such that (9) equals 0. We conclude that the matrix

$$\begin{pmatrix} Z\varphi(x, \omega) & Z\varphi\left(x - \frac{1}{P}, \omega\right) & Z\varphi\left(x - \frac{2}{P}, \omega\right) & \dots & Z\varphi\left(x - \frac{P-1}{P}, \omega\right) \\ Z\varphi\left(x - \frac{1}{Q}, \omega\right) & Z\varphi\left(x - \frac{1}{Q} - \frac{1}{P}, \omega\right) & Z\varphi\left(x - \frac{1}{Q} - \frac{2}{P}, \omega\right) & \dots & Z\varphi\left(x - \frac{1}{Q} - \frac{P-1}{P}, \omega\right) \\ Z\varphi\left(x - \frac{2}{Q}, \omega\right) & Z\varphi\left(x - \frac{2}{Q} - \frac{1}{P}, \omega\right) & Z\varphi\left(x - \frac{2}{Q} - \frac{2}{P}, \omega\right) & \dots & Z\varphi\left(x - \frac{2}{Q} - \frac{P-1}{P}, \omega\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z\varphi\left(x - \frac{Q-1}{Q}, \omega\right) & Z\varphi\left(x - \frac{Q-1}{Q} - \frac{1}{P}, \omega\right) & Z\varphi\left(x - \frac{Q-1}{Q} - \frac{2}{P}, \omega\right) & \dots & Z\varphi\left(x - \frac{Q-1}{Q} - \frac{P-1}{P}, \omega\right) \end{pmatrix}$$

is full rank for a.e.  $(x, \omega) \in [0, 1/P] \times [0, 1]$ , so its  $P$  columns are a spanning set of  $\mathbb{C}^Q$  for a.e.  $(x, \omega) \in [0, 1/P] \times [0, 1]$ . Note that replacing  $x$  by  $x - \frac{p_0}{P}$  in the matrix above corresponds to a circular shift of the columns of the matrix by  $p_0$ , with the possible appearance of a non-zero scalar factor  $e^{2\pi i \omega}$  due to the quasiperiodicity of the Zak transform. This allows us to extend the observation on the columns spanning  $\mathbb{C}^Q$  to hold for almost every  $(x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$ .  $\square$

In the  $Q$  dimensional model, that is, choosing  $\tilde{H}(x, \omega) \in \mathbb{C}^{Q \times Q}$  so that for any  $\mathcal{Z} \in \mathbb{C}^Q$  and any  $(x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$  we have

$$\left( H(x, \omega) \mathcal{Z}^\circ \right)_p = \left( \tilde{H}(x, \omega) \mathcal{Z} \right)_p, \quad p = 0, 1, \dots, Q - 1,$$

we have equivalently (with  $I$  now denoting the identity matrix in  $\mathbb{C}^{Q \times Q}$ )

$$e^{2\pi i (M_1 x - M_2 \omega)} I = \prod_{r=1}^R \tilde{H}(x + ru, \omega + r\eta), \quad \text{a.e. } (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}.$$

Taking  $h(x, \omega) = \det \tilde{H}(x, \omega)$  we conclude

$$e^{2\pi i Q (M_1 x - M_2 \omega)} = \prod_{r=1}^R h(x + ru, \omega + r\eta), \quad \text{a.e. } (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}.$$

It remains to argue that  $h(x, \omega)$  is continuous, since then, Proposition 3 and the  $1/P$  periodicity of  $h(x, \omega)$  in  $x$  and the 1 periodicity in  $\omega$  implies first that  $R$  divides  $QM_2$ . Hence  $RL = QM_2$  for some  $L \in \mathbb{N}$  and  $u = M_2/R = L/Q \in \frac{1}{Q}\mathbb{Z}$ . Second, we have  $RP$  divides  $QM_1$ , that is,  $\eta \frac{Q}{P} = \frac{QM_1}{RP}$  is an integer. By assumption, we have that  $(P, Q) = 1$ , so  $\eta \in P\mathbb{Z}$ . However, since by assumption  $(u, \eta) \notin \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$ , this is a contradiction.

We conclude by showing that  $\tilde{H}$  and therefore  $h$  depends continuously on  $(x, \omega)$ . To this end, observe that  $\varphi \in S_0(\mathbb{R})$  implies that both,  $\mathcal{Z}^\circ \varphi(x, \omega)$  and  $\mathcal{Z} \varphi(x, \omega)$  are continuous in

$(x, \omega)$ . Let  $\Phi(x, \omega) \in \mathbb{C}^{Q \times P}$  be the frame synthesis matrix with columns  $\mathcal{Z}\varphi(x + \frac{p}{P}, \omega)$ ,  $p = 0, \dots, P-1$ . Equation (7) implies that

$$e^{2\pi i \eta x} \mathcal{Z}\varphi(x - u, \omega - \eta) = \tilde{H}(x, \omega) \mathcal{Z}\varphi(x, \omega).$$

Inserting  $x + \frac{p}{P}$  for  $x$  and using that  $\tilde{H}(x, \omega)$  is  $1/P$  periodic in  $x$ , we obtain

$$e^{2\pi i \eta x} \Phi(x - u, \omega - \eta) D(\eta) = \tilde{H}(x, \omega) \Phi(x, \omega),$$

where  $D(\eta)$  is the diagonal matrix with entries  $1, e^{-2\pi i \eta/P}, e^{-2\pi i \eta 2/P}, \dots, e^{-2\pi i \eta (P-1)/P}$ .

The columns of  $\Phi(x, \omega)$  form a frame that depends continuously on  $(x, \omega)$ . Hence, the rame operator  $S(x, \omega) = \Phi(x, \omega)\Phi(x, \omega)^* \in \mathbb{C}^{Q \times Q}$  and its inverse  $S(x, \omega)^{-1}$  depend continuously on  $(x, \omega)$ . Similarly, the matrix consisting of the dual frame elements  $\Psi(x, \omega) = S(x, \omega)^{-1}\Phi(x, \omega)$  depends continuously on  $(x, \omega)$ . Clearly,  $\Psi(x, \omega)^*$  is a right inverse of  $\Phi(x, \omega)$ . The equality

$$e^{2\pi i \eta x} \Phi(x - u, \omega - \eta) D(\eta) \Psi(x, \omega)^* = \tilde{H}(x, \omega) \Phi(x, \omega) \Psi(x, \omega)^* = \tilde{H}(x, \omega)$$

shows that  $\tilde{H}(x, \omega)$  depends continuously on  $(x, \omega)$ . The proof is complete.

#### 4. CONSTRUCTION OF GABOR SPACES WITH ADDITIONAL SHIFT INVARIANCE

In this section, we study the case  $\pi(\frac{1}{R}, 0)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ ,  $\gcd(P, R) = 1$ , and give a complete characterization of those  $\varphi$  which satisfy  $\pi(\frac{1}{R}, 0)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ .

Recall that  $\pi(\frac{1}{R}, 0)\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$  if and only if there exists a sequence  $c = (c_{k, \ell}) \in \ell^2(\mathbb{Z}^2)$  with

$$\begin{aligned} Z\varphi(x - \frac{1}{R}, \omega) &= \sum_{k, \ell \in \mathbb{Z}} c_{k, \ell} e^{2\pi i (P\ell x + k\omega)} Z\varphi(x, \omega) \\ (11) \quad &= h(x, \omega) Z\varphi(x, \omega), \quad (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}. \end{aligned}$$

Our strategy is to construct a quasiperiodic function  $F(x, \omega)$  and a function  $h(x, \omega)$  so that (11) holds with  $F$  in place of  $Z\varphi$ . Then we use the fact that the Zak transform is onto the space of quasiperiodic functions, and, using a Zak transform inversion formula [Grö01], we construct

$$\varphi(x) = \int_0^1 Z\varphi(x, \omega) d\omega = \int_0^1 F(x, \omega) d\omega = \int_u^{1+u} F(x, \omega), \quad \text{a.e. } x \in \mathbb{R}.$$

In order to construct the quasiperiodic function  $F(x, \omega)$ , we shall show that the conditions

- (S)  $F(x - \frac{1}{R}, \omega) = h(x, \omega) F(x, \omega)$ ,  $x \in [1/R, 1]$ ,  $\omega \in [0, 1]$ ;
- (Q)  $e^{2\pi i \omega} = \prod_{r=0}^{R-1} h(x + \frac{r}{R}, \omega)$ ,  $(x, \omega) \in [0, 1/P] \times [0, 1] \cap \text{supp } F$ ;
- (P)  $h(x, \omega)$  is  $1/P$  periodic in  $x$  and 1 periodic in  $\omega$ ,

characterize the pairs  $F(x, \omega) = Z\varphi(x, \omega)$  and  $h(x, \omega)$  that satisfy (11).

First, note that (11) implies that the zero set  $\mathcal{E}$  of  $Z\varphi$  is  $1/R$  periodic. Indeed, clearly  $Z\varphi(x, \omega) = 0$  implies  $Z\varphi(x - \frac{1}{R}, \omega) = 0$ . But also,  $Z\varphi(x, \omega) = 0$  implies

$$0 = Z\varphi(x + 1, \omega) = Z\varphi(x + 1 - \frac{1}{R}, \omega) = Z\varphi(x + 1 - \frac{2}{R}, \omega) = \dots = Z\varphi(x + \frac{1}{R}, \omega).$$

In addition, since  $R \in \mathbb{N}$ , the quasiperiodicity conditions

$$e^{2\pi i M\omega} = \prod_{r=1}^{RM} h(x + \frac{r}{R}, \omega), \quad (x, \omega) \in \text{supp } Z\varphi \subseteq \mathbb{R} \times \widehat{\mathbb{R}},$$

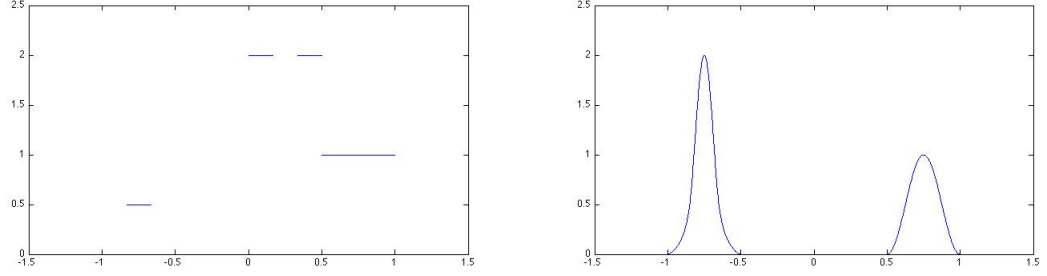


FIGURE 1. Functions as constructed in Example 1 and Example 2

are just the  $M$ -th power of the equation where  $M = 1$ , that is, the set of equations is equivalent to

$$e^{2\pi i\omega} = \prod_{r=1}^R h(x + \frac{r}{R}, \omega), \quad (x, \omega) \in \text{supp } Z\varphi \subseteq \mathbb{R} \times \widehat{\mathbb{R}}.$$

We conclude that the three conditions given above follow from (11). To observe that these conditions are also sufficient, note first that as argued above, quasiperiodicity of  $F$  implies that the zero set of  $F$  is  $1/R$  periodic. Hence, condition (b) extends to all  $(x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$ . Now, (b) together with (a) implies that

$$h(x, \omega)F(x, \omega) = F(x - \frac{1}{R}, \omega) = e^{-2\pi i\omega} F(x + \frac{R-1}{R}, \omega), \quad x \in [0, 1/R] \times [0, 1].$$

Indeed, it suffices to check this on  $\text{supp } F$  where we have

$$\begin{aligned} e^{-2\pi i\omega} F(x + \frac{R-1}{R}, \omega) &= \prod_{r=0}^{R-1} h(x + \frac{r}{R}, \omega) F(x + \frac{R-1}{R}, \omega) \\ &\vdots \\ &= h(x, \omega) h(x + \frac{1}{R}, \omega) F(x + \frac{1}{R}, \omega) \\ &= h(x, \omega) F(x, \omega), \end{aligned}$$

which concludes our proof of sufficiency.

In our first example, we construct a discontinuous window function which generates a Gabor space that features an additional shift invariance.

**Example 1.** We choose  $R = 2$  and  $P = 3$ . Let  $I_k = [k/6, (k+1)/6] \times \widehat{\mathbb{R}}$  for  $k \in \mathbb{Z}$ . We define the function  $h(x, \omega) = 2$  on  $\bigcup_k I_{2k}$  and  $h(x, \omega) = e^{2\pi i\omega}/2$  on  $\bigcup_k I_{2k+1}$ . Clearly,  $h$  satisfies (Q) and (P). We set  $F(x, \omega) = 1$  for  $x \in [1/2, 1]$  and

$$F(x, \omega) = F(x + 1/2 - 1/2, \omega) = h(x + 1/2, \omega) F(x + 1/2, \omega) = h(x + 1/2, \omega), \quad x \in [0, 1/2],$$

and extend the function quasiperiodically. In the following, let  $f = f_{-1/2}^{1/2}$ . Motivated by the Zak transform inversion formula, we define for  $x \in \mathbb{R}$ ,

$$\begin{aligned} \varphi(x) &= \int F(x, \omega) d\omega = \begin{cases} \int e^{2\pi im\omega} = \delta_0(m), & x \in [m + 1/2, m + 1], \\ \int 2e^{2\pi im\omega} = 2\delta_0(m), & x \in [m, m+1/6] \cup [m+1/3, m+1/2], \\ \int \frac{1}{2} e^{2\pi i(m+1)\omega} = \frac{1}{2}\delta_0(m+1), & x \in [m + 1/6, m + 1/3], \end{cases} \\ &= 1/2\chi_{[-5/6, -2/3]} + 2\chi_{[0, 1/6]} + 2\chi_{[1/3, 1/2]} + \chi_{[1/2, 1]}. \end{aligned}$$

Clearly,  $\varphi \notin S_0(\mathbb{R})$ . Moreover, note that  $9/4 \leq \sum_{p=0}^2 \left| Z_\varphi(t - \frac{p}{3}, \nu) \right|^2 \leq 9$  for all  $t$  and  $\nu$  implies that  $(\varphi, \mathbb{Z} \times P\mathbb{Z})$  is a Riesz basis for  $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ .

In addition, we would like to point out once more that  $h(t, \nu) = \sum c_{k,\ell} e^{2\pi i(P\ell x - k\nu)}$  not being continuous implies that  $\pi(1/2, 0)\varphi = \sum c_{k,\ell} \pi(k, P\ell)\varphi$  converges rather slowly, for example, we do not have absolute convergence.

**Remark 6.** Note that the shift-invariant space  $\mathcal{G}(\varphi, \mathbb{Z} \times \{0\})$  is constant on the intervals  $[m + 1/2, m + 1]$ ,  $m \in \mathbb{Z}$ , but the half shift  $\varphi(x - 1/2)$  does not satisfy this property. Hence,  $\pi(1/2, 0)\varphi = T_{1/2}\varphi$  is not a member of the shift-invariant space  $\mathcal{G}(\varphi, \mathbb{Z} \times \{0\})$ , showing that membership of translates to Gabor spaces cannot be reduced to membership of translates to respective shift-invariant spaces.

In the following, we construct a smooth window  $\varphi$  which has an additional shift invariance and which generates therefore not a Riesz basis for the Gabor space it spans. Note that mollifying  $h$  in the example above leads to a continuous function which does not satisfy property (Q).

**Example 2.** We consider again  $R = 2$  and  $P = 3$  and construct a Schwartz class function  $\varphi$  such that  $T_{\frac{1}{2}}\varphi \in \mathcal{G}(\varphi, \mathbb{Z} \times 3\mathbb{Z})$ .

To this end, choose a function  $u(x)$  on  $[0, 1/2]$  with

- (1)  $u$  has only values in  $[\frac{1}{2}, 2]$ ,  $u(0) = 1$  but  $u$  not constant 1;
- (2)  $u$  is smooth;
- (3)  $u(x)u(x + 1/6) = 1$  for  $x \in [0, \frac{1}{3}]$ .

Now, set  $h(x, \omega) = u(x)$  for  $x \in [0, \frac{1}{2}]$  and  $h(x, \omega) = e^{2\pi i\omega}/u(x - 1/2)$  for  $x \in [\frac{1}{2}, 2]$ . So  $h$  periodically extended is smooth away from the set  $\frac{1}{2}\mathbb{Z} \times \widehat{\mathbb{R}}$  and satisfies (Q).

Now, we define  $F(x, \omega) = v(x)$  for  $x \in [1/2, 1]$  where  $v(1/2) = v(1) = 0$ ,  $v(x) \in [0, 1]$ , and  $v$  smooth. Further, define

$$\begin{aligned} F(x, \omega) &= F(x + 1/2 - 1/2, \omega) = h(x + 1/2, \omega)F(x + 1/2, \omega) \\ &= e^{2\pi i\omega}v(x + 1/2)/u(x), \quad x \in [0, 1/2]. \end{aligned}$$

Clearly,  $F$  is smooth away from  $\frac{1}{2}\mathbb{Z} \times \widehat{\mathbb{R}}$ , but by choosing  $v^{(n)}(0) = v^{(n)}(1/2) = 0$  for all  $n \in \mathbb{N}$  ensures that  $F$  is smooth on  $\mathbb{R} \times \widehat{\mathbb{R}}$ .

We compute

$$\begin{aligned} \varphi(x) &= \int_{-1/2}^{1/2} F(x, \omega) d\omega \\ &= \begin{cases} \int e^{2\pi im\omega}v(x) = \delta_0(m)v(x), & x \in [m + 1/2, m + 1], \\ \int \frac{v(x+1/2)}{u(x)} e^{2\pi i(m+1)\omega} = \frac{v(x+1/2)}{u(x)}\delta_0(m + 1), & x \in [m, m + 1/2], \end{cases} \\ &= \begin{cases} \frac{v(x+1/2)}{u(x)}, & x \in [-1, -1/2], \\ v(x), & x \in [1/2, 1], \end{cases} \end{aligned}$$

We conclude that  $\varphi$  is supported on  $[-1, 1]$  and smooth.

## REFERENCES

- [ACH<sup>+</sup>10] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, and U. Molter, *Invariance of a shift-invariant space*, J. Fourier Anal. Appl. **16** (2010), no. 1, 60–75. MR 2587581 (2011a:42052)

- [AKTW12] A. Aldroubi, I. Krishtal, R. Tessera, and H. Wang, *Principal shift-invariant spaces with extra invariance nearest to observed data*, Collect. Math. **63** (2012), no. 3, 393–401. MR 2957978
- [ASW11] A. Aldroubi, Q. Sun, and H. Wang, *Uncertainty principles and Balian-Low type theorems in principal shift-invariant spaces*, Appl. Comput. Harmon. Anal. **30** (2011), no. 3, 337–347. MR 2784568 (2012e:42068)
- [Bal81] R. Balian, *Un principe d'incertitude fort en théorie du signal ou en mécanique quantique*, C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre **292** (1981), no. 20, 1357–1362. MR 644367 (83a:94009)
- [BHW92] J.J. Benedetto, C. Heil, and D.F. Walnut, *Uncertainty principles for time-frequency operators*, Continuous and discrete Fourier transforms, extension problems and Wiener-Hopf equations, Oper. Theory Adv. Appl., vol. 58, Birkhäuser, Basel, 1992, pp. 1–25. MR 1183741 (94a:42041)
- [BHW95] ———, *Differentiation and the Balian-Low theorem*, J. Fourier Anal. Appl. **1** (1995), no. 4, 355–402. MR 1350699 (96f:42002)
- [BHW98] ———, *Gabor systems and the Balian-Low theorem*, Gabor analysis and algorithms, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998, pp. 85–122. MR 1601111 (98j:42016)
- [BW94] J.J. Benedetto and D.F. Walnut, *Gabor frames for  $L^2$  and related spaces*, Wavelets: mathematics and applications, Stud. Adv. Math., CRC, Boca Raton, FL, 1994, pp. 97–162. MR 1247515 (94i:42040)
- [Dau90] I. Daubechies, *The wavelet transform, time frequency localization and signal analysis*, IEEE, Transactions on Information Theory **36** (1990), no. 5, 961–1005.
- [DLL95] I. Daubechies, H.J. Landau, and Z. Landau, *Gabor time-frequency lattices and the Wexler-Raz identity*, J. Fourier Anal. Appl. **1** (1995), no. 4, 437–478. MR 1350701 (96i:42021)
- [Fei81] H.G. Feichtinger, *On a new segal algebra*, Monatsh. Math. **92** (1981), 269–289.
- [FG97] H.G. Feichtinger and K. Gröchenig, *Gabor frames and time-frequency analysis of distributions*, J. Funct. Anal. **146** (1997), no. 2, 464–495. MR 1452000 (98k:42041)
- [FZ98] H.G. Feichtinger and G. Zimmermann, *A Banach space of test functions for Gabor analysis*, Gabor Analysis and Algorithms: Theory and Applications (H.G. Feichtinger and T. Strohmer, eds.), Birkhäuser, Boston, MA, 1998, pp. 123–170.
- [GH04] J.-P. Gabardo and D. Han, *Balian-Low phenomenon for subspace Gabor frames*, J. Math. Phys. **45** (2004), no. 8, 3362–3378. MR 2077516 (2005e:42093)
- [GHHK02] K. Gröchenig, D. Han, C. Heil, and G. Kutyniok, *The Balian-Low theorem for symplectic lattices in higher dimensions*, Appl. Comput. Harmon. Anal. **13** (2002), no. 2, 169–176. MR 1942751 (2003i:42041)
- [Grö01] K. Gröchenig, *Foundations of time-frequency analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2001.
- [Hei07] C. Heil, *History and evolution of the density theorem for Gabor frames*, J. Four. Anal. Appl. **12** (2007), 113–166.
- [Jan08] A. J. E. M. Janssen, *A decay result for certain windows generating orthogonal Gabor bases*, J. Fourier Anal. Appl. **14** (2008), no. 1, 1–15. MR 2379750 (2009a:42047)
- [KP06] W. Kozek and G.E. Pfander, *Identification of operators with bandlimited symbols*, SIAM J. Math. Anal. **37** (2006), no. 3, 867–888.
- [KZZ04] P. Kisilev, M. Zibulevsky, and Y.Y. Zeevi, *A multiscale framework for blind separation of linearly mixed signals*, J. Mach. Learn. Res. **4** (2004), no. 7-8, 1339–1363. MR 2103632
- [Low85] F. Low, *Complete sets of wave packages*, A Passion for Physics—Essays in Honor of Geoffrey Chew (et al. C. DeTar, ed.), World Scientific, 1985, pp. 17–22.
- [Pfa13] G.E. Pfander, *Sampling of operators*, J. Four. Anal. Appl. **19** (2013), no. 3, 612.
- [PW06] G.E. Pfander and D. Walnut, *Measurement of time-variant channels*, IEEE Trans. Info. Theory **52** (2006), no. 11, 4808–4820.
- [ZZ93] M. Zibulski and Y.Y. Zeevi, *Oversampling in the gabor scheme*, IEEE Trans. On Signal Processing **41** (1993), no. 8, 2679 – 2687.
- [ZZ97] ———, *Analysis of multiwindow gabor-type schemes by frame methods analysis of multiwindow gabor-type schemes by frame methods analysis of multiwindow gabor-type schemes by frame methods*, Appl. Comp. Harmonic Analysis **4** (1997), 188 – 221.

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