Boundedness of pseudo-differential operators on L^p, Sobolev, and modulation spaces

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Abstract We introduce new classes of modulation spaces over phase space. By means of the Kohn-Nirenberg correspondence, these spaces induce norms on pseudo-differential operators that bound their operator norms on L^p–spaces, Sobolev spaces, and modulation spaces.

1 Introduction

Pseudo-differential operators are discussed in various areas of mathematics and mathematical physics, for example, in partial differential equations, time-frequency analysis, and quantum mechanics [19, 18, 21, 32, 34]. They are defined as follows.

Let σ be a tempered distribution on phase space $\mathbb{R}^{2d}$, that is, $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ where $\mathcal{S}(\mathbb{R}^{2d})$ denotes the space of Schwartz class functions. The pseudo-differential operator T_σ corresponding to the symbol σ is given by

$$T_\sigma f(x) = \int \sigma(x,\xi)\, \widehat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

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Here, $\widehat{f}$ denotes the Fourier transform of f, namely,

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int f(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

One of the central goals in the study of pseudo-differential operators is to obtain necessary and sufficient conditions for pseudo-differential operators to extend boundedly to function spaces such as $L^p(\mathbb{R}^d)$ [3, 5, 20, 33]. A classical result in this direction is the following.

For $m \in \mathbb{R}$, we let S^m consist of all functions σ in $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for any multi-index (α, β), there is $C_{\alpha,\beta} > 0$ with

$$\left|\left(\partial_x^\beta \partial_\xi^\alpha \sigma\right)(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}.$$

For $\sigma \in S^0(\mathbb{R}^d)$, it is known that T_σ acts boundedly on $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$. A consequence of this result is that if $\sigma \in S^m$, then T_σ is a bounded operator mapping $H^p_{s+m}(\mathbb{R}^d)$ to $H^p_s(\mathbb{R}^d)$, where $H^p_s(\mathbb{R}^d)$ is the Sobolev Spaces of order $s \in \mathbb{R}$; for more details see Wong's book [32]. Similarly, in [33], Wong obtains weighted L^p–boundedness results for pseudo-differential operators with symbols in S^m.

Smoothness and boundedness of symbols though are far from being necessary (nor sufficient) for the L^p-boundedness of pseudo-differential operators. In fact, every symbol $\sigma \in L^2(\mathbb{R}^{2d})$ defines a so-called Hilbert–Schmidt operator and Hilbert-Schmidt operators are bounded, in fact, compact operators on $L^2(\mathbb{R}^d)$. Non-smooth and unbounded symbols have been considered systematically in the framework of modulation spaces, an approach that we continue in this paper.

Modulation spaces were first introduced by Feichtinger in [9] and they have been further developed by him and Gröchenig in [8, 9, 12, 10, 11, 13]. In the following, set $\phi(x) = e^{-\pi\|x\|^2/2}$ and let the dual pair bracket $(\cdot,\cdot)$ be linear in the first argument and antilinear in the second argument.

Definition 1.1 (Modulation spaces over Euclidean space)**.** *Let M_ν denote modulation by $\nu \in \mathbb{R}^d$, namely, $M_\nu f(x) = e^{2\pi i t\cdot\nu} f(x)$, and let T_t be translation by $t \in \mathbb{R}^d$, that is, $T_t f(x) = g(x-t)$.*

The short-time Fourier transform $V_\phi f$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the Gaussian window ϕ is given by

$$V_\phi f(t,\nu) = \mathcal{F}\big(f\, T_t\phi\big)(\nu) = (f, M_\nu T_t \phi) = \int f(x)\ e^{-2\pi i x\nu}\phi(x-t)\, dx\,.$$

The modulation space $M^{pq}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, is a Banach space consisting of those $f \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|f\|_{M^{pq}} = \|V_\phi f\|_{L^{pq}} = \Big(\int \Big(\int |V_\phi f(t,\nu)|^p \, dt \Big)^{1/p} d\nu \Big)^{1/q} < \infty \,,$$

with usual adjustment of the mixed norm space if $p = \infty$ and/or $q = \infty$.

Roughly speaking, distributions in $M^{pq}(\mathbb{R}^d)$ 'decay' at infinity like a function in $L^p(\mathbb{R}^d)$ and have the same local regularity as a function whose Fourier transform is in $L^q(\mathbb{R}^d)$.

The boundedness of pseudo-differential operators on modulation spaces are studied for various classes of symbols, for example, in [5, 7, 15, 16, 27, 28, 30, 31]. In [27, 28] for example, Toft discusses boundedness of pseudo-differential operators on weighted modulation spaces. In [5], Nicola and Cordero describe a class of pseudo-differential operators with symbols σ in modulation spaces for which T_σ is bounded on $L^p(\mathbb{R}^d)$.

The modulation space membership criteria on Kohn–Nirenberg symbols used in [5, 7, 27, 28] do not allow to require different decay in x and ξ of $\sigma(x,\xi)$. In the recently developed sampling theory for operators, though, a separate treatment of the decay of x and ξ was beneficial [17, 23, 24]. In fact, this allows to realize canonical symbol norms of convolution and multiplication operators as modulation space norms on Kohn-Nirenberg symbols. Motivated by this work, we give the following definition.

Definition 1.2 (Modulation spaces over phase space)**.** *The symplectic Fourier transform of $F \in \mathcal{S}(\mathbb{R}^{2d})$ is given by*

$$\widetilde{\mathcal{F}} F(t,\nu) = \int_{\mathbb{R}^{2d}} e^{-2\pi i [(x,\xi),(t,\nu)]} F(x,\xi) \, dx \, d\xi, \tag{1.1}$$

where $[(x,\xi),(t,\nu)]$ is the symplectic form of (x,ξ) and (t,ν) defined by $[(x,\xi),(t,\nu)] = x \cdot \nu - \xi \cdot t$. Analogously, symplectic modulation $\widetilde{M}_{(t,\nu)}$ is $\widetilde{M}_{(t,\nu)} F(x,\xi) = e^{2\pi i [(x,\xi),(t,\nu)]} F(x,\xi)$.

The symplectic short-time Fourier transform $\widetilde{V}_\phi f$ of $F \in \mathcal{S}'(\mathbb{R}^d)$ is given by

$$\begin{aligned} \widetilde{V}_\phi F(x,t,\xi,\nu) &= \widetilde{\mathcal{F}}\big(F \, T_{(x,\xi)} \phi\big)(t,\nu) = (F, \widetilde{M}_{(\nu,t)} T_{(x,\xi)} \phi) \\ &= \iint e^{-2\pi i (\widetilde{x}\nu - \widetilde{\xi} t)} F(\widetilde{x},\widetilde{\xi}) \varphi(\widetilde{x} - x, \widetilde{\xi} - \xi) \, d\widetilde{x} \, d\widetilde{\xi}. \end{aligned} \tag{1.2}$$

The modulation space over phase space $\widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$, $1 \le p_1, p_2, q_1, q_2, \le \infty$, *is the Banach space consisting of those* $F \in \mathcal{S}'(\mathbb{R}^{2d})$ *with*

$$\begin{aligned}\|F\|_{\widetilde{M}^{p_1p_2q_1q_2}} &= \|\widetilde{V}_\phi F\|_{L^{p_1p_2q_1q_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |(\widetilde{V}_\psi F)(x,t,\xi,\nu)|^{p_1}\,dx\Big)^{p_2/p_1}dt\Big)^{q_1/p_2}d\xi\Big)^{q_2/q_1}d\nu\Big)^{1/q_1} \\ &< \infty\,, \end{aligned} \tag{1.3}$$

with usual adjustments if $p_1 = \infty$, $p_2 = \infty$, $q_1 = \infty$, *and/or* $q_2 = \infty$.

Note that the order of the list of variables in (1.2) is crucial as it indicates the order of integration in (1.3). We choose to list first the time variable x followed by the time-shift variable t. The time variables are followed by the frequency variable ξ and the frequency-shift variable ν. Alternative orders of integration were considered, for example, in [2, 5, 27, 28].

Below, $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators mapping the Banach space X to the Banach space Y; $\mathcal{L}(X, Y)$ is equipped with the operator norm. Below, the conjugate exponent of $p \in [1, \infty]$ is denoted by p'. Our main result follows.

Theorem 1.3. *Let* $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1, \infty]$. *Then there exists* $C > 0$ *such that*

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1q_1}, M^{p_2q_2})} \le C\,\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}}, \quad \sigma \in \widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d}), \tag{1.4}$$

if and only if

$$\frac{1}{p_1'} + \frac{1}{p_2} \le \frac{1}{p_3} + \frac{1}{p_4}, \qquad p_4 \le \min\{p_1', p_2\}, \tag{1.5}$$

$$\frac{1}{q_1'} + \frac{1}{q_2} \le \frac{1}{q_3} + \frac{1}{q_4}, \qquad q_4 \le \min\{q_1', q_2\}. \tag{1.6}$$

Theorem 4.1 below is a variant of Theorem 1.3 that involves symbols in weighted modulation spaces.

Observe that (1.5) depends only on the parameters p_i, while (1.6) depends analogously only on the parameters q_i. That is, the conditions on decay in time and on decay in frequency, or, equivalently, on smoothness in frequency and on smoothness in time, on the Kohn-Nirenberg symbol are linked to the respective conditions on domain and range of the operator,

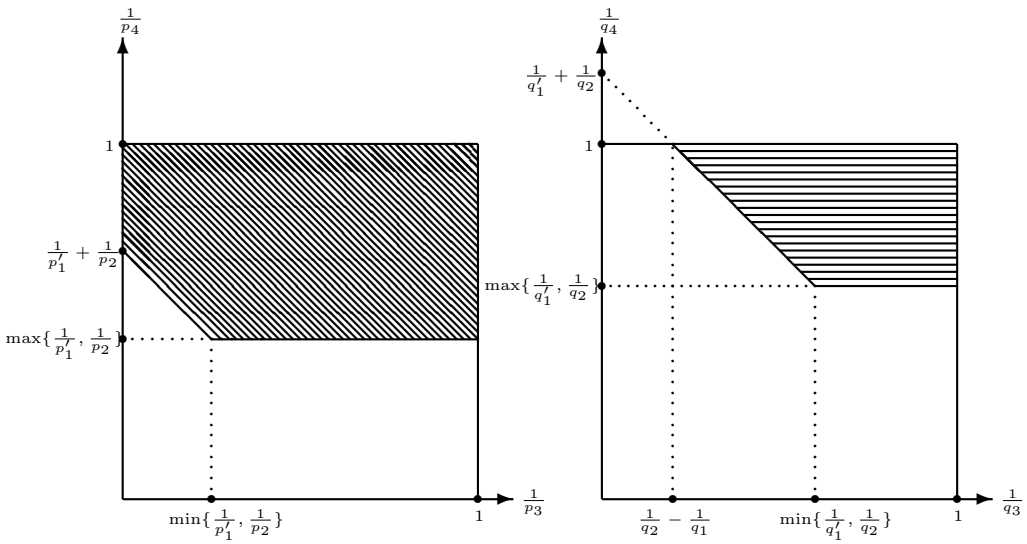

Figure 1: For fixed p_1, p_2 and q_1, q_2, we mark the regions of $(\frac{1}{p_3}, \frac{1}{p_4})$ and $(\frac{1}{q_3}, \frac{1}{q_4})$ for which every $\sigma \in \widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})$ induces a bounded operator $T_\sigma : M^{p_1 q_1}(\mathbb{R}^d) \to M^{p_2 q_2}(\mathbb{R}^d)$. In fact, for $(\frac{1}{p_3}, \frac{1}{p_4})$ and $(\frac{1}{q_3}, \frac{1}{q_4})$ in the hashed region, there exists $C > 0$ with $\|T_\sigma\|_{\mathcal{L}(M^{p_1 q_1}, M^{p_2 q_2})} \leq C \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}}$. The conditions on the time decay parameters p_1, p_2, p_3, p_4 are independent of the conditions on the frequency decay parameters q_1, q_2, q_3, q_4.

but time and frequency remain independent of one another. See Figure 1 for an illustration of conditions (1.5) and (1.6).

An L^p–boundedness result for the introduced classes of pseudo-differential operators follows.

Corollary 1.4. *Let $p, p_3, p_4, q, q_3, q_4 \in [1, \infty]$. Assume*

$$\frac{1}{p'} + \frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4}, \quad p_4 \leq \min\{p', q\},$$

and

$$\begin{cases} \frac{1}{p} + \frac{1}{q} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p, q\}, \quad \text{if} \quad p, q \in [1, 2], \\ \frac{1}{p} + \frac{1}{q'} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p, q'\}, \quad \text{if} \quad 1 \leq p \leq 2 \leq q, \\ \frac{1}{p'} + \frac{1}{q'} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p', q'\}, \quad \text{if} \quad 2 \leq \min\{p, q\}, \\ \frac{1}{p'} + \frac{1}{q} & \leq \frac{1}{q_3} + \frac{1}{q_4}, \quad q_4 \leq \min\{p', q\}, \quad \text{if} \quad 1 \leq q \leq 2 \leq p. \end{cases}$$

Then $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded and there exists a constant $C > 0$ such that

$$\|T_\sigma\|_{\mathcal{L}(L^p, L^q)} \leq C \, \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}}, \quad \sigma \in \widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d}).$$

Corollary 1.4 encompasses, for example, the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$, namely

$$HS\big(L^2(\mathbb{R}^d)\big) = \big\{T_\sigma : \sigma \in \widetilde{M}^{2,2,2,2}(\mathbb{R}^{2d}) = L^2(\mathbb{R}^{2d})\big\} \subset \mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)).$$

Moreover, Corollary 1.4 reconfirms also L^2–boundedness of Sjöstrand class operators [25, 26],

$$Sj \subset \{T_\sigma : \sigma \in \widetilde{M}^{\infty,1,\infty,1}(\mathbb{R}^{2d})\} \subset \mathcal{L}\Big(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\Big).$$

Using the weighted version of Theorem 1.3, namely, Theorem 4.1, we get the following boundedness result for Sobolev spaces.

Corollary 1.5. *Let $p_1, p_2, p_3, p_4 \in [1, \infty]$ and $s \in \mathbb{R}$. Let w be a moderate weight function on $\mathbb{R}^{4d}$ satisfying*

$$w(x, t, \nu, \xi) \leq \big(1 + |\xi|^2\big)^{s/2} \big(1 + |\nu + \xi|^2\big)^{s/2}, \quad x, t, \nu, \xi \in \mathbb{R}^d.$$

Assume that

$$\frac{1}{p_1'}+\frac{1}{p_2}\le\frac{1}{p_3}+\frac{1}{p_4},\quad p_4\le\min\{p_1',p_2\}.$$

Then

$$\|T_\sigma\|_{\mathcal{L}(H_s^{p_1},H_s^{p_2})}\le C\,\|\sigma\|_{\widetilde{M}_w^{p_3,p_4,1,1}},\quad \sigma\in\widetilde{M}_w^{p_3,p_4,1,1}(\mathbb{R}^{2d}),$$

for some constant $C>0$.

The paper is structured as follows. Section 2 discusses mixed norm spaces and modulation spaces over Euclidean and over phase space in some detail. In Section 3, our boundedness results for pseudo-differential operators with symbols in modulation spaces over phase space are compared to results in the literature. Finally, in Section 4 we prove our main results, Theorem 1.3, Corollary 1.4, and Theorem 4.1.

2 Background on modulation spaces

In the following, x,ξ,t,ν denote d-dimensional Euclidean variables. If not indicated differently, integration is with respect to the Lebesgue measure on $\mathbb{R}^d$.

Let $r=(r_1,r_2,\dots,r_n)$ where $1\le r_i<\infty$, $i=1,2,\dots,n$. The mixed norm space $L^r(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|F\|_{L^r}=\Big(\int_{\mathbb{R}}\dots\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|F(x_1,\dots,x_n)|^{r_1}\,dx_1\Big)^{r_1/r_2}dx_2\dots\Big)^{r_n/r_{n-1}}dx_n\Big)^{1/r_n}$$

is finite [1]. $L^r(\mathbb{R}^n)$ is a Banach space with norm $\|\cdot\|_{L^r}$. Similarly, we define $L^r(\mathbb{R}^n)$ where $r_i=\infty$ for some indices i.

If $n=2d$, $r_1=r_2=\cdots=r_d=p$ and $r_{d+1}=\cdots=r_{2d}=q$, then we denote $L^r(\mathbb{R}^{2d})$ by $L^{pq}(\mathbb{R}^{2d})$. Similarly, if $n=4d$ and $r_1=r_2=\cdots=r_d=p_1$, $r_{d+1}=\cdots=r_{2d}=p_2$, $r_{2d+1}=\cdots=r_{3d}=p_3$ and $r_{3d+1}=\cdots=r_{4d}=p_4$, we write $L^{p_1p_2p_3p_4}(\mathbb{R}^{4d})=L^r(\mathbb{R}^{4d})$.

Let w be a nonnegative measurable function on $\mathbb{R}^n$. We define $L^r_w(\mathbb{R}^n)$ to be the space all f on $\mathbb{R}^n$ for which wf is in $L^r(\mathbb{R}^n)$. $L^r_w(\mathbb{R}^n)$ is a Banach space with norm given by

$$\|f\|_{L^r_w}=\|wf\|_{L^r}.$$

In time-frequency analysis, it is advantageous to consider moderate weight functions w. To define these, let $\mathbb{R}_0^+$ be the set of all nonnegative points in $\mathbb{R}$. Any locally integrable function $v : \mathbb{R}^n \to \mathbb{R}_0^+$ with

$$v(x+y) \leq v(x)v(y)$$

is called submultiplicative. Moreover, if $w : \mathbb{R}^n \to \mathbb{R}_0^+$ is locally integrable with

$$w(x+y) \leq Cw(x)v(y),$$

$C > 0$, and v submultiplicative, then w is called moderate.

The short-time Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to the window $\psi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$V_\psi f(x,\xi) = \mathcal{F}(fT_x\overline{\psi})(\xi) = (f, M_\xi T_x \psi)$$

where M_ξ and T_x denote modulation and translation as defined above.

With $\phi(x) = e^{-\pi\|x\|^2/2}$, w moderate on $\mathbb{R}^{2d}$, and $p, q \in [1,\infty]$, the modulation space $M_w^{pq}(\mathbb{R}^d)$ is the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$V_\phi f \in L_w^{pq}(\mathbb{R}^{2d}).$$

with respective Banach space norm. Clearly, if $w \equiv 1$, then $M_w^{pq}(\mathbb{R}^d) = M^{pq}(\mathbb{R}^d)$. Moreover, for any $s \in \mathbb{R}$ let

$$w_s(x,\xi) = \left(1 + |\xi|^2\right)^{s/2}$$

and denote $M_{w_s}^{pq}(\mathbb{R}^d)$ by $M_s^{pq}(\mathbb{R}^d)$.

Note that replacing the Gaussian function ϕ in the definition of modulation spaces by any other $\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ defines the same space and an equivalent norm, a fact that will be used extensively below.

Recall that the Sobolev space $H_s^p(\mathbb{R}^d)$ consist of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ for which $\|u\|_{H_s^p} = \|T_{w_s}u\|_{L^p} < \infty$ [27]. For any $s \in \mathbb{R}$ and $1 \leq q \leq p \leq r \leq q' \leq \infty$ we have

$$M_s^{pq}(\mathbb{R}^d) \subseteq H_s^r(\mathbb{R}^d), \tag{2.1}$$

and for some $C > 0$,

$$\|f\|_{H_s^r} \leq C\|f\|_{M_s^{pq}}, \quad f \in M_s^{pq}(\mathbb{R}^d).$$

Similarly, $1 \leq q' \leq r \leq p \leq q \leq \infty$ implies

$$H_s^r(\mathbb{R}^d) \subseteq M_s^{pq}(\mathbb{R}^d), \tag{2.2}$$

and for some constant $C > 0$,

$$\|f\|_{M_s^{pq}} \leq C\|f\|_{H_s^r}, \quad f \in H_s^r(\mathbb{R}^d).$$

Let $\mathcal{F}L^p(\mathbb{R}^d)$ be the space of all tempered distributions f in $\mathcal{S}'(\mathbb{R}^d)$ for which there exists a function $h \in L^p(\mathbb{R}^d)$ such that $\hat{h} = f$. Then $\mathcal{F}L^p(\mathbb{R}^d)$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{F}L^p} = \|h\|_{L^p}.$$

The following lemma shows that modulation space norms of compactly supported or bandlimited functions can be estimated using $\mathcal{F}L^p$ and L^p norms respectively [22, 6, 8, 29].

Proposition 2.1. *For $K \subset \mathbb{R}^d$ compact and $p, q \in [1, \infty]$, there are constants $A, B, C, D > 0$ with*

(i) $A\|f\|_{\mathcal{F}L^q} \leq \|f\|_{M^{pq}} \leq B\|f\|_{\mathcal{F}L^q}, \quad f \in \mathcal{S}'(\mathbb{R}^d)$ *with supp* $f \subseteq K$*;*

(ii) $C\|f\|_{L^p} \leq \|f\|_{M^{pq}} \leq D\|f\|_{L^p}, \quad f \in \mathcal{S}'(\mathbb{R}^d)$ *with supp* $\widehat{f} \subseteq K$.

In the following, we shall denote norm equivalences as in statement (i) above by

$$\|f\|_{\mathcal{F}L^q} \asymp \|f\|_{M^{pq}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad \operatorname{supp} f \subseteq K.$$

Similarly, statement (ii) becomes

$$\|f\|_{L^p} \asymp \|f\|_{M^{pq}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad \operatorname{supp} \widehat{f} \subseteq K.$$

The symplectic Fourier transform of $F \in \mathcal{S}(\mathbb{R}^{2d})$ given in (1.1) is a $2d$-dimensional Fourier transform followed by a rotation of phase space by $\frac{\pi}{2}$. This implies that the symplectic Fourier transform shares most properties with the Fourier transform, for example, Proposition 2.1 remains true when replacing the Fourier transform by the symplectic Fourier transform.

Let $p_1, p_2, q_1, q_2 \in [1, \infty]$ and let w be a v–moderate weight function on $\mathbb{R}^{4d}$. The weighted modulation space over phase space $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ is the set of all tempered distributions $F \in \mathcal{S}'(\mathbb{R}^{2d})$ for which $\widetilde{V}_\psi F \in L_w^{p_1p_2q_1q_2}(\mathbb{R}^{4d})$.

Recapitulate that for $F \in \mathcal{S}'(\mathbb{R}^{2d})$, we have $\widetilde{V}_\psi F(x,t,\xi,\nu) = V_\psi F(x,\xi,\nu,-t)$,

$$\begin{aligned}\|F\|_{\widetilde{M}^{p_1p_2q_1q_2}} &= \|\widetilde{V}_\phi F\|_{L^{p_1p_2q_1q_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |\widetilde{V}_\psi F(x,t,\xi,\nu)|^{p_1}\,dx\Big)^{p_2/p_1}dt\Big)^{q_1/p_2}d\xi\Big)^{q_2/q_1}d\nu\Big)^{1/q_1},\end{aligned}$$

and

$$\begin{aligned}\|F\|_{M^{p_1q_1q_2p_2}} &= \|V_\phi F\|_{L^{p_1q_1q_2p_2}} \\ &= \Big(\int\Big(\int\Big(\int\Big(\int |V_\psi F(x,\xi,\nu,t)|^{p_1}\,dx\Big)^{q_1/p_1}d\xi\Big)^{q_2/q_1}d\nu\Big)^{p_1/q_1}dt\Big)^{1/p_1},\end{aligned}$$

with usual adjustments if $p_1 = \infty$, $p_2 = \infty$, $q_1 = \infty$, and/or $q_2 = \infty$. This shows that the definition of $\widetilde{M}^{p_1,p_2,q_1,q_4}(\mathbb{R}^{2d})$ is based on changing the order of integration and on relabeling the integration exponents accordingly. Mixed L^p spaces are sensitive towards the order of integration, and, hence $\widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \not\subseteq M^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ and $M^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \not\subseteq \widetilde{M}^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ in general. But for $1 \le p \le q \le \infty$, Minkowski's inequality

$$\Big(\int\Big(\int |F(x,y)|^p\,dx\Big)^{q/p}dy\Big)^p \le \Big(\int\Big(\int |F(x,y)|^q\,dy\Big)^{p/q}dx\Big)^q$$

(with adjustments for $p = \infty$ and/or $q = \infty$ holds and implies the following.

Proposition 2.2. *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$ and w be a moderate weight function on $\mathbb{R}^{4d}$.*

(a) If $p_2 \le \min\{q_1,q_2\}$, then $M_w^{p_1q_1q_2p_2}(\mathbb{R}^{2d}) \subseteq \widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ and $\|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}} \le \|\sigma\|_{M_w^{p_1q_1q_2p_2}}$.

(b) If $\max\{q_1,q_2\} \le p_2$, then $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \subseteq M_w^{p_1q_1q_2p_2}(\mathbb{R}^{2d})$ and $\|\sigma\|_{M_w^{p_1q_1q_2p_2}} \le \|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}}$.

Note that results similar to ours could also be achieved using symbols in $M_w^{p_3p_4q_3q_4}(\mathbb{R}^{2d})$, but the so obtained results would be weaker and they would necessitate the additional condition $p_4 \le \min\{q_3,q_4\}$.

The modulation space over phase space$\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ shares most of the properties of ordinary modulation spaces. For example, if $p_1 \le \widetilde{p}_1$, $p_2 \le \widetilde{p}_2$, $q_1 \le \widetilde{q}_1$ and $q_2 \le \widetilde{q}_2$, then

$$\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d}) \subseteq \widetilde{M}_w^{\widetilde{p}_1\widetilde{p}_2\widetilde{q}_1\widetilde{q}_2}(\mathbb{R}^{2d}), \tag{2.3}$$

and

$$\|\sigma\|_{\widetilde{M}_w^{\tilde{p}_1,\tilde{p}_2,\tilde{q}_1,\tilde{q}_2}} \le \|\sigma\|_{\widetilde{M}_w^{p_1p_2q_1q_2}}, \quad \sigma \in \widetilde{M}_w^{p_1,p_2,q_2,q_2}(\mathbb{R}^{2d}).$$

Furthermore, let $p_1, p_2, q_1, q_2 \in [1,\infty]$. Then the dual of $\widetilde{M}_w^{p_1p_2q_1q_2}(\mathbb{R}^{2d})$ is $\widetilde{M}_w^{p_1'p_2'q_1'q_2'}(\mathbb{R}^{2d})$ where p_1', p_2', q_1', q_2' are conjugate exponenets of p_1, p_2, q_1, q_2 respectively.

The proofs of these results for modulation spaces over phase space are similar to the ones for the ordinary modulation spaces [14], and are omitted.

3 Comparison of Theorem 1.3 to results in the literature

Cordero and Nicola as well as Toft proved the following theorem on M^{pq}–boundedness for the class of pseudo-differential operators with symbols in $M^{s_1s_1s_2s_2}(\mathbb{R}^{2d})$, see Theorem 5.2 in [5] and Theorem 4.3 in [27].

Theorem 3.1. *Let $p, q, s_1, s_2 \in [1,\infty]$. Then for some $C > 0$,*

$$\|T_\sigma\|_{\mathcal{L}(M^{pq},M^{pq})} \le C\,\|\sigma\|_{M^{s_1,s_1,s_2,s_2}}, \quad \sigma \in M^{s_1,s_1,s_2,s_2}(\mathbb{R}^{2d}), \tag{3.1}$$

if and only if

$$s_2 \le \min\{p, p', q, q', s_1'\}.$$

Roughly speaking, to apply Theorem 3.1, we need to ensure that $\sigma(x,\xi)$ has L^s 'decay' in x and ξ and that $\mathcal{F}\sigma(\nu,-t) = \mathcal{F}_s\sigma(t,\nu)$ has $L^{\min\{p,p',q,q',s'\}}$ 'decay' in t and ν. To apply Theorem 1.3, it suffices to ensure that $\sigma(x,\xi)$ has L^{s_1} 'decay' in x and L^{s_2} 'decay'ξ, and that $\mathcal{F}_s\sigma(t,\nu)$ has $L^{\min\{p,p',s_1'\}}$ 'decay' in t and $L^{\min\{q,q',s_2\}}$ 'decay' in ν.

Using embeddings such as (2.3), we observe that indeed Theorem 3.1 provides boundedness of T_σ if and only if

$$\sigma \in \bigcup_{s=\max\{p,p',q,q'\}}^{\infty} M^{s,s,s',s'} \subseteq \bigcup_{s=\max\{p,p',q,q'\}}^{\infty} \widetilde{M}^{s,s',s,s'} \tag{3.2}$$

while Theorem 1.3 provides boundedness of T_σ if and only if

$$\sigma \in \bigcup_{s_1=\max\{p,p'\}}^{\infty} \bigcup_{s_2=\max\{q,q'\}}^{\infty} \widetilde{M}^{s_1,s_1',s_2,s_2'}.$$

To obtain the set inclusion in (3.2), we used Theorem 2.2 and the fact that $s \geq \max\{p, p'\}$ implies $s \geq 2 \geq s'$.

As $L^2 = M^{2,2}$, Theorem 3.1 implies the following L^2–boundedness result.

Corollary 3.2. *Let $r, s \in [1, \infty]$. Then for some $C > 0$,*

$$\|T_\sigma\|_{\mathcal{L}(L^2,L^2)} \leq C \, \|\sigma\|_{M^{r,r,s,s}}, \quad \sigma \in M^{r,r,s,s}(\mathbb{R}^{2d}),$$

if and only if

$$s \leq \min\{2, r'\}.$$

Corollary 3.2 has been obtained earlier in 2003 by Gröchenig and Heil [15]. As comparison, we formulate the respective consequence of Theorem 1.3.

Corollary 3.3. *For $r, s \in [2, \infty]$, there exists a constant $C > 0$ such that*

$$\|T_\sigma\|_{\mathcal{L}(L^2,L^2)} \leq C \, \|\sigma\|_{\widetilde{M}^{r,r',s,s'}}, \quad \sigma \in \widetilde{M}^{r,r',s,s'}(\mathbb{R}^{2d}).$$

As example, note that Theorem 3.1 does not imply that $T_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded for $\sigma \in M^{\infty,2,2,1}(\mathbb{R}^{2d})$. But as $M^{\infty,2,2,1}(\mathbb{R}^{2d}) \subseteq \widetilde{M}^{\infty,1,2,2}(\mathbb{R}^{2d})$, Theorem 1.3 indeed implies boundedness of T_σ in this case.

For compositions of product and convolution operators, Theorem 1.3 implies the following result.

Corollary 3.4. *For $p, q \in [2, \infty]$, let $h_1 \in M^{p,q'}(\mathbb{R}^d)$ and $h_2 \in M^{p',q}(\mathbb{R}^d)$. Define*

$$Tf = h_1 \cdot (h_2 * f), \quad f \in L^2(\mathbb{R}^d),$$

and

$$Hf = (h_1 \cdot f) * h_2, \quad f \in L^2(\mathbb{R}).$$

Then T and H are bounded operators on L^2 and moreover, there exist positive constants C and C' such that

$$\|T\|_{\mathcal{L}(L^2,L^2)} \leq C \, \|h_1\|_{M^{p,q'}} \|h_2\|_{M^{p',q}},$$

and

$$\|H\|_{\mathcal{L}(L^2,L^2)} \leq C' \, \|h_1\|_{M^{p,q'}} \|h_2\|_{M^{p',q}}.$$

The proof of Corollary 3.4 follows immediately from Corollary 3.3, Lemma 4.9 and Lemma 4.10. Note that not separately, the convolution and multiplication operators above may not be bounded operators.

4 Proof of Theorem 1.3, Corollary 1.4, and Theorem 4.1

4.1 Proof of Theorem 4.1 and thereby of (1.5) and (1.6) implies (1.4) in Theorem 1.3

In this section we prove the weighted version of one implication of Theorem 1.3, that is the following theorem.

Theorem 4.1. *Let w_1, w_2 be moderate weight functions on $\mathbb{R}^{2d}$ and w be a moderate weight function on $\mathbb{R}^{4d}$ that satisfies*

$$w(x,t,\nu,\xi) \le w_1(x-t,\xi)w_2(x,\nu+\xi). \tag{4.1}$$

Let $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1,\infty]$ be such that

$$\begin{aligned} \tfrac{1}{p_1'} + \tfrac{1}{p_2} \le \tfrac{1}{p_3} + \tfrac{1}{p_4}, \quad p_4 \le \min\{p_1', p_2\}, \\ \tfrac{1}{q_1'} + \tfrac{1}{q_2} \le \tfrac{1}{q_3} + \tfrac{1}{q_4}, \quad q_4 \le \min\{q_1', q_2\}. \end{aligned}$$

Then there exists a constant $C > 0$ such that

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1q_1}_{w_1}, M^{p_2q_2}_{w_2})} \le C\,\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}_w}, \quad \sigma \in \widetilde{M}^{p_3p_4q_3q_4}_w(\mathbb{R}^{2d}).$$

To prove Theorem 4.1 we need some preparation. For functions f and g in $\mathcal{S}(\mathbb{R}^d)$, the Rihaczek transform $R(f,g)$ of f and g is defined by

$$R(f,g)(x,\xi) = e^{2\pi i x\cdot\xi}\hat{f}(\xi)\overline{g(x)}.$$

For $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$, pseudo-differential operators are related to Rihaczek transforms by

$$\big(T_\sigma f, g\big) = \big(\sigma, \overline{R(f,g)}\big)$$

for all functions f and g in $\mathcal{S}(\mathbb{R}^d)$. We define A, T_A by

$$(T_A F)(x,t) = F(A(x,t)) = F(x-t,x)\,.$$

Then

$$\overline{R(f,g)}(x,\xi) = \mathcal{F}_{t\to\xi}\big(T_A(\overline{f}\otimes g)(x,\cdot)\big),$$

where

$$\mathcal{F}_{t\to\xi}f(\cdot + x) = \int e^{-2\pi i t\xi} f(t+x)\,dt.$$

Lemma 4.2. *Let φ be a real valued Schwartz function on $\mathbb{R}^d$. Then for all f and g in $\mathcal{S}(\mathbb{R}^d)$*

$$V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi) = \overline{V_\varphi f(x-t,\xi)}\,V_\varphi g(x,\nu+\xi).$$

Proof We compute

$$\begin{aligned}
&V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi)\\
&= \iint e^{-2\pi i(\widetilde{x}\nu+\widetilde{t}\xi)}T_A(\overline{f}\otimes g)(\widetilde{x},\widetilde{t})T_A(\varphi\otimes\varphi)(\widetilde{x}-x,\widetilde{t}-t)\,d\widetilde{x}\,d\widetilde{t}\\
&= \int\Big(\int e^{-2\pi i\widetilde{t}\xi}\overline{f}(\widetilde{x}-\widetilde{t})\varphi(\widetilde{x}-x-\widetilde{t}+t)\,d\widetilde{t}\Big)e^{-2\pi i\widetilde{x}\nu}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\\
&= \iint \overline{f}(s)g(\widetilde{x})e^{-2\pi i\nu\widetilde{x}-2\pi i\xi(\widetilde{x}-s)}\varphi(s-(x-t))\varphi(\widetilde{x}-x)\,d\widetilde{x}\,ds\\
&= \Big(\overline{\int e^{-2\pi i\xi s}f(s)\varphi(s-(x-t))\,ds}\Big)\Big(\int e^{-2\pi i(\nu+\xi)\widetilde{x}}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\Big)\\
&= \overline{V_\varphi f(x-t,\xi)}\;V_\varphi g(x,\nu+\xi).
\end{aligned}$$

□

Lemma 4.3. *Let $\varphi\in\mathcal{S}(\mathbb{R}^d)$ be a nonzero even real valued Schwartz function on $\mathbb{R}^d$. Then for all f and g in $\mathcal{S}(\mathbb{R}^d)$*

$$V_{\overline{R(\varphi,\varphi)}}\overline{R(f,g)}\,(x,\xi,\nu,t) = e^{-2\pi i\xi t}\;V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,-t,\nu,\xi)\,.$$

Proof For all f and g in $\mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}
&V_{\overline{R(\varphi,\varphi)}}\overline{R(f,g)}(x,\xi,\nu,t)\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})}\overline{R(f,g)(\widetilde{x},\widetilde{\xi})}R(\varphi,\varphi)(\widetilde{x}-x,\widetilde{\xi}-\xi)\,d\widetilde{x}\,d\widetilde{\xi}\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})}\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)g(\widetilde{x})\overline{\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}-\xi}\big(\varphi(\widetilde{x}-x-\cdot)\big)}\varphi(\widetilde{x}-x)\,d\widetilde{x}\,d\widetilde{\xi}\\
&= \iint e^{-2\pi i(\nu\widetilde{x}+t\widetilde{\xi})}\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)g(\widetilde{x})\mathcal{F}_{\widetilde{t}\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big)\varphi(\widetilde{x}-x)\,d\widetilde{x}\,d\widetilde{\xi}.
\end{aligned}\tag{4.2}$$

On the other hand, Parseval's identity gives

$$\begin{aligned}
&V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(x,t,\nu,\xi)\\
&= \iint e^{-2\pi i(\widetilde{x}\nu+\widetilde{t}\xi)}T_A(\overline{f}\otimes g)(\widetilde{x},\widetilde{t})T_A(\varphi\otimes\varphi)(\widetilde{x}-x,\widetilde{t}-t)\,d\widetilde{x}\,d\widetilde{t}\\
&= \int\Big(\int e^{-2\pi i\widetilde{t}\xi}\overline{f}(\widetilde{x}-\widetilde{t})\varphi(\widetilde{x}-x-\widetilde{t}+t)\,d\widetilde{t}\Big)e^{-2\pi i\widetilde{x}\nu}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\\
&= \iint \mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)\mathcal{F}^{-1}_{\widetilde{t}\to\widetilde{\xi}}\big(e^{-2\pi i\widetilde{t}\xi}\varphi(\widetilde{x}-x+t-\cdot)\big)e^{-2\pi i\widetilde{x}\nu}g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{\xi}\,d\widetilde{x}.
\end{aligned}$$

But,

$$\mathcal{F}^{-1}_{\widetilde{t}\to\widetilde{\xi}}\big(e^{-2\pi i\widetilde{t}\xi}\varphi(\widetilde{x}-x+t-\cdot)\big)=e^{-2\pi it(\xi-\widetilde{\xi})}\mathcal{F}_{\gamma\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big),$$

therefore,

$$\begin{aligned}
V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(x,t,\nu,\xi) \;=\; & e^{-2\pi it\xi}\iint e^{2\pi i(t\widetilde{\xi}-v\widetilde{x})}\mathcal{F}_{\widetilde{t}\to\widetilde{\xi}}\big(\overline{f}(\widetilde{x}-\cdot)\big)\;\cdot\\
&\mathcal{F}_{\widetilde{t}\to\xi-\widetilde{\xi}}\big(\varphi(\widetilde{x}-x-\cdot)\big)g(\widetilde{x})\varphi(\widetilde{x}-x)\,d\widetilde{x}\,d\widetilde{\xi}.
\end{aligned}$$

Combining this identity with (4.2) completes the proof. □

Proposition 4.4. *Let w_1, w_2, w be moderate functions that satisfy*

$$w(x,t,\nu,\xi)\le w_1(x-t,\xi)w_2(x,\nu+\xi).$$

Let φ be a nonzero real valued Schwartz function on $\mathbb{R}^d$ and define

$$\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\xi,\nu)=V_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\nu,\xi) \tag{4.3}$$

for all $f,g\in\mathcal{S}(\mathbb{R}^d)$ and $x,t,\xi,\nu\in\mathbb{R}^d$. If $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4\in[1,\infty]$ satisfy

$$\begin{aligned}
&\tfrac{1}{p_1}+\tfrac{1}{p_2}=\tfrac{1}{p_3}+\tfrac{1}{p_4},\quad p_3\le\min\{p_1,p_2,p_4\},\\
&\tfrac{1}{q_1}+\tfrac{1}{q_2}=\tfrac{1}{q_3}+\tfrac{1}{q_4},\quad q_3\le\min\{q_1,q_2,q_4\},
\end{aligned} \tag{4.4}$$

then

$$\|\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\|_{L^{p_3p_4q_3q_4}_w}\le\|f\|_{M^{p_1q_1}_{w_1}}\|g\|_{M^{p_2\cdot q_2}_{w_2}}.$$

Proof By Lemma 4.2, we have

$$\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\,(x,t,\xi,\nu) = \overline{V_\varphi f(x-t,\xi)}\; V_\varphi g(x,\nu+\xi).$$

So, by (4.1), for $t,\xi,\nu\in\mathbb{R}^d$,

$$\begin{aligned}
&\|w(\cdot,t,\xi,\nu)\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(\cdot,t,\xi,\nu)\|_{L^{p_3}}\\
&\quad\le\ \Big(\int |w_1(x-t,\xi)\Big(V_\varphi f\Big)(x-t,\xi)|^{p_3}|w_2(x,\nu+\xi)\Big(V_\varphi g\Big)(x,\nu+\xi)|^{p_3}\,dx\Big)^{1/p_3}\\
&\quad=\ \Big(|w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3} * |w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\,(t)\Big)^{1/p_3}.
\end{aligned}$$

Then, (4.4) implies

$$\frac{1}{r_1}+\frac{1}{s_1} = 1+\frac{1}{a_1},$$

with $r_1 = p_2/p_3 \ge 1$, $s_1 = p_1/p_3 \ge 1$ and $a_1 = p_4/p_3 \ge 1$, hence, we can apply Young's inequality and obtain

$$\begin{aligned}
&\|w(\cdot,\cdot,\xi,\nu)\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)(\cdot,\cdot,\xi,\nu)\|_{L^{p_3,p_4}}\\
&\quad=\ \||w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3} * |w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\|_{L^{a_1}}^{1/p_3}\\
&\quad\le\ \||w_2(\cdot,\nu+\xi)V_\varphi g(\cdot,\nu+\xi)|^{p_3}\|_{L^{r_1}}^{1/p_3}\ \ \||w_1(\cdot,\xi)V_\varphi f(\cdot,\xi)|^{p_3}\|_{L^{s_1}}^{1/p_3}. \quad (4.5)
\end{aligned}$$

To estimate (4.5) further, we note that integrating with respect to ξ can be again considered a convolution. In fact (4.4) leads to

$$\frac{1}{r_2}+\frac{1}{s_2} = 1+\frac{1}{a_2},$$

where $r_2 = q_2/q_3$, $s_2 = q_1/q_3$ and $a_2 = q_4/q_3$. Young's inequality then implies

$$\begin{aligned}
&\|w\mathcal{V}_{T_A(\varphi\otimes\varphi)}T_A(\overline{f}\otimes g)\|_{L^{p_3p_4q_3q_4}}\\
&\quad\le\ \Big(\int\Big(\int |w_2(x,y)V_\varphi g(x,y)|^{p_3r_1}\,dx\Big)^{(r_2q_3)/(p_3r_1)}dy\Big)^{1/(r_2q_3)}\cdot\\
&\qquad\Big(\int\Big(\int |w_1(x,y)V_\varphi f(x,y)|^{p_3s_1}\,dx\Big)^{(s_2q_3)/(p_3s_1)}dy\Big)^{(1/s_2q_3)}\\
&\quad=\ \|f\|_{M^{p_1q_1}_{w_1}}\|g\|_{M^{p_2q_2}_{w_2}},
\end{aligned}$$

which completes the proof. □

Now, we are ready to give sufficient conditions on the boundedness of pseudo-differential operators with symbols in $\widetilde{M}^{p_3,p_4,q_3,q_4}(\mathbb{R}^{2d})$.

Lemma 4.5. *Let w_1, w_2, w be moderate weight functions that satisfy (4.1). Let $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1, \infty]$ be such that*

$$\begin{aligned} \tfrac{1}{p_3} &\in \left[\tfrac{1}{p_1'} + \tfrac{1}{p_2} - \tfrac{1}{p_4}, \min\{\tfrac{1}{p_1'}, \tfrac{1}{p_2}, \tfrac{1}{p_4}\} \right], \\ \tfrac{1}{q_3} &\in \left[\tfrac{1}{q_1'} + \tfrac{1}{q_2} - \tfrac{1}{q_4}, \min\{\tfrac{1}{q_1'}, \tfrac{1}{q_2}, \tfrac{1}{q_4}\} \right]. \end{aligned} \tag{4.6}$$

Then there exists a constant $C > 0$ such that

$$\|T_\sigma\|_{\mathcal{L}(M^{p_1q_1}_{w_1}, M^{p_2q_2}_{w_2})} \le C \, \|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}_w}, \quad \sigma \in \widetilde{M}^{p_3p_4q_3q_4}_w(\mathbb{R}^{2d}). \tag{4.7}$$

Proof Let us first assume $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1, \infty]$ satisfy (4.6) and in addition

$$\frac{1}{p_1'} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1'} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{4.8}$$

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Since the dual of $\widetilde{M}^{p_3p_4q_3q_4}_w(\mathbb{R}^{2d})$ is $\widetilde{M}^{p_3',p_4',q_3',q_4'}_w(\mathbb{R}^{2d})$, it follows that

$$\begin{aligned} |(T_\sigma f, g)| &= |(\sigma, \overline{R(f,g)})| \\ &\le \|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}_w} \|\overline{R(f,g)}\|_{\widetilde{M}^{p_3',p_4',q_3',q_4'}_w}. \end{aligned}$$

To obtain (4.7), it is enough to show that there exists $C > 0$ such that

$$\|\overline{R(f,g)}\|_{\widetilde{M}^{p_3',p_4',q_3',q_4'}_w} \le C \, \|f\|_{M^{p_1q_1}_{w_1}} \|g\|_{M^{p_2',q_2'}_{w_2}}.$$

Let φ be a nonzero real valued even function in $\mathcal{S}(\mathbb{R}^d)$. Then by Lemma 4.3,

$$\begin{aligned} \left| \mathcal{V}_{\overline{R(\varphi,\varphi)}} \overline{R(f,g)} \, (x,t,\xi,\nu) \right| &= \left| V_{\overline{R(\varphi,\varphi)}} \overline{R(f,g)} \, (x,\xi,\nu,-t) \right| \\ &= \left| V_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g) \, (x,t,\nu,\xi) \right| \\ &= \left| \mathcal{V}_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g) \, (x,t,\xi,\nu) \right|. \end{aligned}$$

where $\mathcal{V}_{T_A(\varphi\otimes\varphi)}$ is defined in (4.3). Therefore, by Proposition 4.4, we have

$$\begin{aligned} \|R(f,g)\|_{\widetilde{M}^{p_3',p_4',q_3',q_4'}_w} &= \|\mathcal{V}_{T_A(\varphi\otimes\varphi)} T_A(\overline{f}\otimes g)\|_{L^{p_3',p_4',q_3',q_4'}_w} \\ &\le \|f\|_{M^{p_1q_1}_{w_1}} \|g\|_{M^{p_2',q_2'}_{w_2}}. \end{aligned}$$

To obtain (4.7) in the general case, that is $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [1, \infty]$ satisfy (4.6) but not necessarily (4.8), set

$$\frac{1}{\widetilde{p}_2} = \frac{1}{p_3} + \frac{1}{p_4} - \frac{1}{p_1'} \quad \text{and} \quad \frac{1}{\widetilde{q}_2} = \frac{1}{q_3} + \frac{1}{q_4} - \frac{1}{q_1'}.$$

Then it is easy to see that $\widetilde{p}_2 \leq p_2$, $\widetilde{q}_2 \leq q_2$ and $p_1, \widetilde{p}_2, p_3, p_4, q_1, \widetilde{q}_2, q_3, q_4 \in [1, \infty]$ satisfy (4.6). Hence

$$\|T_\sigma f\|_{M^{p_2 q_2}_{w_2}} \leq C \, \|T_\sigma f\|_{M^{\tilde{p}_2, \tilde{q}_2}_{w_2}} \leq \|f\|_{M^{p_1 q_1}_{w_1}} \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}_{w}},$$

for some $C > 0$. □

Proof of Theorem 4.1: Let $f \in \mathcal{S}(\mathbb{R}^d)$. Set

$$\frac{1}{\widetilde{p}_3} = \frac{1}{p_1'} + \frac{1}{p_2} - \frac{1}{p_4} \quad \text{and} \quad \frac{1}{\widetilde{q}_3} = \frac{1}{q_1'} + \frac{1}{q_2} - \frac{1}{q_4}.$$

Then it is easy to see that

$$\widetilde{p}_3 \geq p_3, \quad \widetilde{q}_3 \geq q_3.$$

Furthermore, $\{p_1, p_2, \widetilde{p}_3, p_4, q_1, q_2, \widetilde{q}_3, q_4\}$ satisfies (4.6), therefore there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} \|T_\sigma f\|_{M^{p_2 q_2}_{w_2}} &\leq C_1 \|f\|_{M^{p_1 q_1}_{w_1}} \|\sigma\|_{\widetilde{M}^{\tilde{p}_3, p_4, \tilde{q}_3, q_4}_{w}} \\ &\leq C_2 \|f\|_{M^{p_1 q_1}_{w_1}} \|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}_{w}}. \end{aligned}$$

4.2 Proof of Corollary 1.4

Let $1 \leq p, q \leq 2$. By Theorem 1.3, $T_\sigma : M^{p,p'} \to M^q$ is bounded. Using the bounded embeddings $M^p \subset L^p \subset M^{p,p'}$ for all $1 \leq p \leq 2$ (for more details see [8]), it follows that $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded. Similarly, using $M^{p,p'} \subset L^p \subset M^p$ for all $q \geq 2$, we can prove $T_\sigma : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded for p, p_3, p_4, q, q_3, q_4 satisfying (b) or (c) or (d) in Corollary 1.4. □

4.3 Proof of (1.4) implies (1.5), (1.6) in Theorem 1.3

To show necessity of (1.5) and (1.6) in Theorem 1.3, we shall use two mixed L^p norms on phase space, namely,

$$\|F\|_{L^{pq}} = \Big(\int \Big(\int |F(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q},$$

and

$$\|F\|_{\widetilde{L}^{pq}} = \Big(\int \Big(\int |F(x,\xi)|^q \, d\xi \Big)^{p/q} \, dx \Big)^{1/p},$$

for $p, q \in [1, \infty)$. For $p = \infty$ and/or $q = \infty$ we make the usual adjustment.

Similarly, we can define $\widetilde{M}^{pq}(\mathbb{R}^d)$ to be the space of all functions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{\widetilde{M}^{pq}} = \|V_\varphi f\|_{\widetilde{L}^{pq}} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Note that it can be easily checked that

$$\|f\|_{\widetilde{M}^{pq}} = \|\widehat{f}\|_{M^{qp}}.$$

Below, we use an idea from the proof of Proposition 2.1 given in [22] to prove the following lemma.

Lemma 4.6. *Let $K \subset \mathbb{R}^{2d}$ be compact. Then*

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}} \asymp \|\sigma\|_{\mathcal{F}\widetilde{L}^{q_4 p_4}}, \quad \sigma \in \mathcal{S}'(\mathbb{R}^{2d}), \quad \operatorname{supp} \sigma \subset K.$$

Proof Choose $r > 0$ with $\operatorname{supp} \sigma \subseteq B_r^{2d}(0)$, where

$$B_r^{2d}(0) = \{x \in \mathbb{R}^{2d} : \|x\| \le r\}$$

is the Euclidean unit ball in $\mathbb{R}^{2d}$ with center 0, radius r and Lebesgue measure $|B_r^{2d}(0)|$. Let $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ with *supp* $\psi \subset B_r^{2d}(0)$. Then it is easy to see that

$$\big|\widetilde{V}_\psi \sigma\big|(x,t,\xi,\nu) = \big|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}\big|(x,\xi),$$

where

$$\widetilde{\psi}(x,\xi) = \psi(-x,-\xi).$$

Therefore, for fixed t, ν we have

$$\begin{aligned} \operatorname{supp}\Big(\Big|\widetilde{V}_\psi \sigma\Big| (\cdot,t,\cdot,\nu)\Big) &\subseteq \operatorname{supp}(\sigma) + \mathit{supp}\ (M_{\nu,-t}\overline{\widetilde{\psi}}) \\ &\subseteq B_r^{2d}(0) + B_r^{2d}(0) \subseteq B_{2r}^{2d}(0). \end{aligned} \tag{4.9}$$

Let $\xi \in B_{2r}^d(0)$. Then by (4.9),

$$\begin{aligned} \|\widetilde{V}_\psi \sigma(\cdot,t,\xi,\nu)\|_{L^{p_3}(\mathbb{R}^d)}^{p_3} &= \int_{B_{2r}^d(0)} \big|\widetilde{V}_\psi \sigma\big|^{p_3}(x,t,\xi,\nu)\, dx \\ &\le |B_{2r}^d(0)| \, \big\|\widetilde{V}_\psi \sigma(\cdot,t,\xi,\nu)\big\|_{L^\infty} = |B_{2r}^d(0)| \, \big\|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}(\cdot,\xi)\big\|_{L^\infty} \\ &\le |B_{2r}^d(0)| \, \big\|\sigma * M_{\nu,-t}\overline{\widetilde{\psi}}\big\|_{L^\infty} \le |B_{2r}^d(0)| \, \big\|\mathcal{F}^{-1}\big(\widehat{\sigma}\, T_{\nu,-t}\overline{\widehat{\psi}}\big)\big\|_{L^\infty} \\ &\le |B_{2r}^d(0)| \, \big\|\widehat{\sigma} T_{\nu,-t}\overline{\widehat{\psi}}\big\|_{L^1} = |B_{2r}^d(0)| \, \big(|\widehat{\sigma}| * |\widehat{\psi}|\big)(-\nu,t) \end{aligned} \tag{4.10}$$

On the other hand, if $\xi \in \mathbb{R}^d \setminus B^d_{2r}(0)$, then by (4.9),

$$\|\widetilde{V}_\psi \sigma(\cdot, t, \xi, \nu)\|_{L^{p_3}} = 0. \tag{4.11}$$

Therefore, (4.10) and (4.11) imply

$$\begin{aligned}
\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}} &= \left\|\widetilde{V}_\psi \sigma\right\|_{L^{p_3 p_4 q_3 q_4}} \\
&\leq |B^d_{2r}(0)|^{1/p_3} \int \Big(\int_{B^d_{2r}(0)} \Big(\int \Big((|\widehat{\sigma}| * |\widehat{\psi}|(-\nu, t))^{p_4}\, dt \Big)^{q_3/p_4} d\xi \Big)^{q_4/q_3} d\nu \Big)^{1/q_4} \\
&\leq |B^d_{2r}(0)|^{(1/p_3)+(1/q_3)} \Big(\int \Big(\int (|\widehat{\sigma}| * |\widehat{\psi}|(-\nu, t))^{p_4}\, dt \Big)^{q_4/p_4} d\nu \Big)^{1/q_4} \\
&\leq |B^d_{2r}(0)|^{(1/p_3)+(1/q_3)} \left\| |\widehat{\sigma}| * |\widehat{\psi}| \right\|_{\widetilde{L}^{q_4, p_4}} \\
&\leq |B^d_{2r}(0)|^{(1/p_3)+(1/q_3)} \|\widehat{\sigma}\|_{\widetilde{L}^{q_4,p_4}} \|\widehat{\psi}\|_{\widetilde{L}^{1,1}} \leq C \|\widehat{\sigma}\|_{\widetilde{L}^{q_4,p_4}}.
\end{aligned}$$

Now, let $\psi \in C^\infty(\mathbb{R}^{2d})$ be compactly supported with $\psi \equiv 1$ on $B^{2d}_{2r}(0)$. Let $\chi_{B^{2d}_r(0)}$ be the characteristic function on $B^{2d}_r(0)$. Then using *supp* $\sigma \subseteq B^{2d}_r(0)$, it follows that for all $x, t, \xi, \nu \in \mathbb{R}^d$,

$$\begin{aligned}
&\chi_{B^{2d}_{2r}(0)}(x, \xi)\ \widetilde{V}_\psi \sigma(x, t, \xi, \nu) \\
&\quad = \chi_{B^{2d}_{2r}(0)}(x, \xi) \int_{B^{2d}_r(0)} \sigma(\widetilde{x}, \widetilde{\xi})\, e^{-2\pi i(\widetilde{x}\nu - \widetilde{\xi} t)}\, \psi(\widetilde{x} - x, \widetilde{\xi} - \xi)\, d\widetilde{x}\, d\widetilde{\xi} \\
&\quad = \chi_{B^{2d}_{2r}(0)}(x, \xi) \int_{B^{2d}_r(0)} \sigma(\widetilde{x}, \widetilde{\xi})\, e^{-2\pi i(\widetilde{x}\nu - \widetilde{\xi} t)}\ d\widetilde{x}\, d\widetilde{\xi} \\
&\quad = \chi_{B^{2d}_{2r}(0)}(x, \xi) \mathcal{F}\sigma(\nu, -t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}} = \left\|\widetilde{V}_\psi \sigma\right\|_{L^{p_3 p_4 q_3 q_4}} &\geq \left\|\chi_{B^{2d}_{2r}(0)}\, \widetilde{V}_\psi \sigma\right\|_{L^{p_3 p_4 q_3 q_4}} \\
&= \Big(\int \Big(\int \Big(\int \Big(\int \left|\chi_{B^{2d}_{2r}(0)}(x, \xi)\, \mathcal{F}\sigma(\nu, -t)\right|^{p_3} dx \Big)^{p_4/p_3} dt \Big)^{q_3/p_4} d\xi \Big)^{q_4/q_3} d\nu \Big)^{1/q_4} \\
&= \left\|\chi_{B^{2d}_{2r}(0)}\right\|_{L^{p_3 q_3}}\ \|\sigma\|_{\mathcal{F}\widetilde{L}^{q_4, p_4}}
\end{aligned}$$

which completes the proof. □

Lemma 4.7. *Let $\lambda > 0$ and $\varphi_\lambda(x) = e^{-\pi\lambda|x|^2}$. Then for $\lambda \geq 1$,*

$$\|\varphi_\lambda\|_{M^{pq}} \asymp \|\varphi_\lambda\|_{\widetilde{M}^{pq}} \asymp \lambda^{-d/q'},$$

and

$$\|\varphi_{\lambda^{-1}}\|_{M^{pq}} \asymp \|\varphi_{\lambda^{-1}}\|_{\widetilde{M}^{pq}} \asymp \lambda^{d/p}.$$

The proof of Lemma 4.7 is an immediate corollary of Lemma 3.2 in [4] and is omitted here.

Lemma 4.8. *Let $K \subset \mathbb{R}^d$ be compact. For $h \in C^\infty(\mathbb{R}^d)$ and $\lambda \geq 1$ set $h_\lambda(x) = h(x)e^{-\pi i\lambda|x|^2}$. Then for all $p, q \in [1, \infty]$,*

$$\|h_\lambda\|_{M^{pq}} \asymp \|\widehat{h}_\lambda\|_{L^q} \asymp \lambda^{d/q-d/2}, \quad h \in C^\infty(\mathbb{R}^d), \quad \operatorname{supp} h \subset K.$$

Lemma 4.8 is well known and its proof can be found in, for example, [5].

Lemma 4.9. *Let $h_1, h_2 \in \mathcal{S}(\mathbb{R}^d)$ and*

$$\eta(t, \nu) = e^{-2\pi it\nu} h_2(t)\widehat{h}_1(\nu), \quad t, \nu \in \mathbb{R}^d.$$

If $\sigma = \widetilde{\mathcal{F}}\eta$. Then we have

$$\sigma(x, \xi) = (M_\xi h_2 * h_1)(x) \tag{4.12}$$

and

$$T_\sigma f = (h_1 f) * h_2, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{4.13}$$

Moreover,

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}} \|h_2\|_{M^{p_4,q_3}}.$$

Proof Clearly, (4.12) and (4.13) hold. Now, let φ be any nonzero real valued Schwartz function on $\mathbb{R}^d$. Let

$$\psi(t, \nu) = \varphi(t)\varphi(\nu)e^{-2\pi it\nu}.$$

and define

$$\widetilde{\psi}(x, \xi) = \widetilde{\mathcal{F}}\psi(-x, -\xi).$$

Then

$$\begin{aligned} \left|\widetilde{V}_{\widetilde{\psi}}\sigma(x, t, \xi, \nu)\right| &= \left|\left(\sigma, M_{\nu,-t}T_{x,\xi}\widetilde{\psi}\right)\right| \\ &= \left|\left(\widetilde{\mathcal{F}}\eta, \widetilde{\mathcal{F}}(T_{-t,\nu}M_{-\xi,x}\psi)\right)\right|. \end{aligned}$$

Now since $\widetilde{\mathcal{F}}$ is a unitary operator, it follows that

$$\begin{aligned}
\left|\left(\widetilde{V}_{\widetilde{\psi}}\sigma\right)(x,t,\xi,\nu)\right| &= \left|\left(\eta, T_{-t,\nu}M_{-\xi,x}\psi\right)\right| \\
&= \left|\iint \eta(\widetilde{t},\widetilde{\nu})\, e^{2\pi i\xi(\widetilde{t}+t)}\, e^{-2\pi i x(v-\widetilde{\nu})}\overline{\psi}(t+\widetilde{t},\widetilde{\nu}-\nu)\, d\widetilde{t}\, d\widetilde{\nu}\right| \\
&= \left|\iint \widehat{h}_1(\widetilde{\nu})h_2(\widetilde{t})\varphi(\widetilde{\nu}-\nu)\varphi(\widetilde{t}+t)\, e^{-2\pi i\widetilde{\nu}(x-t)}\, e^{2\pi i\widetilde{t}(\nu-\xi)}\, d\widetilde{t}\, d\widetilde{\nu}\right| \\
&= |(V_\varphi\widehat{h}_1)(\nu,x-t)|\ |(V_\varphi h_2)(-t,\nu-\xi)|.
\end{aligned}$$

Hence,

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}}\|h_2\|_{M^{p_4,q_3}}.$$

□

Similarly, we can prove the following.

Lemma 4.10. *Let Let $h_1, h_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma = h_1 \otimes \widehat{h}_2$. Then*

$$T_\sigma f = h_1 \cdot (h_2 * f), \quad f \in \mathcal{S}(\mathbb{R}^d)$$

and

$$\|h_1 \otimes \widehat{h}_2\|_{\widetilde{M}^{p_3p_4q_3q_4}} = \|h_1\|_{M^{p_3,q_4}}\|\widehat{h}_2\|_{\widetilde{M}^{q_3,p_4}}.$$

Proof of (1.4) implies (1.5) and (1.6) in Theorem 1.3: Let $h \in C^\infty(\mathbb{R}^d)$ be chosen with compact support and $h(0) = 1$ and $h(x) \geq 0$ for all $x \in \mathbb{R}^d$. Then for any $\lambda \geq 1$, we define h_λ and σ_λ respectively by

$$h_\lambda(x) = h(x)e^{-\pi i\lambda|x|^2}.$$

and

$$\sigma_\lambda(x,\xi) = h\otimes h_\lambda(x,\xi) = h(x)h_\lambda(\xi).$$

Let $f_\lambda = \mathcal{F}^{-1}\overline{h_\lambda}$. Then $f_\lambda \in \mathcal{S}(\mathbb{R}^d)$ and

$$T_{\sigma_\lambda}f_\lambda(x) = \int e^{2\pi ix\xi}h(x)|h(\xi)|^2\, d\xi.$$

So, $T_{\sigma_\lambda}f_\lambda$ is independent of λ. Since σ_λ has compact support, by Lemma 4.6 and Lemma 4.8

$$\begin{aligned}
\|\sigma_\lambda\|_{\widetilde{M}^{p_3p_4q_3q_4}} &\asymp \|\mathcal{F}\sigma_\lambda\|_{\widetilde{L}^{q_4,p_4}} \\
&= \|\widehat{h}\|_{L^{q_4}(\mathbb{R}^d)}\|\widehat{h}_\lambda\|_{L^{p_4}(\mathbb{R}^d)} \\
&\asymp \lambda^{(d/p_4)-(d/2)}. \qquad (4.14)
\end{aligned}$$

Moreover, by Lemma 2.1 and Lemma 4.8, since $\mathcal{F}f_\lambda$ has compact support,

$$\|f_\lambda\|_{M^{p_1q_1}(\mathbb{R}^d)} = \|f_\lambda\|_{L^{p_1}(\mathbb{R}^d)} \asymp \lambda^{(d/p_1)-(d/2)}. \tag{4.15}$$

Hence by (1.4), (4.14) and (4.15), there exists $C > 0$ such that for all $\lambda \geq 1$

$$\|T_\sigma f_\lambda\|_{M^{p_2q_2}(\mathbb{R}^d)} \leq C\, \lambda^{(d/p_4)+(d/p_1)-d}.$$

But $\|T_\sigma f_\lambda\|_{M^{p_2q_2}(\mathbb{R}^d)}$ is nonzero and independent of λ, therefore $\frac{d}{p_4}+\frac{d}{p_1}-d \geq 0$, and $p_4 \leq p_1'$.
To prove $q_4 \leq q_1'$, we let $h_1 = \overline{f} = h_\lambda$ and $h_2 \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{h}_2$ is compactly supported, independent of λ and

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \neq 0.$$

Let $\sigma = \widetilde{\mathcal{F}}\eta$ where

$$\eta(t,\nu) = \widehat{h}_1(\nu) h_2(t) e^{-2\pi i t\nu}.$$

Then by Lemma 4.9 and (1.4)

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \leq C\, \|\widehat{h}_1\|_{L^{q_4}(\mathbb{R}^d)} \|h_2\|_{L^{p_4}(\mathbb{R}^d)} \|\widehat{f}\|_{L^{q_1}(\mathbb{R}^d)},$$

for some constant $C > 0$. So, by Lemma 4.8 for all $\lambda \geq 1$

$$\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)} \leq C\, \lambda^{(d/q_4)-(d/2)} \lambda^{(d/q_1)-(d/2)},$$

but $\|(h_1 f) * h_2\|_{L^{p_2}(\mathbb{R}^d)}$ is nonzero and independent of λ, therefore $(d/q_4) + (d/q_1) - d \geq 0$ and, hence, $q_4 \leq q_1'$.

Now, let $h_1 = f = \varphi_\lambda$ and $h_2 = \varphi_{\lambda^{-1}}$, where φ_λ and $\varphi_{\lambda^{-1}}$ are defined in Lemma 4.7. If we let $\sigma = h_1 \otimes h_2$. Then by Lemma 4.7 and Lemma 4.10, for $\lambda \geq 1$ we have

$$\|\sigma\|_{\widetilde{M}^{p_3p_4q_3q_4}(\mathbb{R}^{2d})} \asymp \lambda^{d/q_3 - d/q_4'}$$

and $\|f\|_{M^{p_1q_1}(\mathbb{R}^d)} \asymp \lambda^{-d/q_1'}$. On the other hand $T_\sigma f$ is also a Gaussian function and it can be easily checked that

$$\|T_\sigma f\|_{M^{p_2q_2}(\mathbb{R}^d)} \asymp \lambda^{-d/q_2'}.$$

Therefore by (1.4)

$$\lambda^{d/q_3 - d/q_4' - d/q_1' + d/q_2'} \geq 1$$

for all $\lambda \geq 1$. Hence, we get

$$\frac{1}{q_1'} + \frac{1}{q_2} \leq \frac{1}{q_3} + \frac{1}{q_4}.$$

Similarly, by letting $h_1 = f = \varphi_{\lambda^{-1}}$ and $h_2 = \varphi_\lambda$, we get

$$\frac{1}{p_1'} + \frac{1}{p_2} \leq \frac{1}{p_3} + \frac{1}{p_4}.$$

Again assume σ has the form given in Lemma 4.9. Let $h(x) = f(x) = e^{-\pi|x|^2/2}$ and $h_2 = \varphi_{\lambda^{-1}}$. Then T_σ is also a Gaussian function, moreover by Lemma 4.7 and (1.4) for all $\lambda \geq 1$

$$\lambda^{d/p_4 - d/p_2} \geq C,$$

for some $C > 0$. Hence $p_4 \leq p_2$.
To prove $q_4 \leq q_2$, we let

$$\sigma(x, \xi) = e^{2\pi i x\xi} h_1(x) h_2(\xi),$$

where h_1 and h_2 are compactly supported Schwartz functions on $\mathbb{R}^d$. Then σ is compactly supported and therefore by Lemma 4.6,

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})} = \|\mathcal{F}\sigma\|_{L^{p_4, q_4}(\mathbb{R}^{2d})}.$$

On the other hand, by an easy calculation, we have

$$\big|\mathcal{F}\sigma\big|(\nu, t) = \big|V_{h_1}\widehat{h}_2\big|(t, \nu) = \big|V_{\widehat{h_2}}\overline{h_1}\big|(t, \nu).$$

Therefore,

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})} \leq C_{h_2} \|\widehat{h}_1\|_{L^{q_4}(\mathbb{R}^d)}, \tag{4.16}$$

and

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})} \leq C_{h_1} \|\widehat{h}_2\|_{L^{p_4}(\mathbb{R}^d)},$$

where C_{h_1} and C_{h_2} are positive constants depending on h_1 and h_2 respectively. Let $h_1 = h_\lambda$ and h_2 be any compactly supported function and f be a Schwartz function on $\mathbb{R}^d$ and both h_2 and $\widehat{f}$ be independent of λ such that $(h_2, \overline{\widehat{f}}) \neq 0$. Then

$$\begin{aligned} \|T_\sigma f\|_{M^{p_2 q_2}(\mathbb{R}^d)} &= \|h_1\|_{M^{p_2 q_2}(\mathbb{R}^d)} |(h_2, \overline{\widehat{f}})| \\ &= |(h_2, \overline{\widehat{f}})| \|\widehat{h}_1\|_{L^{q_2}(\mathbb{R}^d)} \asymp \lambda^{(d/q_2) - (d/2)}, \end{aligned} \tag{4.17}$$

and by (4.16)

$$\|\sigma\|_{\widetilde{M}^{p_3 p_4 q_3 q_4}(\mathbb{R}^{2d})} \leq C_{h_2}\, \lambda^{(d/q_4)-(d/2)}.$$

Hence, (4.17) and (1.4) imply

$$\lambda^{(d/q_4)-(d/q_2)} \geq C,$$

where $C > 0$ is independent of $\lambda \geq 1$. Hence $(d/q_4) - (d/q_2) \geq 0$ which implies that $q_4 \leq q_2$. □

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